ON THE ABSOLUTE SUMMABILITY OF FOURIER SERIES (II)

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1. Introduction. Let $f(t)$ be a function integrable L over the interval $(0,2\pi)$ and periodic with period 2π. Let its Fourier series be

$$\frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) \equiv \sum_{n=0}^{\infty} c_n(t). \tag{1.1}$$

And let $\varphi_x(t)$ denote

$$\frac{1}{2} \{f(x+t) + f(x-t)\},$$

then

$$\varphi_x(t) \sim \sum_{n=0}^{\infty} c_n(x) \cos nt. \tag{1.2}$$

If we denote by $s_n^\alpha(t)$ the nth (C, α) mean $\alpha > -1$, of the sequence

$$s_n(t) = s_n^0(t) = \sum_{\nu=0}^{n} c_\nu(t).$$

Following T. M. Flett [5], the Fourier series (1.1) is called summable $|C, \alpha|_k$ at the point $t = x$, where $\alpha > -1$ and $k \geqq 1$, if the series

$$\sum_{n=1}^{\infty} n^{-1} |s_n^\alpha(x) - s_n^{\alpha-1}(x)|^k \tag{1.3}$$

is convergent.

About this summability, T. Tsuchikura and the author [9] essentially obtained the following theorem.

THEOREM A. *If* $1 < p \leqq 2$, $f(t)$ *is integrable* L^p *throughout the interval* $(0, 2\pi)$ *and for* $k \geqq 1$

$$\sum_{n=0}^{\infty} \left(\int_{\pi/2^{n+1}}^{\pi/2^n} \frac{|\varphi_x(t)|^p}{t} dt \right)^{k/p} < \infty \tag{1.4}$$

then the Fourier series (1.1) *is summable* $|C, \alpha|_k$ *at the point* $t = x$, *where* $\alpha > \sup(1/p, 1/k')$.

If the condition

$$\Phi_x^{(p)}(t) = \int_0^t |\varphi_x(u)|^p du = O\left\{t\Big/\left(\log \frac{1}{t}\right)^{p/k+\varepsilon}\right\},\ \varepsilon > 0, \tag{1.5}$$

is satisfied, then the condition (1.4) holds.[1] But the condition $f(t) \in L^p(0, 2\pi)$ is indispensable.

On the orther hand, if $f(t) \in L^p(0, 2\pi)$, $1 < p \leqq 2$ and $1 < k \leqq 2$, we have the following properties:

(i) *For $p \geqq k$ and $\alpha = 1/p + \varepsilon$, $\varepsilon > 0$, since $p \leqq k'$,*

$$\sum_{n=1}^{\infty} \frac{1}{n} |n^{1-\alpha} c_n(x)|^k = \sum_{n=1}^{\infty} \frac{1}{n^{1+k\varepsilon}} |n^{1-1/p} c_n(x)|^k$$

$$\leqq \left(\sum_{n=1}^{\infty} \frac{1}{n} |n^{1/p'} c_n(x)|^{p'}\right)^{k/p'} \left(\sum_{n=1}^{\infty} \frac{1}{n^{1+\varepsilon'}}\right)^{1-k/p'} \leqq A\left(\int_0^{2\pi} |f(t)|^p dt\right)^{k/p},$$

where $\varepsilon' = kp'\varepsilon/(p' - k)$ and A is a absolute constant.

(ii) *For $p \leqq k$, $p \leqq k'$ and $\alpha \geqq 1/p$*

$$\sum_{n=1}^{\infty} \frac{1}{n} |n^{1-\alpha} c_n(x)|^k \leqq \sum_{n=1}^{\infty} n^{-k(1/k+1/p-1)} |c_n(x)|^k \leqq A\left(\int_0^{2\pi} |f(t)|^p dt\right)^{k/p},$$

by H.L. Pitt [10].

(iii) *For $p \leqq k$, $p \geqq k'$ and $\alpha \geqq 1/k'$,*

$$\sum_{n=1}^{\infty} \frac{1}{n} |n^{1-\alpha} c_n(x)|^k \leqq \sum_{n=1}^{\infty} |c_n(x)|^k$$

$$\leqq A\left(\int_0^{2\pi} |f(t)|^{k'} dt\right)^{k/k'} \leqq A\left(\int_0^{2\pi} |f(t)|^p dt\right)^{k/p}.$$

Hence, it seems reasonable to conjecture that, if the condition (1.5) and

$$\sum_{n=1}^{\infty} \frac{1}{n} |n^{1-\alpha} c_n(x)|^k < \infty \tag{1.6}$$

are satisfied, the result of theorem A holds.
In this note we prove this conjecture.

2. We first prove the following theorem which is an analogue of a theorem of Bosanque-Offord [1] and of H.C. Chow [4].

THEOREM 1. *If* (1.6) *and*

1) For the case $k=1$, see T. Tsuchikura [11].

$$\Phi_x(t) = \int_0^t \{\varphi_x(u) - s\}\, du = O\left\{t\Big/\left(\log \frac{1}{t}\right)^{\rho}\right\}, \tag{2.1}$$

where $k > 1$, $1/k' \leqq \alpha < 1$ *and* $\rho > 1/k$, *necessary and sufficient condition that*

$$\sum_{n=1}^{\infty} \frac{1}{n} \,|s_n^{\alpha-1}(x) - s|^k < \infty \tag{2.2}$$

should holds is that

$$\sum_{n=1}^{\infty} \frac{1}{n} \left| n^{1-\alpha} \int_0^{\delta} \{\varphi_x(t) - s\} \left(1 - \frac{t}{\delta}\right) \frac{\sin(n, \alpha - 1; t)}{t^{\alpha}}\, dt \right|^k < \infty, \tag{2.3}$$

where δ *is any positive number less than* π *and*

$$(n, \alpha; t) = \left\{n + \frac{1}{2}(\alpha + 1)\right\} t - \frac{\alpha}{2}\pi.$$

LEMMA 1. *Let* $G_n^{\alpha}(t)$ *denote the* (C,α) *mean of the sequence*

$$\pi^{-1} + 2\pi^{-1} \sum_{\nu=1}^{n} \cos \nu t \qquad \textit{where } -1 < \alpha < 0,$$

then, for $0 < t < \pi$, *we have*

$$G_n^{\alpha}(t) = g_n^{\alpha}(t) + h_n^{\alpha}(t), \tag{2.4}$$

where

$$g_n^{\alpha}(t) = 2 \sin(n, \alpha; t)/\pi A_n^{\alpha} \left(2 \sin \frac{t}{2}\right)^{\alpha+1}, \tag{2.5}$$

$$|G_n^{\alpha}(t)| = O(n), \qquad \left| \frac{d}{dt} G_n^{\alpha}(t) \right| = O(n^2), \tag{2.6}$$

$$|h_n^{\alpha}(t)| = O(n^{-1}t^{-2}), \quad \left| \frac{d}{dt} h_n^{\alpha}(t) \right| = O(n^{-1}t^{-3}), \tag{2.7}$$

where the O *holds uniformly in* $0 < t < \pi$.

This is due to J. J. Gergen.

PROOF OF THEOREM 1. We may suppose without loss of generality that $c_0(x) = 0$ and $s = 0$

$$s_n^{\alpha-1}(x) = \int_0^{\pi} \varphi_x(t) G_n^{\alpha-1}(t)\, dt = \int_0^{\pi} \varphi_x(t) g_n^{\alpha-1}(t)\, dt + \int_0^{\pi} \varphi_x(t) h_n^{\alpha-1}(t)\, dt$$
$$= I_1(n) + I_2(n), \tag{2.8}$$

say. Then

$$I_2(n) = [\Phi_x(t) h_n^{\alpha-1}(t)]_0^\pi - \int_0^\pi \Phi_x(t) \frac{d}{dt} h_n^{\alpha-1}(t) dt = I_2'(n) - I_2''(n).$$

It is easy to see that

$$I_2'(n) = O(n^{-1}).$$

Using (2. 1) and (2. 4) —— (2. 7), we get

$$I_2''(n) = \int_0^{\pi/n} \Phi_x(t) \frac{d}{dt} G_n^{\alpha-1}(t) dt - \int_0^{\pi/n} \Phi_x(t) \frac{d}{dt} g_n^{\alpha-1}(t) dt + \int_{\pi/n}^\pi \Phi_x(t) \frac{d}{dt} h_n^{\alpha-1}(t) dt$$

$$= O\left\{\int_0^{\pi/n} \frac{n^2 t}{\left(\log \frac{1}{t}\right)^\rho} dt\right\} + O\left\{\int_0^{\pi/n} \frac{t}{\left(\log \frac{1}{t}\right)^\rho} (n^{-\alpha+1} t^{-\alpha-1} + n^{-\alpha+2} t^{-\alpha}) dt\right\}$$

$$+ O\left\{\int_{\pi/n}^\pi \frac{t n^{-1} t^{-3}}{\left(\log \frac{1}{t}\right)^\rho} dt\right\} = O\{1/(\log n)^\rho\}.$$

Thus we have, since $\rho > 1/k$,

$$\sum_{n=1}^\infty \frac{|I_2(n)|^k}{n} \leqq A \sum_{n=1}^\infty \frac{1}{n^{1+k}} + A \sum_{n=2}^\infty \frac{1}{n(\log n)^{\rho k}} < \infty. \tag{2. 9}$$

Hence, by (2. 8) and (2. 9), (2. 2) holds if and only if

$$\sum_{n=1}^\infty \frac{|I_1(n)|^k}{n} < \infty. \tag{2. 10}$$

Let

$$k(t) = \frac{1}{\left(2 \sin \frac{t}{2}\right)^\alpha} - \frac{1}{t^\alpha} \qquad (0 < t \leqq \pi),\ k(0) = 0.$$

Then

$$\frac{\pi}{2} A_n^{\alpha-1} I_1(n) = \int_0^\pi \varphi_x(t) \frac{\sin(n, \alpha-1;\ t)}{t^\alpha} dt + \int_0^\pi \varphi_x(t) k(t) \sin(n, \alpha-1;\ t) dt$$

$$= J_1(n) + J_2(n), \text{ say.}$$

It was proved by Bosanque and Offord [1] that

$$J_2(n) = O\left\{\sum{}' \frac{|c_\nu(x)|}{(n-\nu)^2}\right\} + O(|c_n(x)|), \tag{2. 11}$$

where $\sum'$ denotes summation over $1 \leqq \nu \leqq n-1,\ n+1 \leqq \nu < \infty$

We write

$$K_1(n) = \sum_{\nu=1}^{n-1} \frac{|c_\nu(x)|}{(n-\nu)^2},\quad K_2(n) = \sum_{\nu=n+1}^{2n} \frac{|c_\nu(x)|}{(\nu-n)^2} \text{ and}$$

$$K_3(n) = \sum_{\nu=2n+1}^{\infty} \frac{|c_\nu(x)|}{(\nu-n)^2}.$$

Then, by Minkowski's inequality, we get

$$\sum_{n=2}^{\infty} \frac{1}{n} |n^{1-\alpha}K_1(n)|^k = \sum_{n=2}^{\infty} \frac{1}{n}\left(n^{1-\alpha} \sum_{\nu=1}^{n-1} \frac{|c_{n-\nu}(x)|}{\nu^2}\right)^k$$

$$\leqq \left\{\sum_{\nu=1}^{\infty}\left(\sum_{n=\nu+1}^{\infty}\left(\frac{1}{n^{\alpha-1+1/k}} \frac{|c_{n-\nu}(x)|}{\nu^2}\right)^k\right)^{1/k}\right\}^k$$

$$= \left\{\sum_{\nu=1}^{\infty} \frac{1}{\nu^2}\left(\sum_{n=1}^{\infty} \frac{|c_n(x)|^k}{(n+\nu)^{(\alpha-1/k')k}}\right)^{1/k}\right\}^k. \tag{2.12}$$

Since $(n+\nu)^{-(\alpha-1/k')k} \leqq n^{-(\alpha-1/k')k}$ for $\alpha \geqq 1/k'$, the right-hand expression of (2.12) is not greater than

$$\left\{\sum_{\nu=1}^{\infty} \frac{1}{\nu^2}\left(\sum_{n=1}^{\infty} \frac{|c_n(x)|^k}{n^{1+(\alpha-1)k}}\right)^{1/k}\right\}^k \leqq A \sum_{n=1}^{\infty} \frac{1}{n} |n^{1-\alpha}c_n(x)|^k. \tag{2.13}$$

Moreover, by Minkowski's inequality,

$$\sum_{n=1}^{\infty} \frac{1}{n} |n^{1-\alpha}K_2(n)|^k = \sum_{n=1}^{\infty} \frac{1}{n}\left(n^{1-\alpha} \sum_{\nu=n+1}^{2n} \frac{|c_\nu(x)|}{(\nu-n)^2}\right)^k$$

$$= \sum_{n=1}^{\infty} \frac{1}{n}\left(n^{1-\alpha} \sum_{\nu=1}^{n} \frac{|c_{n+\nu}(x)|}{\nu^2}\right)^k \leqq \left\{\sum_{\nu=1}^{\infty} \frac{1}{\nu^2}\left(\sum_{n=\nu}^{\infty} \frac{|c_{n+\nu}(x)|^k}{n^{(\alpha-1/k')k}}\right)^{1/k}\right\}^k$$

$$= \left\{\sum_{\nu=1}^{\infty} \frac{1}{\nu^2}\left(\sum_{n=2\nu}^{\infty} \frac{|c_n(x)|^k}{(n-\nu)^{(\alpha-1/k')k}}\right)^{1/k}\right\}^k$$

$$= \left\{\sum_{\nu=1}^{\infty} \frac{1}{\nu^2}\left(\sum_{n=2\nu}^{\infty} \frac{|c_n(x)|^k}{n\,(n-\nu)^{(\alpha-1)k}} \frac{n}{n-\nu}\right)^{1/k}\right\}^k. \tag{2.14}$$

Since $\alpha < 1$, we have $(n-\nu)^{-(\alpha-1)k} < n^{-(\alpha-1)k}$ and $n/(n-\nu) < 2$ for $n \geqq 2\nu$. Hence, it follows that the right side of (2.14) is not greater than

$$A\left\{\sum_{\nu=1}^{\infty} \frac{1}{\nu^2}\left(\sum_{n=1}^{\infty} \frac{|c_n(x)|^k}{n^{1+(\alpha-1)k}}\right)^{1/k}\right\}^k \leqq A \sum_{n=1}^{\infty} \frac{1}{n} |n^{1-\alpha}c_n(x)|^k, \tag{2.15}$$

and

$$\sum_{n=1}^{\infty}\frac{1}{n}\,|n^{1-\alpha}K_3(n)|^k=\sum_{n=1}^{\infty}\frac{1}{n}\Big(n^{1-\alpha}\sum_{\nu=2n+1}^{\infty}\frac{|c_\nu(x)|}{(\nu-n)^2}\Big)^k$$
$$=\sum_{n=1}^{\infty}\frac{1}{n}\Big(n^{1-\alpha}\sum_{\nu=n+1}^{\infty}\frac{|c_{n+\nu}(x)|}{\nu^2}\Big)^k\leqq\Big\{\sum_{\nu=1}^{\infty}\frac{1}{\nu^2}\Big(\sum_{n=1}^{\nu-1}\frac{|c_{n+\nu}(x)|^k}{n^{(\alpha-1+1/k)k}}\Big)^{1/k}\Big\}^k$$
$$=\Big\{\sum_{\nu=2}^{\infty}\frac{1}{\nu^2}\Big(\sum_{n=\nu+1}^{2\nu-1}\frac{|c_n(x)|^k}{(n-\nu)^{(\alpha-1+1/k)k}}\Big)^{1/k}\Big\}^k\leqq A\Big\{\sum_{\nu=2}^{\infty}\frac{1}{\nu^2}\Big(\sum_{n=\nu+1}^{2\nu-1}|c_n(x)|^k\Big)^{1/k}\Big\}^k$$
$$\leqq A\Big\{\sum_{\nu=2}^{\infty}\frac{\nu^{\alpha-1+1/k}}{\nu^2}\Big(\sum_{n=\nu+1}^{2\nu-1}\frac{|c_n(x)|^k}{n^{(\alpha-1+1/k)k}}\Big)^{1/k}\Big\}^k\leqq\sum_{n=1}^{\infty}\frac{1}{n}\,|n^{1-\alpha}c_n(x)|^k, \tag{2.16}$$

since $\sum_{\nu=2}^{\infty}\frac{1}{\nu^{2-\alpha+1/k'}}<\infty$.

Accordingly, by (2. 11), (2. 13), (2. 15) and (2. 16), we have

$$\sum_{n=1}^{\infty}\frac{1}{n}\,|n^{1-\alpha}J_2(n)|^k<\infty. \tag{2.17}$$

Next, we consider $J_1(n)$.

Let $0<\delta<\pi$ and

$$\chi(t)=\begin{cases}t^{-\alpha} & (\delta\leqq t\leqq\pi)\\ \delta^{-1}t^{1-\alpha} & (0\leqq t\leqq\delta).\end{cases}$$

Then

$$J_1(n)=\int_0^{\delta}\varphi_x(t)\Big(1-\frac{t}{\delta}\Big)\frac{\sin(n,\alpha-1;t)}{t^{\alpha}}\,dt+\int_0^{\pi}\varphi_x(t)\chi(t)\sin(n,\alpha-1;t)dt$$
$$=L_1(n)+L_2(n), \tag{2.18}$$

say. It was also proved by Bosanquet and Offord in [1] that

$$L_2(n)=O\Big\{\sum{}'\frac{|c_\nu(x)|}{(n-\nu)^{2-\alpha}}\Big\}+O(|c_n(x)|),$$

where $\sum'$ has the same meaning as before.

If we write, as before,

$$L_2(n)=M_1(n)+M_2(n)+M_3(n)+O(|c_n(x)|),$$

we get, by the same process as used in establishing (2. 13) and (2.15),

$$\sum_{n=1}^{\infty}\frac{1}{n}\,|n^{1-\alpha}M_i(n)|^k\leqq A\sum_{n=1}^{\infty}\frac{1}{n}\,|n^{1-\alpha}c_n(x)|^k,\ (i=1,2). \tag{2.19}$$

Also

$$\sum_{n=1}^{\infty}\frac{1}{n}\,|n^{1-\alpha}M_3(n)| = \sum_{n=1}^{\infty}\frac{1}{n}\Big(n^{1-\alpha}\sum_{\nu=2n+1}^{\infty}\frac{|c_\nu(x)|}{(\nu-n)^{2-\alpha}}\Big)^k$$

$$=\sum_{n=1}^{\infty}\frac{1}{n^{1+(\alpha-1)k}}\Big(\sum_{\nu=2n+1}^{\infty}\frac{|c_\nu(x)|}{(\nu-n)^{1/k'+\epsilon+1+1/k-\alpha-\epsilon}}\Big)^k,\quad (0<\varepsilon<1-\alpha),$$

$$\leqq\sum_{n=1}^{\infty}\frac{1}{n^{1+(\alpha-1)k}}\Big(\sum_{\nu=2n+1}^{\infty}\frac{|c_\nu(x)|^k}{(\nu-n)^{(1+1/k-\alpha-\epsilon)k}}\Big)\Big(\sum_{\nu=2n+1}^{\infty}\frac{1}{(\nu-n)^{1+k'\epsilon}}\Big)^{k/k'}$$

(by Hölder's inequality, where $1/k+1/k'=1$)

$$\leqq A\sum_{n=1}^{\infty}\frac{1}{n^{1+(\alpha-1)k+k\epsilon}}\sum_{\nu=2n+1}^{\infty}\frac{|c_\nu(x)|^k}{(\nu-n)^{(1+1/k-\alpha-\epsilon)k}}$$

$$\leqq A\sum_{\nu=3}^{\infty}|c_\nu(x)|^k\sum_{n=1}^{[\frac{1}{2}(\nu-1)]}\frac{1}{n^{1+(\alpha-1+\epsilon)k}(\nu-n)^{(1+1/k-\alpha-\epsilon)k}}$$

$$\leqq A\sum_{\nu=3}^{\infty}\frac{|c_\nu(x)|^k}{\nu^{(1+1/k-\alpha-\epsilon)k}}\sum_{n=1}^{[\frac{1}{2}(\nu-1)]}\frac{1}{n^{1+(\alpha-1+\epsilon)k}}$$

$$\leqq A\sum_{\nu=3}^{\infty}\frac{|c_\nu(x)|^k}{\nu}\leqq A\sum_{\nu=1}^{\infty}\frac{1}{\nu}\,|\nu^{1-\alpha}c_\nu(x)|^k. \tag{2.20}$$

Thus, by (2. 19) and (2. 20),

$$\sum_{n=1}^{\infty}\frac{1}{n}\,|n^{1-\alpha}L_2(n)|^k<\infty.$$

Therefore, by (2. 17) and (2. 18), (2. 10) holds if and only if

$$\sum_{n=1}^{\infty}\frac{1}{n}\,|n^{1-\alpha}L_1(n)|^k<\infty.$$

The theorem is thus proved.

3. THEOREM 2. *Let* $1<p\leqq 2$, $k>1$, $1/k+1/k'=1$ *and* $1>\alpha>\sup(1/p,1/k')$. *If*

$$\sum_{n=1}^{\infty}\frac{1}{n}\,|n^{1-\alpha}c_n(x)|^k<\infty \tag{1.6}$$

and

$$\Phi_x^{(p)}(t)=\int_0^t|\varphi_x(t)|^p du = O\Big\{t\Big/\Big(\log\frac{1}{t}\Big)^\rho\Big\}, \tag{1.5}$$

as $t\to+0$, *where* $\rho>p/k$, *then the Fourier series* (1. 1) *is summable* $|C,\alpha|_k$ *at the point* $t=x$.

PROOF. By T. M. Flett [6, Theorem 5], it is sufficient to prove that

$$\sum_{n=1}^{\infty} \frac{1}{n} |s_n^{\alpha-1}(x)|^p < \infty. \tag{3.1}$$

Since, by Hölder's inequality,

$$\Phi_x^{(1)}(t) = \int_0^t |\varphi_x(u)| du = O\left\{t\Big/\left(\log \frac{1}{t}\right)^{\rho/p}\right\}, \tag{3.2}$$

(3, 1) holds if and only if

$$\sum_{n=1}^{\infty} \frac{1}{n} \left| n^{1-\alpha} \int_0^\delta \varphi_x(t)\left(1 - \frac{t}{\delta}\right) \frac{\sin(n, \alpha - 1; t)}{t^\alpha} dt \right|^k < \infty, \tag{3.3}$$

where $0 < \delta < \pi$.

$$\sum_{n=1}^{\infty} \frac{1}{n} \left| n^{1-\alpha} \int_0^\delta \varphi_x(t) \left(1 - \frac{t}{\delta}\right) \frac{\sin(n, \alpha - 1; t)}{t^\alpha} dt \right|^k$$
$$\leqq A \sum_{n=0}^{\infty} \frac{1}{2^{n(\alpha-1+1/k)k}} \sum_{j=2^n}^{2^{n+1}-1} \left| \int_0^\delta \varphi_x(t) \left(1 - \frac{t}{\delta}\right) \frac{\sin(j, \alpha-1; t)}{t^\alpha} dt \right|^k$$
$$\leqq A \sum_{n=0}^{\infty} \frac{1}{2^{n(\alpha-1+1/k)k}} \sum_{j=2^n}^{2^{n+1}-1} \left| \int_0^{\delta/2^n} \right|^k + A \sum_{n=0}^{\infty} \frac{1}{2^{n(\alpha-1+1/k)k}} \sum_{j=2^n}^{2^{n+1}-1} \left| \int_{\delta/2^n}^{\delta} \right|^k$$
$$= N_1 + N_2,$$

say. By the integration by part and (3. 2), we have

$$\left| \int_0^{\delta/2^n} \varphi_x(t)\left(1 - \frac{t}{\delta}\right) \frac{\sin(j, \alpha - 1; t)}{t^\alpha} dt \right| \leqq \int_0^{\delta/2^n} \frac{|\varphi_x(t)|}{t^\alpha} dt$$
$$= [\Phi_x^{(1)}(t) t^{-\alpha}]_0^{\delta/2^n} + \alpha \int_0^{\delta/2^n} \frac{\Phi_x^{(1)}(t)}{t^{1+\alpha}} dt = O(2^{n(\alpha-1)} n^{-\rho/p})$$

Hence we get

$$N_1 \leqq A \sum_{n=1}^{\infty} \frac{2^{n+nk(\alpha-1)}}{2^{n(\alpha-1+1/k)k} n^{\rho k/p}} = A \sum_{n=1}^{\infty} \frac{1}{n^{\rho k/p}} < \infty. \tag{3.4}$$

Next we consider N_2. Let

$$F(t) = \begin{cases} \varphi_x(t)(1 - t/\delta) & (\delta/2^n \leqq t \leqq \delta) \\ 0 & (0 \leqq t < \delta/2^n, \ \delta \leqq t \leqq \pi). \end{cases}$$

We have now to distinguish three cases.

Case I. $k \geqq p$, $k' \geqq p$,

Case II. $k \geqq p$, $k' \leqq p$,

Case III. $k < p$.

Case I. By Hölder's inequality, we have

$$N_2 \leqq A \sum_{n=0}^{\infty} \frac{1}{2^{n(\alpha-1+1/k)k}} \left(\sum_{j=2^n}^{2^{n+1}-1} \left| \int_0^\pi F(t) \frac{\sin(j, \alpha-1; t)}{t^\alpha} dt \right|^{p'} \right)^{k/p'} \left(\sum_{j=2^n}^{2^{n+1}-1} 1 \right)^{1-k/p'}$$

$$= A \sum_{n=0}^{\infty} \frac{1}{2^{n(\alpha-1/p)k}} \left(\sum_{j=2^n}^{2^{n+1}-1} \left| \int_0^\pi F(t) \frac{\sin(j, \alpha-1; t)}{t^\alpha} dt \right|^{p'} \right)^{k/p'}$$

$$\leqq A \sum_{n=0}^{\infty} \frac{1}{2^{n(\alpha-1/p)k}} \left(\int_{\delta/2^n}^{\delta} \frac{|\varphi_x(t)|^p}{t^{\alpha p}} dt \right)^{k/p}$$

(by the theorem of Hausdorff-Young)

$$\leqq A \sum_{n=0}^{\infty} \frac{1}{2^{n(\alpha-1/p)k}} \left\{ [\Phi_x^{(p)}(t) t^{-\alpha p}]_{\delta/2^n}^{\delta} + \alpha p \int_{\delta/2^n}^{\delta} \frac{\Phi_x^{(p)}(t)}{t^{\alpha p+1}} dt \right\}^{k/p}$$

$$\leqq A \sum_{n=0}^{\infty} \frac{1}{2^{n(\alpha-1/p)k}} \left\{ \left[t^{1-\alpha p} \Big/ \left(\log \frac{1}{t} \right) \right]_{\delta/2^n}^{\delta} + \alpha p \int_{\delta/2^n}^{\delta} \frac{dt}{t^{\alpha p} \left(\log \frac{1}{t} \right)^{\rho}} \right\}^{k/p}$$

$$\leqq A \sum_{n=1}^{\infty} \frac{1}{2^{n(\alpha-1/p)k}} 2^{n(\alpha-1/p)k} n^{-\rho k/p} < \infty. \tag{3.5}$$

(since $\alpha p > 1$ and $\rho > p/k$).

Case II. In this case, $1 < k' \leqq 2$. Hence by Hausdorff-Young's inequality, we get

$$N_2 \leqq A \sum_{n=0}^{\infty} \frac{1}{2^{n(\alpha-1+1/k)k}} \sum_{j=2^n}^{2^{n+1}-1} \left| \int_0^\pi F(t) \frac{\sin(j, \alpha-1; t)}{t^\alpha} dt \right|^k$$

$$\leqq A \sum_{n=0}^{\infty} \frac{1}{2^{n(\alpha-1/k')k}} \left(\int_{\delta/2^n}^{\delta} |\varphi_x(t)|^{k'} \frac{dt}{t^{\alpha k'}} \right)^{k/k'}$$

$$\leqq A \sum_{n=0}^{\infty} \frac{1}{2^{n(\alpha-1/k')k}} \left\{ [\Phi_x^{(k')}(t) t^{-\alpha k'}]_{\delta/2^n}^{\delta} + \alpha k' \int_{\delta/2^n}^{\delta} \frac{\Phi_x^{(k')}(t)}{t^{\alpha k'+1}} dt \right\}^{k/k'} \tag{3.6}$$

Since, in this case, $\alpha > 1/k'$ and

$$\Phi_x^{(k')}(t) = \int_0^t |\varphi_x(u)|^{k'} du \leqq \left(\int_0^t |\varphi_x(u)|^p du \right)^{k'/p} \left(\int_0^t du \right)^{1-k'/p}$$

$$= O\left\{ t \Big/ \left(\log \frac{1}{t} \right)^{\rho k'/p} \right\},$$

(3.6) is not greater than

$$A \sum_{n=1}^{\infty} \frac{1}{2^{n(\alpha-1/k')k}} 2^{n(\alpha-1/k')k} (\log 2^n)^{-\rho k/p} \leqq A \sum_{n=1}^{\infty} \frac{1}{n^{\rho k/p}} < \infty. \tag{3.7}$$

Case III. Since $p < k'$, the estimation is quite similar to that of case 1. This shows together with (3.5) and (3.7) that the theorem is completed.

THEOREM 3. *If* $1 < p \leqq k$, (1.5) *and* (1.6) *and*

$$\int^{\delta} \frac{|\varphi_x(t)|^p}{t} dt < \infty, \tag{3.8}$$

then the Fourier series (1.2) *is summable* $|C, \alpha|_k$ *at the point* $t = x$, *where* $\alpha = \sup(1/p, 1/k')$. (cf. H.C. Chow [2].)

LEMMA 1. (T. M. Flett [5, Lemma 14]). *Let* $r \geqq k > 1$, $\mu = \alpha + \sup(1/p, 1/k')$, *and let*

$$B_n = \int_0^{\pi} \chi(t) t^{-\mu} e^{nit} dt \ (n = 1, 2, \cdots\cdots).$$

Then

$$\left\{\sum_{n=1}^{\infty} n^{r(\alpha-\mu+1)-1} |B_n|^r\right\}^{1/r} \leqq A\left\{\int^{\pi} |\chi(t)|^k t^{-1-k\alpha} dt\right\}^{1/k}.$$

PROOF OF THEOREM 3. We write

$$\int_0^{\delta} \varphi_x(t)\left(1 - \frac{t}{\delta}\right)\frac{\sin(n, \alpha - 1; t)}{t^{\alpha}} dt = \int^{\delta/n} + \int_{\delta/n}^{\delta} = P_1(n) + P_2(n),$$

say, where $\alpha = \sup(1/p, 1/k')$.

Then, it is easy to see that

$$P_1(n) = O\{n^{\alpha-1}(\log n)^{-\rho/p}\}.$$

Hence, we obtain

$$\sum_{n=1}^{\infty} \frac{1}{n} |n^{1-\alpha} P_1(n)|^k \leqq A \sum_{n=2}^{\infty} \frac{1}{n(\log n)^{\rho k/p}} < \infty. \tag{3.9}$$

Using Lemma 1, we have

$$\sum_{n=1}^{\infty} \frac{1}{n} |n^{1-\alpha} P_2(n)|^k = \sum_{n=1}^{\infty} n^{k(1-\alpha)-1} \left| \int_{\delta/n}^{\delta} \varphi(t)\left(1 - \frac{t}{\delta}\right)\frac{\sin(n, \alpha - 1; t)}{t^{\alpha}} dt \right|^k$$

$$\leqq \left(\int_0^{\delta} \frac{|\varphi_x(t)|^p}{t} dt\right)^{k/p} < \infty, \text{ by (3.8)}. \tag{3.10}$$

Then, by (3.8), (3.9) and Theorem 1, we get the required result.

4. In this section we consider the theorems of the summability factor of $|C, \alpha|_k$ at a point.

THEOREM 4. *If* $k > 1$, $1/k' \leqq \alpha < 1$,

$$\int_0^t \{\varphi_x(u) - s\}du = O\left\{t\Big/\left(\log \frac{1}{t}\right)^\rho\right\} \tag{4.1}$$

and

$$\sum_{n=1}^{\infty} \frac{1}{n\{\log(n+1)\}^{\gamma k}} |n^{1-\alpha}c_n(x)|^k < \infty, \tag{4.2}$$

where $\rho + \gamma > 1/k$, $\gamma \geqq 0$, *then the necessary and sufficient condition that*

$$\sum_{n=1}^{\infty} \frac{1}{n\{\log(n+1)\}^{\gamma k}} |s_n^{\alpha-1}(x) - s|^k < \infty \tag{4.3}$$

should hold is that

$$\sum_{n=1}^{\infty} \frac{1}{n\{\log(n+1)\}^{\gamma k}} \left| n^{1-\alpha}\int_0^\delta (\varphi_x(t) - s)\left(1 - \frac{t}{\delta}\right)\frac{\sin(n, \alpha - 1; t)}{t^\alpha} dt\right|^k < \infty.$$

For $\gamma = 0$, the theorem is identical to Theorem 1.
For the case $\gamma > 0$, we can prove by the same process as used in establishing Theorem 1.

THEOREM 5.[2] *If* $1 < p \leqq 2$, $\alpha = \sup(1/p, 1/k')$, (4.2)[3] *and*

$$\Phi_x^{(p)}(t) = \int_0^t |\varphi_x(u)|^p du = O\left\{t\Big/\left(\log \frac{1}{t}\right)^\rho\right\}, \tag{4.4}$$

$\sum_{n=1}^{\infty} \frac{c_n(x)}{\{\log(n+1)\}^\gamma}$ *is summable* $|C, \alpha|_k$ *at the point* $t = x$, *where* $\rho > \sup(p/k, p/p')$ *and* $\gamma = 1/p$ *for* $p < k$ *or* $\rho = 1 - \varepsilon$ *for sufficiently small* $\varepsilon \geqq 0$, *and* $\gamma > 1/k$ *for* $p \geqq k$, *respectively.*

We need two lemmas.

LEMMA 2. *If* $0 < \beta < 1$ *and* $\{\lambda_n\}$ *is a sequence of positive numbers such that* $\Delta\lambda_n = \lambda_n - \lambda_{n+1} = O(\lambda_n/n)$ *and* λ_n/n *is non-increasing, and if the series* $\sum_{n=1}^{\infty} \lambda_n^k |t_n^\beta(x)|^k/n < \infty$, *then the series* $\sum_{n=1}^{\infty} \lambda_n c_n(x)$ *is summable* $|C, \beta|_k$ *where* $k \geqq 1$.

PROOF. If $k = 1$ this lemma is due to C. H. Chow [2]. The proof runs similar to that of Chow but for the sake of completeness we prove here. Let

2) It is obvious that the condition $f(t) \in L^p(0, 2\pi)$ implies (4.2).
3) The theorems of summablity $|C, \alpha|_k$ concerned with almost all point t corresponding to Theorems 1 and 5 are known (Flett [5], [7]).

$t_n^\alpha(x)$, $\tau_n^\alpha(x)$ are the (C, α) means of $\{nc_n(x)\}$, $\{n\lambda_n c_n(x)\}$, respectively, where $\alpha > -1$.

We have to prove the series $\sum_{n=1}^{\infty} |\tau_n^\beta(x)|^k/n$ is convergent.

$$\begin{aligned} A_n^\beta \tau_n^\beta &= \sum_{\nu=1}^{n} A_{n-\nu}^{\beta-1}\lambda_\nu \nu c_\nu(x) = \sum_{\nu=1}^{n} A_{n-\nu}^{\beta-1}\lambda_\nu \sum_{\mu=1}^{\nu} A_{\nu-\mu}^{-\beta-1} A_\mu t_\mu^\beta(x) \\ &= \sum_{\mu=1}^{n} A_\mu^\beta t_\mu^\beta \sum_{\nu=\mu}^{n} A_{n-\nu}^{\beta-1} A_{\nu-\mu}^{-\beta-1}\lambda_\nu \\ &= \sum_{\mu=1}^{n} A_\mu^\beta t_\mu^\beta \sum_{\nu=0}^{N} A_{N-\nu}^{\beta-1} A_\nu^{-\beta-1}\lambda_{n-N+\nu} \qquad (N \equiv n-\mu) \\ &= A_n^\beta t_n^\beta \lambda_n + \sum_{\mu=1}^{n-1} A_\mu^\beta t_\mu^\beta \sum_{\nu=0}^{N} A_\nu^{\beta-1} A_\nu^{-\beta-1}\lambda_{n-N+\nu}. \end{aligned}$$

Now, let

$$B_{N,\nu} = \sum_{k=0}^{\nu} A_{N-k}^{\beta-1} A_k^{-\beta-1},$$

so that

$$B_{N,N} = \sum_{k=0}^{N} A_{N-k}^{\beta-1} A_k^{-\beta-1} = \begin{cases} 1 & \text{when } N=0 \\ 0 & \text{when } N \geqq 1. \end{cases}$$

Writing $B_{N,-1} = 0$,

$$\sum_{\nu=0}^{N} A_{N-\nu}^{\beta-1} A_\nu^{-\beta-1}\lambda_{n-N+\nu} = \sum_{\nu=0}^{N} B_{N,\nu}\Delta\lambda_{n-N+\nu}.$$

Hence, for $N \geqq 1$,

$$\begin{aligned} \sum_{\nu=0}^{N} |B_{N,\nu}| &= \sum_{\nu=0}^{N} \left| \sum_{k=0}^{\nu} A_{N-k}^{\beta-1} A_k^{-\beta-1} \right| = \sum_{\nu=0}^{N} \left| -\sum_{k=\nu+1}^{N} A_{N-k}^{\beta-1} A_k^{-\beta-1} \right| \\ &= -\sum_{\nu=0}^{N}\sum_{k=\nu}^{N} A_{N-k}^{\beta-1} A_k^{-\beta-1} = -\sum_{k=0}^{N} A_{N-k}^{\beta-1} A_k^{-\beta-1}\,(k+1) = \beta. \end{aligned}$$

Thus we get, for $N \geqq 1$,

$$\begin{aligned} \left| \sum_{\nu=0}^{N} A_{N-\nu}^{\beta-1} A_\nu^{-\beta-1}\lambda_{n-N+\nu} \right| &\leqq \sum_{\nu=0}^{N} |B_{N,\nu}|\,|\Delta\lambda_{n-N+\nu}| \\ &= \sum_{\nu=0}^{N} |B_{N,\nu}|\, O\!\left(\frac{\lambda_{n-N+\nu}}{n-N+\nu}\right) = O(\lambda_\mu/\mu), \end{aligned}$$

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