A NOTE ON HOLOMORPHICALLY PROJECTIVE TRANSFORMATIONS OF A KÄHLERIAN SPACE WITH PARALLEL RICCI TENSOR

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Introduction. Recently we have discussed infinitesimal holomorphically projective transformations in Kählerian spaces and obtained, for instance, the following theorems[1]:

If a Kählerian space with parallel Ricci tensor admits an analytic infinitesimal HP-transformation which is not affine, then it is a Kähler-Einstein space.

In a complete Kähler-Einstein space with $R<0$, if the length of the associated vector of an analytic infinitesimal HP-transformation is bounded, then the transformation is affine.

In connection with these problems, we shall study complete Kählerian spaces holomorphically projective related and having parallel Ricci tensors. Then we shall prove theorems stated at the end of §1.

T.Nagano [1] has recently studied the corresponding problems concerning complete Riemannian spaces which are projectively related and have parallel Ricci tensors.

1. Preliminaries. Let M be a Kählerian space of real dimension $n=2m>2$ and $(g_{ji},\ \varphi_i{}^h)$ be its structure.[2] Denoting by $\left\{ {h \atop ji} \right\}$ the Christoffel symbols constructed from the metric tensor g_{ji}, we call a curve $x^h=x^h(\tau)$ a *holomorphically planar curve* or briefly an *H-plane curve*, if the curve satisfies the differential equations

$$(1.1)\qquad \frac{d^2x^h}{d\tau^2}+\left\{ {h \atop ji} \right\}\frac{dx^j}{d\tau}\frac{dx^i}{d\tau}=\alpha\frac{dx^h}{d\tau}+\beta\varphi_r{}^h\frac{dx^r}{d\tau},$$

where α and β are certain functions along the curve. A curve $x^h=x^h(\tau)$ is an H-plane curve if and only if the plane elements determined by two vectors

1) Cf., Tachibana, S. and S. Ishihara [2]. The number in brackets refers to the Bibliography at the end of the paper.

2) As to notations we follow Tachibana, S. and S. Ishihara [2]. Indices $h,i,j,r,s,t,\ldots\ldots$ run over $1,\ldots\ldots,n$.

$dx^h/d\tau$ and $\varphi_r{}^h(dx^r/d\tau)$ are parallel along the curve.

We suppose that there is given in M another Riemannian metric $\overline{g}_{ji}$ such that the pair $(\overline{g}_{ji}, \varphi_i{}^h)$ is a Kählerian structure and that we have

$$\overline{\left\{ {h \atop ji} \right\}} = \left\{ {h \atop ji} \right\} + \rho_j\delta_i^h + \rho_i\delta_j^h - \widetilde{\rho}_j\varphi_i{}^h - \widetilde{\rho}_i\varphi_j{}^h, \tag{1.2}$$

where $\overline{\left\{ {h \atop ji} \right\}}$ are the Christoffel symbols constructed from $\overline{g}_{ji}$, ρ_i is a vector field and $\widetilde{\rho}_i = \varphi_i{}^r\rho_r$. If this is the case, the two Kählerian metrics are said to be *holomorphically projective ralated* or briefly *HP-related.*

It is known that g_{ji} and $\overline{g}_{ji}$ are HP-related if and only if they have all H-plane curves in common.

From (1.2) we know that ρ_i is necessarily gradient. Especially if $\rho_i = 0$, then the metrics are said to be *affinely related.*

Now we shall give some formulas which will be useful later. The curvature tensor of g_{ji} is defined by

$$R_{kji}{}^h = \partial_k \left\{ {h \atop ji} \right\} - \partial_j \left\{ {h \atop ki} \right\} + \left\{ {h \atop kr} \right\}\left\{ {r \atop ji} \right\} - \left\{ {h \atop jr} \right\}\left\{ {r \atop ki} \right\}, \qquad \partial_j = \partial/\partial x^j.$$

If we denote by $\overline{R}_{kji}{}^h$ the curvature tensor of $\overline{g}_{ji}$, taking account of (1.2) we have

$$\begin{aligned} \overline{R}_{kji}{}^h = R_{kji}{}^h &+ \rho_{ki}\delta_j^h - \rho_{ji}\delta_k^h - (\rho_{kr}\varphi_j{}^h - \rho_{jr}\varphi_k{}^h)\varphi_i{}^r \\ &- (\rho_{kr}\varphi_j{}^r - \rho_{jr}\varphi_k{}^r)\varphi_i{}^h, \end{aligned} \tag{1.3}$$

where ρ_{ji} is a symmetric tensor defined by

$$\rho_{ji} = \nabla_j\rho_i - \rho_j\rho_i + \widetilde{\rho}_j\widetilde{\rho}_i. \tag{1.4}$$

Contracting h and k in (1.3), we have

$$\overline{R}_{ji} = R_{ji} - n\rho_{ji} - 2\varphi_j{}^s\varphi_i{}^r\rho_{sr}, \tag{1.5}$$

where R_{ji} and $\overline{R}_{ji}$ denote the Ricci tensors of g_{ji} and $\overline{g}_{ji}$ respectively.

On the other hand we know that R_{ji} and $\overline{R}_{ji}$ are hybrid, i.e. that it holds

$$R_{ji} = R_{sr}\varphi_j{}^s\varphi_i{}^r, \qquad \overline{R}_{ji} = \overline{R}_{sr}\varphi_j{}^s\varphi_i{}^r.$$

Taking account of this fact, we obtain by means of (1.5)

$$(n-2)(\rho_{ji} - \rho_{sr}\varphi_j{}^s\varphi_i{}^r) = 0,$$

which implies because of $n > 2$ that ρ_{ji} is hybrid, i.e. that it holds

$$\rho_{ji} = \rho_{sr}\varphi_j{}^s\varphi_i{}^r. \tag{1.6}$$

Substituting (1.6) into (1.5) we find

$$\bar{R}_{ji} = R_{ji} - (n+2)\rho_{ji}. \tag{1.7}$$

Next it follows that from (1.4) and (1.6)

$$\nabla_j\tilde{\rho}_i + \nabla_i\tilde{\rho}_j = 2(\rho_j\tilde{\rho}_i + \tilde{\rho}_j\rho_i). \tag{1.8}$$

We shall here state the theorems which will be proved in this paper.

THEOREM 1. *Let g_{ji} and $\bar{g}_{ji}$ be two complete Kählerian metrics on a complex manifold with real dimension > 2 whose Ricci tensors R_{ji} and $\bar{R}_{ji}$ are parallel. If g_{ji} and $\bar{g}_{ji}$ are holomorphically projective related, then*

1) *in the case when the Ricci form $R(\xi) = R_{ji}\xi^j\xi^i$ of g_{ji} is negative semi-definite, the two metrics are affinely related and hence their Ricci tensors coincide; or*

2) *in the case when $R(\xi)$ is positive semi-definite, so is the Ricci form $\bar{R}(\xi)$ of $\bar{g}_{ji}$ also.*

Especially for an Einstein space we have

THEOREM 2. *Let g_{ji} and $\bar{g}_{ji}$ be two complete Kähler-Einstein metrics on a complex manifold with real dimension > 2. If g_{ji} and $\bar{g}_{ji}$ are holomorphically projective related, then the scalar curvature R and $\bar{R}$ satisfys*

1) $R = 0,\quad \bar{R} = 0,$ *or*

2) $R < 0,\quad \bar{R} < 0,$ *or*

3) $R > 0,\quad \bar{R} > 0.$

In the cases 1) *and* 2), *the two metrics are affinely related.*

In a Kählerian space consider a point transformation f and denote by $\bar{g}_{ji}$ the induced Kählerian metric. If g_{ji} and $\bar{g}_{ji}$ are holomorphically projective related, then we shall call f a *holomorphically projective transformation.* If a point transformation preserves the complex structure, then it is called to be *analytic.* From these definitions and Theorems we have

COROLLARY. *Let f be an analytic transformation of a complete Kählerian space with real dimension > 2 whose Ricci tensor is parallel and negative semi-definite. If f is holomorphically projective, then it is necessarily affine.*

COROLLARY. *In a complete Kähler-Einstein space with real dimension > 2 whose scalar curvature R is non-positive, an analytic holomorphically projective transformation is necessarily affine.*

2. ρ-geodesics. Keeping notations and assumptions in §1, if we consider a geodesic $\mathfrak{g}$: $x^h = x^h(s)$, s being an affine parameter, then we have

$$\frac{d^2x^h}{ds^2} + \left\{ {h \atop ji} \right\} \frac{dx^j}{ds}\frac{dx^i}{ds} = 0. \tag{2.1}$$

We shall define a scalar function $f(s)$ along the geodesic by

$$f(s) = \tilde{\rho}_i \frac{dx^i}{ds}.$$

Then by virtue of (2. 1) we obtain

$$\frac{df}{ds} = \nabla_j \tilde{\rho}_i \frac{dx^j}{ds}\frac{dx^i}{ds} = \frac{1}{2}(\nabla_j \tilde{\rho}_i + \nabla_i \tilde{\rho}_j)\frac{dx^j}{ds}\frac{dx^i}{ds},$$

which implies together with (1. 8)

$$\frac{df}{ds} = 2(\rho_r \frac{dx^r}{ds}) f.$$

The last equation shows that the function f vanishes identically along the geodesic $\mathfrak{g}$, if there exists a zero point of f on $\mathfrak{g}$. Consequently we know that there exists a geodesic along which the function $f(s)$ vanishes identically. We call such a geodesic a *ρ-geodesic* with respect to g_{ji}. We define a ρ-geodesic with respect to $\bar{g}_{ji}$ in the same way. We have here the following

LEMMA 1. *A ρ-geodesic with respect to g_{ji} ($\bar{g}_{ji}$) is at the same time a ρ-geodesic with respect to $\bar{g}_{ji}$ (g_{ji}).*

PROOF. We have for a curve $x^h = x^h(s)$

$$\frac{d^2x^h}{ds^2} + \overline{\left\{ {h \atop ji} \right\}} \frac{dx^j}{ds}\frac{dx^i}{ds} = \frac{d^2x^h}{ds^2} + \left\{ {h \atop ji} \right\} \frac{dx^j}{ds}\frac{dx^i}{ds} + 2\rho_r \frac{dx^r}{ds}\frac{dx^h}{ds} - 2\tilde{\rho}_r \frac{dx^r}{ds} \varphi_s{}^h \frac{dx^s}{ds}$$

by virtue of (1 2). If the curve is a ρ-geodesic $\mathfrak{g}$, s being an affine parameter, then we have

$$\frac{d^2x^h}{dx^2} + \overline{\left\{ {h \atop ji} \right\}} \frac{dx^j}{ds}\frac{dx^i}{ds} = 2\rho_r \frac{dx^r}{ds}\frac{dx^h}{ds}, \tag{2.2}$$

which shows that $\mathfrak{g}$ is also a geodesic with respect to $\bar{g}_{ji}$. This proves the lemma. Q. E. D.

Let $\bar{s}$ be an affine parameter with respect to $\bar{g}_{ji}$, then we have

$$\bar{s} = C \int_0^s A(u)du + B, \tag{2.3}$$

$$A(u) = \exp\left(\int_0^u \rho_i dx^i\right), \tag{2.4}$$

C and B being constant, where the line integral of (2. 4) is taken along $\mathfrak{g}$. If we put $C = 1$ and $B = 0$, then we have

$$(2.5) \qquad \bar{s} = \int_0^s A(u)du,$$

$A(u)$ being given by (2.4). Now we shall prove the following

LEMMA 2. *If the parameter* $\bar{s}$ *defined by* (2.5) *has the form* $\bar{s} = as$, a *being constant, along any* ρ-*geodesic, then the two metric* g_{ji} *and* $\bar{g}_{ji}$ *are affinely related.*

PROOF. Take a ρ-geodesic $\mathfrak{g}$ arbitrarily and assume that the parameter $\bar{s}$ defined by (2, 5) along $\mathfrak{g}$ has the form $\bar{s} = as$ with a constant a. By virtue of (2.5), we know that the function $A(u)$ is constant along $\mathfrak{g}$. This implies $\rho_i(dx^i/ds) = 0$. Take an arbitrary point P in the space. Then since the last equation holds for any ρ-geodesic passing through P, we have $\rho_i v^i = 0$ at P for any vector v^i such that $\widetilde{\rho}_i v^i = 0$. Hence we can conclude that ρ_i vanishes at P, because we have $g_{ji}\rho^j\widetilde{\rho}^i = 0$. The point P being arbitrary, ρ_i vanishes identically. Thus the two metrics are affinely related. Q. E. D.

3. H-projective parameters. Keep notaions and assumptions as above. We shall next introduce the notion of H-projective parameters along any geodesics and prove that the metrics g_{ji} and $\bar{g}_{ji}$ have H-projective parameters in common along ρ-geodesics. This fact will play an important role in the proof of Theorem 1.

Let us consider a geodesic $\mathfrak{g}$ and we shall define a parameter t as a solution of the equation

$$(3.1) \qquad \{t\}_s = \frac{2}{n+2} R_{ji} \frac{dx^j}{ds}\frac{dx^i}{ds},$$

where s is an affine parameter of $\mathfrak{g}$ and $\{t\}_s$ is the Schwarzian derivative, i.e.

$$\{t\}_s = \frac{\dfrac{d^3t}{ds^3}}{\dfrac{dt}{ds}} - \frac{3}{2}\left(\frac{\dfrac{d^2t}{ds^2}}{\dfrac{dt}{ds}}\right)^2.$$

We call such a parameter t an *H-projective parameter along* $\mathfrak{g}$ *with respect to* g_{ji}. It is well known that the general solution of (3.1) is given by

$$t = \frac{a\tau(s) + b}{c\tau(s) + d}, \qquad ad - bc \neq 0,$$

$\tau(s)$ being a solution of (3.1), where a, b, c and d are constant. We shall prove the following

LEMMA 3. *Two HP-related Kählerian metrics have all H-projective* *parameters in common along ρ-geodesics.*

PROOF. Consider a ρ-geodesic $x^h = x^h(s)$, s being an affine parameter with respect to g_{ji}. If we take $\bar{s}$ defined by (2. 5), an affine parameter with respect to $\bar{g}_{ji}$, then it follows that

$$\{\bar{s}\}_s = 2\left[\nabla_j\rho_i \frac{dx^j}{ds}\frac{dx^i}{ds} - \left(\rho_i \frac{dx^i}{ds}\right)^2\right] \tag{3.2}$$

by virtue of (2. 4) and (2. 5). If we substitute (3. 2) into the well known identity

$$\{t\}_{\bar{s}}\left(\frac{d\bar{s}}{ds}\right)^2 = \{t\}_s - \{\bar{s}\}_s,$$

then we find

$$\begin{aligned}\{t\}_{\bar{s}}\left(\frac{d\bar{s}}{ds}\right)^2 &= \frac{2}{n+2}[R_{ji} - (n+2)\rho_{ji}]\frac{dx^j}{ds}\frac{dx^i}{ds}\\ &= \frac{2}{n+2}\bar{R}_{ji}\frac{dx^j}{ds}\frac{dx^i}{ds},\end{aligned}$$

which implies

$$\{t\}_{\bar{s}} = \frac{2}{n+2}\bar{R}_{ji}\frac{dx^j}{ds}\frac{dx^i}{ds}.$$

This proves the lemma. Q. E. D.

4. The proof of Theorem 1. In this section we shall assume that two metrics g_{ji} and $\bar{g}_{ji}$ are both complete and have parallel Ricci tensors. By the assumption we have

$$\nabla_k R_{ji} = 0, \qquad \bar{\nabla}_k\bar{R}_{ji} = 0,$$

where ∇_k and $\bar{\nabla}_k$ are operators of covariant differentiation with respect to g_{ji} and $\bar{g}_{ji}$ respectively.

Consider a ρ-geodesic $\mathfrak{g}$: $x^h = x^h(s)$, s being an affine parameter, then from (2. 5)

$$\bar{s} = \int_0^s A(u)du, \qquad A(u) = \exp\left(\int_0^u \rho_i dx^i\right),$$

is an affine parameter with respect to $\bar{g}_{ji}$. Evidently we have that $\bar{s}$ is an increasing function of s and that $s = 0$ if and only if $\bar{s} = 0$. Thus, taking account of the completeness, we find immediately

$$s = 0 \text{ if and only if } \bar{s} = 0,$$
$$s \to +\infty \text{ if and only if } \bar{s} \to +\infty, \tag{4.1}$$
$$s \to -\infty \text{ if and only if } \bar{s} \to -\infty.$$

Now if we put

$$K(\mathfrak{g}) = \frac{2}{n+2} R_{ji} \frac{dx^j}{ds} \frac{dx^i}{ds}, \quad \overline{K}(\mathfrak{g}) = \frac{2}{n+2} \overline{R}_{ji} \frac{dx^j}{d\bar{s}} \frac{dx^i}{d\bar{s}},$$

then they are constant along $\mathfrak{g}$, because R_{ji} and $\overline{R}_{ji}$ are parallel. An H-projective parameter t of $\mathfrak{g}$ satisfies the equations

$$\{t\}_s = K(\mathfrak{g}), \tag{4.2}$$

$$\{t\}_{\bar{s}} = \overline{K}(\mathfrak{g}). \tag{4.3}$$

In the first place we consider the case $K(\mathfrak{g}) = 0$. In this case the affine parameter s itself satisfies (4.2), i.e. $t = s$ is a solution of (4.2), and hence a solution of (4.3) by virtue of Lemma 3. Now we shall consider the following three cases.

i) When $\overline{K}(\mathfrak{g}) = 0$, since $t = \bar{s}$ is a solution of (4.3), we have

$$\bar{s} = \frac{as + b}{cs + d}, \quad ad - bc \neq 0,$$

by virtue of the arguments in §3. Taking account of (4.1), we have $\bar{s} = as$.

ii) When $\overline{K}(\mathfrak{g}) > 0$, if we put $\sqrt{\overline{K}(\mathfrak{g})/2} = \bar{\lambda}$, then $\bar{t} = (1/\bar{\lambda})\tan \bar{\lambda}\bar{s}$ is a solution of (4.3). Hence we have

$$\frac{1}{\bar{\lambda}} \tan \bar{\lambda}\bar{s} = \frac{as + b}{cs + d}, \ ad - bc \neq 0,$$

which contradicts to (4.1).

iii) When $\overline{K}(\mathfrak{g}) < 0$, if we put $\sqrt{-\overline{K}(\mathfrak{g})/2} = \bar{\lambda}$, then $\bar{t} = (1/\bar{\lambda}) \tanh \bar{\lambda}\bar{s}$ is a solution of (4.3). The same arguments as in ii) lead to a contradiction.

Next we consider the case $K(\mathfrak{g}) < 0$. In this case if we put $\sqrt{-K(\mathfrak{g})/2} = \lambda$, then $t = (1/\lambda) \tanh \lambda s$ is a solution of (4.2). Let us consider the following two cases.

iv) When $\overline{K}(\mathfrak{g}) < 0$, since $\bar{t}$ defined in ii) is a solution of (4.3), we have

$$\bar{t} = \frac{at + b}{ct + d}, \quad ad - bc \neq 0, \tag{4.4}$$

which contradicts to (4.1).

v) When $\overline{K}(\mathfrak{g}) > 0$, since $\bar{t}$ defined in iii) is a solution of (4.3), we have

(4.4). As we have $\bar{t} = 0$ at $t = 0$, it holds that $b = 0$. If we consider the limiting cases $s \to \pm\infty$, then we know that $a = d$ and $c = 0$, from which we get $\bar{t} = t$. Hence we have $(1/\bar{\lambda}) \tanh \bar{\lambda}\bar{s} = (1/\lambda) \tanh \lambda s$. Again taking account of that $s \to \pm\infty$, we have $\bar{\lambda} = \lambda$ and hence $\bar{s} = s$.

In the same way we have a contradiction in the following cases: $K(\mathfrak{g}) < 0$, $\bar{K}(\mathfrak{g}) = 0$; $K(\mathfrak{g}) > 0$, $\bar{K}(\mathfrak{g}) = 0$; $K(\mathfrak{g}) > 0$, $\bar{K}(\mathfrak{g}) < 0$.

From the above arguments we know that for a ρ-geodesic $\mathfrak{g}$ there are only three possible cases:

1) $K(\mathfrak{g}) = \bar{K}(\mathfrak{g}) = 0$; $\bar{s} = as$,

2) $K(\mathfrak{g}) < 0$, $\bar{K}(\mathfrak{g}) < 0$; $\bar{s} = s$,

3) $K(\mathfrak{g}) > 0$, $\bar{K}(\mathfrak{g}) > 0$.

Therefore taking account of Lemma 2 and semi-definiteness of $R(\xi)$, we can prove Theorem 1. Q. E. D.

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