ON AUTOMORPHISMS OF CERTAIN COMPACT ALMOST-HERMITIAN SPACES

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0. An infinitesimal isometry of a compact almost-Kählerian space is analytic and hence is an automorphism. But an infinitesimal analytic transformation is not necessarily isometric. Concerning this the following theorem is known. In a compact Kählerian space, an infinitesimal analytic transformation which leaves invariant the form defining Chern class of degree 2 is an automorphism. In this paper we shall generalize this theorem to a certain compact almost-Hermitian space which is called an O^*-space.

In §1 we shall give some preliminary facts for later use. In §2 we shall associate in an almost-Hermitian space a 2-form to a connection called a canonical connection, and prove that the form is closed. This form corresponds to the form of Chern. In §3 we shall deal with an almost-Hermitian space which will be called an A-space and discuss an infinitesimal analytic transformation which leaves invariant the form. The main theorem will be given in the last section.

1. Almost-complex spaces. Consider an n dimensional space[1)] which admits a tensor field $\varphi_i^{\ h}$. A tensor is called *pure* (*hybrid*) with respect to its two indices, if it commutes (anti-commutes) with $\varphi_i^{\ h}$ in these indices. For instance, $\xi_{ji}^{\ \ h}$ is pure (hybrid) with respect to i and h, if

$$\varphi_i^{\ r}\xi_{jr}^{\ \ h} = \varphi_r^{\ h}\xi_{ji}^{\ \ r}, \qquad (\varphi_i^{\ r}\xi_{jr}^{\ \ h} = -\varphi_r^{\ h}\xi_{ji}^{\ \ r}).$$

If a tensor is pure (hybrid) with respect to all pairs of its indices, then it is called a pure (hybrid) tensor.

For simplicity, we denote by $\mathfrak{p}(i,h)$ ($\mathfrak{h}(i,h)$) the fact that the tensor in consideration is pure (hybrid) with respect to i and h.

The following facts are known or easily proved.

If $\xi_{ji}^{\ \ h}$ *is* $\mathfrak{p}(j,h)$ ($\mathfrak{h}(j,h)$) *and also is* $\mathfrak{p}(i,h)$ ($\mathfrak{h}(i,h)$), *then it is* $\mathfrak{p}(j,i)$.

If $\xi_{ji}^{\ \ h}$ *is* $\mathfrak{p}(j,h)$ *and is* $\mathfrak{h}(i,h)$, *then it is* $\mathfrak{h}(j,i)$

If ξ_{kji} *and* $\eta_l^{\ ji}$ *are both* $\mathfrak{p}(j,i)$ ($\mathfrak{h}(j,i)$), *then* $\xi_{kjr}\eta_l^{\ ri}$ *is* $\mathfrak{p}(j,i)$.

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1) Throughout this paper we shall consider spaces which are manifolds with the differentiability class C^∞. Indices run from 1 to n. We follow the notations of Tachibana, S., [7], [8], [9]. The number in brackets [ ] refers to the Bibliography at the end of the paper.

If ξ_{kji} *is* $\mathfrak{p}(j,i)$ *and* η_l^{ji} *is* $\mathfrak{h}(j,i)$, *then* $\xi_{kjr}\eta_l^{ri}$ *is* $\mathfrak{h}(j,i)$.

If a tensor field φ_i^h satisfies

$$\varphi_i^r\varphi_r^h = -\delta_i^h, \tag{1.1}$$

then the tensor assigns to the space in consideration an almost-complex structure.

In the following we shall only concern ourselves with a space admitting a fixed almost-complex structure.

In an almost-complex space i.e. a space with a fixed almost-complex structure φ_i^h, the following fact is important.

If ξ_{ji}^h *is* $\mathfrak{h}(i,h)$, *then* $\xi_{jr}^r = 0$ *holds good.*

By making use of these facts, we can perform some calculation effectively. For instance we can easily check that if ξ_{kr}^s is $\mathfrak{h}(r,s)$, then $\xi_{kr}^s\xi_{js}^t\xi_{it}^r = 0$.

A vector field v^i is called *contravariant almost-analytic*[2] (or for brevity *analytic*), if it satisfies

$$\underset{v}{\pounds}\varphi_i^h = v^r\partial_r\varphi_i^h - \varphi_i^r\,\partial_r v^h + \varphi_r^h\partial_i v^r = 0, \tag{1.2}$$

where $\underset{v}{\pounds}$ denotes the operator of Lie derivation.[3]

2. Almost-Hermitian spaces. An almost-complex space admits always an almost-Hermitian metric[4]. An almost-Hermitian metric is by definition a positive definite Riemannian metric tensor g_{ji} which is hybrid.

By an almost-Hermitian space we shall mean a space with a fixed almost-Hermitian structure (φ_i^h, g_{ji}).

In an almost-Hermitian space, an affine connection defined by

$$\Gamma_{ji}^h = \left\{ {h \atop ji} \right\} + T_{ji}^h,\ T_{ji}^h = (-1/2)\varphi_r^h\nabla_j\varphi_i^r,$$

will be called a canonical connection, where ∇_j denotes the operator of Riemannian derivation.

It is known that the metric tensor g_{ji} and the almost-complex structure φ_i^h are covariantly constant with respect to the canonical connection.[5]

Denoting the curvature tensor formed from Γ_{ji}^h and the Riemannian curvature tensor by K_{kji}^h and R_{kji}^h respectively, we have

$$K_{kji}^h = R_{kji}^h + \nabla_k T_{ji}^h - \nabla_j T_{ki}^h + T_{kr}^h T_{ji}^r - T_{jr}^h T_{ki}^r.$$

2) Tachibana, S., [9].
3) Yano, K., [11].
4) Cf., Frölicher, A., [2], Obata, M., [5].
5) Cf., Lichnerowicz, A., [4], Obata, M., [5], Tachibana, S., [10], Yano, K. [11].

Taking account of (1.1) we have

$$K_{kji}{}^{h} = (1/2)(R_{kji}{}^{h} - R_{kjt}{}^{r}\varphi_r{}^{h}\varphi_i{}^{t}) - (1/4)(\nabla_k \varphi_r{}^{h}\nabla_j\varphi_i{}^{r} - \nabla_j\varphi_r{}^{h}\nabla_k\varphi_i{}^{r}).$$

Now we define skew-symmetric tensors $\hat{K}_{kj}$ and $\hat{R}_{kj}$ by

$$\hat{K}_{kj} = 2K_{kjr}{}^{t}\varphi_t{}^{r} \text{ and } \hat{R}_{kj} = 2R_{kjr}{}^{t}\varphi_t{}^{r},$$

so we can easily obtain the following equation:

$$\hat{K}_{kj} = \hat{R}_{kj} - \varphi_s{}^{t}\nabla_k\varphi_r{}^{s}\nabla_j\varphi_t{}^{r}. \tag{2.1}$$

If the space is (pseudo-) Kählerian, then $\hat{K}_{kj}$ and $\hat{R}_{kj}$ both reduce to $4\varphi_k{}^{r}R_{rj}$, where R_{rj} is the Ricci tensor. In this case the differential form $\hat{K} = \hat{K}_{kj}\, dx^k \wedge dx^j$ is nothing but the form defining Chern class of degree 2.

In an almost-Hermitian space, $\varphi_i{}^{h}$ itself is pure but $\nabla_k\varphi_i{}^{h}$ is $\mathfrak{h}(i, h)$. On taking account of this fact, we shall prove the following

THEOREM 1. *In an almost-Hermitian space, the differential form* $\hat{K} = \hat{K}_{kj}\, dx^k \wedge dx^j$ *is closed.*

PROOF. From (2.1) we have

$$\nabla_i\hat{K}_{kj} = \nabla_i\hat{R}_{kj} - \nabla_i(\varphi_s{}^{t}\nabla_k\varphi_r{}^{s}\nabla_j\varphi_t{}^{r}).$$

Now we denote by $\mathfrak{S}\{a_{kji}\}$ the cyclic sum of a given tensor a_{kji}, i.e.

$$\mathfrak{S}\{a_{kji}\} = a_{kji} + a_{jik} + a_{ikj}.$$

With this notation, we have

$$\mathfrak{S}\{\nabla_i\hat{R}_{kj}\} = 2\mathfrak{S}\{\nabla_i(R_{kjr}{}^{t}\varphi_t{}^{r})\} = 2\mathfrak{S}\{R_{kjr}{}^{t}\nabla_i\varphi_t{}^{r}\}$$

by virtue of the Bianchi's identity.

On the other hand, if we put

$$\nabla_i(\varphi_s{}^{t}\nabla_k\varphi_r{}^{s}\nabla_j\varphi_t{}^{r}) = b_1 + b_2 + b_3,$$

then we have

$$b_1 \equiv \nabla_i\varphi_s{}^{t}\nabla_k\varphi_r{}^{s}\nabla_j\varphi_t{}^{r} = 0$$

by the hybridity $\mathfrak{h}(i,h)$ of $\nabla_j\varphi_i{}^{h}$ and the arguments in §1 and have

$$\begin{aligned} b_2 &\equiv \varphi_s{}^{t}\nabla_j\varphi_t{}^{r}\nabla_i\nabla_k\varphi_r{}^{s}, \\ b_3 &\equiv \varphi_s{}^{t}\nabla_k\varphi_r{}^{s}\nabla_i\nabla_j\varphi_t{}^{r} = \varphi_t{}^{r}\nabla_k\varphi_s{}^{t}\nabla_i\nabla_j\varphi_r{}^{s} \\ &= -\varphi_s{}^{t}\nabla_k\varphi_t{}^{r}\nabla_i\nabla_j\varphi_r{}^{s}. \end{aligned}$$

Hence it follows that

$$\mathfrak{S}\{b_1 + b_2 + b_3\} = \mathfrak{S}\{\varphi_s{}^{t}\nabla_j\varphi_t{}^{r}\ (\nabla_i\nabla_k\varphi_r{}^{s} - \nabla_k\nabla_i\varphi_r{}^{s})\}$$

$$= \mathfrak{S}(\varphi_s{}^t\nabla_j\varphi_t{}^r\ (R_{ikp}{}^s\varphi_r{}^p - R_{ikr}{}^p\varphi_p{}^s)\}$$
$$= 2\,\mathfrak{S}\{R_{ikr}{}^p\nabla_j\varphi_p{}^r\}.$$

Thus we get $\mathfrak{S}\{\nabla_i\hat{K}_{kj}\} = 0.$ q.e.d.

In the next place we shall prove the following

THEOREM 2. *In an almost-Hermitian space, the equation*

$$\underset{v}{\pounds}\hat{K}_{kj} = 2\,(\nabla_k t_j - \nabla_j t_k)$$

is valid for an analytic vector v^i, *where we put* $t_j = t_{jr}{}^s\varphi_s{}^r$ *and*

$$t_{ji}{}^h = \underset{v}{\pounds}\left\{ {h \atop ji} \right\} = \nabla_j\nabla_i v^h + R_{tji}{}^h v^t.\text{[6]}$$

PROOF. It is well known that the equations

$$\underset{v}{\pounds}R_{kji}{}^h = \nabla_k t_{ji}{}^h - \nabla_j t_{ki}{}^h, \tag{2.2}$$
$$\underset{v}{\pounds}\nabla_k\varphi_r{}^s - \nabla_k\underset{v}{\pounds}\varphi_r{}^s = t_{kp}{}^s\varphi_r{}^p - t_{kr}{}^p\varphi_p{}^s \text{[7]}$$

are valid for any vector field v^i.

Since v^i is analytic, from the last equation we have

$$\underset{v}{\pounds}\nabla_k\varphi_r{}^s = t_{kp}{}^s\varphi_r{}^p - t_{kr}{}^p\varphi_p{}^s. \tag{2.3}$$

From (2. 2) we have

$$\underset{v}{\pounds}\hat{R}_{kj} = 2\,\varphi_s{}^r(\nabla_k t_{jr}{}^s - \nabla_j t_{kr}{}^s) \tag{2.4}$$

and from (2. 3) and (1. 1) we have

$$-\varphi_s{}^t\underset{v}{\pounds}(\nabla_k\varphi_r{}^s\nabla_j\varphi_t{}^r) = 2\,(t_{jr}{}^s\nabla_k\varphi_s{}^r - t_{kr}{}^s\nabla_j\varphi_s{}^r). \tag{2.5}$$

Thus by virtue of (2. 1), (2. 4) and (2. 5) we get

$$\underset{v}{\pounds}\hat{K}_{kj} = 2\{\nabla_k(t_{jr}{}^s\varphi_s{}^r) - \nabla_j(t_{kr}{}^s\varphi_s{}^r)\}.$$ q. e. d.

3. *A*-spaces. We have known that in an almost-Kählerian space and a *K*-space the equation

$$\nabla_r\varphi_i{}^r = 0 \tag{3.1}$$

is valid.[8] An Hermitian space in which (3. 1) holds was first introduced by M.Apte [1]. Recently we dealed with an almost-Hermitian space satisfying (3.1) and obtained some results[9] and S.Kotō [3] also obtained interesting results.

By an *A*-space we shall mean such a space i.e. an almost-Hermitian space

6) Yano, K., [11], p. 9.
7) Yano, K., [11], p. 16-17.
8) Tachibana, S., [7], [8].
9) Tachibana, S., [8], [9].

in which (3. 1) is valid.[10]

In the first place we have

LEMMA 3. 1.[11] *In a compact A-space, if scalar function ρ satisfies $\partial_i\rho = \pm \varphi_i^{\ r}u_r$, where u_r is closed*[12], *then ρ is constant.*

PROOF. By the assumption we have $\nabla_i\rho = \pm \varphi_i^{\ r}u_r$. Hence it holds that $g^{rt}\nabla_r\nabla_t\rho = 0$ by making use of the skew-symmetry of φ^{ir}. Since the space is compact, we obtain the lemma. q e. d.

Consider an analytic vector v^i in an A-space. From (2. 3) we have

$$\underset{v}{\pounds}\nabla_k\varphi_r^{\ s} = t_{kp}^{\ \ s}\varphi_r^{\ p} - t_{kr}^{\ \ p}\varphi_p^{\ s}.$$

Contracting k and s and taking account of (3. 1) and $t_{ji}^{\ \ h} = t_{ij}^{\ \ h}$, we get $t_i = t_{pr}^{\ \ r}\varphi_i^{\ p}$. On the other hand we have $t_{pr}^{\ \ r} = \partial_p(\nabla_t v^t)$ for any vector field v^i. Hence we obtain

LEMMA 3. 2. *In an A-space, the relaiton $t_i = \varphi_i^{\ r}\partial_r(\nabla_t v^t)$ holds good for an analytic vector v^i.*

By virtue of these lemmas, we can prove the following

THEOREM 3. *In a compact A-space, if an analytic vector v^i satisfies*

$$\underset{v}{\pounds}\hat{K}_{kj} = 0, \tag{3.2}$$

then the infinitesimal transformation v^i is volume-preserving, i.e., $\nabla_t v^t = 0$.

PROOF. Let v^i be an analytic vector satisfying (3.2). From Lemma 3. 2 we have $\partial_i(\nabla_t v^t) = -\varphi_i^{\ r}t_r$. On the other hand, from (3. 2) and Theorem 2 we have that t_j is closed. Hence Lemma 3.1 implies that $\nabla_t v^t = \text{const.}$, from which we have the theorem. q. e. d.

4. O^*-spaces. In an almost-Hermitian space, $\varphi_{ji} = \varphi_j^{\ r}g_{ri}$ is hybrid but $\nabla_k\varphi_{ji}$ is $\mathfrak{p}(j, i)$.

S. Kotō [3] defined a certain almost-Hermitian space, called O^*-space, and discussed such a space. By definition an O^*-space is an almost-Hermitian space such that $\nabla_k\varphi_{ji}$ is a pure tensor. He also proved that almost-Kählerian spaces and K-spaces are O^*-spaces and O^**-spaces are* A-spaces. These relations are indicated by the following diagramm:

A-space ——— O^*-space < almost-Kählerian space / K-space

10) An A-Space is called an almost semi-Kählerian space in [3].

11) This lemma is a generalization of Lemma 1. 1 in [8]. Professor K. Yano proved the lemma without the assumption (3. 1), personal communication.

12) We suppose a covariant vector as a form in a natural way.

Since an O^*-space is an A-space, the discussions in §3 are applicable to O^*-spaces.

In an O^*-space, an analytic vector has a remarkable property which will be given in the following Lemma 4. 1.

LEMMA 4. 1. (Kotō [3]) *In an O^*-space, an analytic vector v^i satisfies*

$$g^{rs}t_{rs}{}^h = g^{rs}\nabla_r\nabla_s v^h + R_r{}^h v^r = 0.$$

For the completeness we shall give its proof.

PROOF In the first place we notice that the purity (hybridity) of a tensor is preserved by Lie derivation with respect to an analytic vector. Hence if v^i is an analytic vector, then in an O^*-space $\underset{v}{£}\nabla_k\varphi_j{}^h$ is $\mathfrak{p}(k,j)$.

From (2. 3) we have

$$\underset{v}{£}\nabla_k\varphi_j{}^h = t_{kp}{}^h\varphi_j{}^p - t_{kj}{}^p\varphi_p{}^h.$$

Transvecting this with $\varphi^{jk} = \varphi_r{}^k g^{rj}$ which is hybrid, we get

$$\varphi^{jk}\underset{v}{£}\nabla_k\varphi_j{}^h = g^{kp}t_{kp}{}^h - \varphi^{jk}t_{kj}{}^p\varphi_p{}^h.$$

If we take account of the arguments in §1 and the symmetry of $t_{ji}{}^h$, then we have $g^{kp}t_{kp}{}^h = 0$. q. e. d.

LEMMA 4. 2. (Kotō [3]) *In a compact O^*-space, if an analytic vector v^i satisfies $\nabla_i v^i = 0$, then it is a Killing vector and hence an automorphism.*[13)]

This follows directly from Lemma 4. 1 and the well known theorem on Killing vectors.

By virtue of Theorem 3 and Lemma 4. 2 we obtain

THEOREM 4.[14)] *In a compact O^*-space, if an analytic vector v^i satisfies*

$$\underset{v}{£}\hat{K}_{kj} = 0,$$

then it is an automorphism.

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13) By an automorphism we mean an infinitesimal isometry which is analytic.

14) This is a generalization of Corollary in Lichnerowicz, A., [4], p. 148. We remark that Theorem 1 does not play any role in the proof of Theorem 4.

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