APPLICATIONS OF FIBRE BUNDLES TO THE CERTAIN CLASS OF C^*-ALGEBRAS

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Introduction. It has been long discussed whether the sets of pure states of C^*-algebras are compact or not and some negative examples are found in the literature (cf. [5], [6], [13], [18]). However the question that which C^*-algebras have this property is remained unknown and it is the first motivation of our present paper to remove this obscurity. The result is the following one: *Let A be a C^*-algebra. If the set of pure states of A is compact and that of primitive ideals which are the kernels of one-dimensional irreducible representations forms an open set in the structure space of A, then A is isomorphic to the C^*-sum of a finite number of homogemeous C^*-algebras.*

A C^*-algebra is called *n-dimensionally homogeneous* if each irreducible representation of the algebra is n-dimensional. Such C^*-algebras were partly studied (without assuming a unit) in Kaplansky [10], [11] and Fell [4]. However, only a few results are known about the structure of these algebras. On the other hand, these algebras play an essential rôle in the construction of the composition series of GCR algebras. Thus the main part of the present paper is devoted to develop the structure theory of homogeneous C^*-algebras. Our method is somewhat different from the one usually employed in the literature. We use the theory of fibre bundles and illustrate the structure of homogeneous C^*-algebras in terms of fibre bundles.

Let A be an n-dimensionally homogeneous C^*-algebra and denote by $\Omega(A)$ the structure space of A. Let M_n and G be the $n \times n$ full matrix algebra and the group of all $*$-automorphisms of M_n. Then A defines a fibre bundle $\mathfrak{B}(A)$, called the structure bundle of A, over $\Omega(A)$ with fibre M_n and group G and A is represented as the C^*-algebra constructed by all cross-sections in $\mathfrak{B}(A)$. It is shown that the $*$-isomorphic relation between two n-dimensionally homogeneous C^*-algebras are equivalent to the equivalence relation between their structure bundles. Moreover, using the theory of bundles we can show that two algebraically isomorphic homogeneous C^*-algebras are necessarily $*$-isomorphic. Next we shall prove that the bundle $\mathfrak{B}$ over an arbitrary compact Hausdorff space with fibre M_n and group G yields necessarily an n-dimensionally homogeneous C^*-algebra A and $\mathfrak{B}(A)$ is equivalent to $\mathfrak{B}$. Thus there exist non-isomorphic n-dimensionally homogeneous C^*-algebras as much as non-equivalent fibre bundles over the compact Hausdorff spaces with fibre M_n and group G.

A typical example of a homogeneous C^*-algebra is the C^*-tensor product of a commutative C^*-algebra and M_n, but an arbitrary n-dimensionally homogeneous C^*-algebra does not necessarily belong to this type, contrary to the case of W^*-algebras. Our result says that an n-dimensionally homogeneous C^*-algebra belongs to this type if and only if its structure bundle equivalent to a product bundle.

Finally, we study the $*$-automorphisms of homogeneous C^*-algebras leaving the center elementwise fixed and give the necessary and sufficient condition that the automorphism is inner.

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1. The structure of C*-algebras whose sets of pure states are compact.

Let A be a C^*-algebra. We always assume a unit. We call a C^*-algebra A *n-dimensionally homogeneous* (sometimes abbreviated as *n-homogeneous C^*-algebra*) if any irreducible representation of A is n-dimensional. We assume that A is acting on a fixed Hilbert space H_0. and we shall denote by $\widetilde{A}$ the weak closure of A on H_0. We always denote by $Q(A)$, $P(A)$, $S(A)$ and $\Omega(A)$ the set of pure states, the pure state space (weak closure of $Q(A)$), the state space and the structure space of A, respectively. $B(H)$ and C_H mean the ring of all bounded linear operators and that of all completely continuous operators on a Hilbert space H.

The following theorem almost clarifies the structure of C*-algebras having compact set of pure states.

THEOREM 1. *Let A be a C^*-algebra. If the set of pure states of A is compact and that of primitive ideals which are the kernels of one-dimensional irreducible representatious forms an open set in the structure space of A, then A is isomorphic to the C^*-sum of a finite number of homogeneous C^*-algebras. The converse is also true.*

We divide the proof into several steps and at first assume that A has no one-dimensional representations.

LEMMA 1.1. *Let π be an arbitrary irreducible representation of A on a Hilbert space H, then H is finite dimensional and $\pi(A) = C_H = B(H)$.*

PROOF. Since $Q(A)$ is compact, one easily verifies that the homomorphic image of A also has this property. Therefore, by [6: Theorem 2], we see that $\pi(A)\cap C_H \neq \{0\}$, so that $\pi(A) \supset C_H$ by the same theorem. Suppose $\pi(A) \supsetneqq C_H$, then there exists a state φ with the property $\varphi(C_H)=0$. Using the same theorem quoted above one can see that $a\omega_\xi + (1-a)\varphi \in P(\pi(A))$ for $0<a<1$ and a vector state ω_ξ. Therefore $Q(\pi(A))$ is not compact, which is a contradiction. We have $\pi(A)=C_H$. Hence C_H contains a unit, and H must be finite dimensional. Thus, $\pi(A)=C_H=B(H)$.

LEMMA 1. 2. *Let π be an irreducible representation of $\tilde{A}$ on a Hilbert space H, then $\pi|A$, the restriction of π to A, is also irreducible and $\pi(\tilde{A})=\pi(A)$*

PROOF. Take a vector state ω_ξ of **H**, then $\omega_\xi|\pi(\tilde{A})$ is a pure state. Hence ${}^t\pi(\omega_\xi|\pi(\tilde{A}))$ is a pure state of $\tilde{A}$. As A is σ-weakly dense in $\tilde{A}$, by [6: Theorem 5], $P(\tilde{A})|A = P(A) = Q(A)$, hence ${}^t\pi(\omega_\xi|\pi(\tilde{A}))|A = {}^t\pi(\omega_\xi|\pi(A)) \in Q(A)$. It follows that $\omega_\xi|\pi(A) \in Q(\pi(A))$. Since ω_ξ is an arbitrary vector state of H and A has no one-dimensional representations, this means that $\pi|A$ is an irreducible representation of A. Therefore, by the above lemma, $\pi(A)=C_H=B(H)$ whence $\pi(A)=\pi(\tilde{A})$.

LEMMA 1. 3. *$\tilde{A}$ is a W*-algebra of finite type 1 whose homogeneous components are finite.*

PROOF. This follows immediately from Lemma 1. 2. and [11: Theorem 9. 1.].

By Lemma 1, 3 we see that $\tilde{A}=\sum_{i=1}^{m}\tilde{A}z_i$ where z_i are orthogonal central projections and each $\tilde{A}z_i$ is an n_i-dimensionally homogeneous W*-algebra. We assume that $n_1<n_2\cdots<n_m$.

LEMMA 1. 4. *Az_i is an n_i-dimensionally homogeneous C*-algebra.*

PROOF. Let π be an irreducible representation of Az_i on Hilbert space H. Since every pure state of Az_i can be extended to a pure state of $\tilde{A}z_i$, it follows from the correspondence between states and representations of $\tilde{A}z_i$ that π can be extended to an irreducible repreentation $\tilde{\pi}$ of $\tilde{A}z_i$ on a Hilbert space $H'\supseteqq H$. On the other hand, as Az_i is the homomorphic image of A, $Q(Az_i)$ is compact and Az_i is weakly dense in $\tilde{A}z_i$. Hence, applying Lemma 1. 2 to this couple of algebras, one can see that $\tilde{\pi}|Az_i$ is an n_i-dimensional irreducible representation of Az_i on H'. Therefore we get $H=H'$ and π is an n_i-dimensional irreducible representation. This completes the proof.

Now, consider a primitive ideal P of a GCR algebra. Since P is the kernel of some irreducible representation of the algebra, by Dixmier [3: Theorem 5], the dimension of the irreducible representation whose kernel is P is unique (the hypothesis that the algebra is separable is unnecessary in this part of the theorem). Thus, without the ambiguity, we can say that a primitive ideal P is the kernel of a ( )-dimensional irreducible representation.

LEMMA 1.5. *There is a one-to-one correspondence between the family of all primitive ideals in A which are kernels of n_k-dimensional irreducible representations and that of all primitive ideals in Az_k.*

PROOF. Let P be a primitive ideal of A whfch is the kernel of an n_k-dimensional irreducible representation. Take a pure state φ associated with P and consider the pure state extension $\bar{\varphi}$ of φ to $\tilde{A}$. Denote by π_φ and $\pi_{\bar{\varphi}}$ the canonical representations of φ and $\bar{\varphi}$. As $\bar{\varphi} \in P(\tilde{A}) = \bigcup_{i=1}^{m} P(\tilde{A}z_i)$ (where we identify $P(\tilde{A}z_i)$ with its natural embedding in $P(\tilde{A})$ (cf. [6]), there exists a number j for $\bar{\varphi} \in P(\tilde{A}z_j)$. Therefore the representation $\pi_{\bar{\varphi}}$ is the composition of the mapping $\tilde{A} \to \tilde{A}z_j$ and an irreducible representation π' of $\tilde{A}z_j$. By Lemma 1.2, $\pi_{\bar{\varphi}}|A$ is an irreducible representation of A, and we can identify π_φ and $\pi_{\bar{\varphi}}|A$. Hence π_φ is the composition of the map $A \to Az_j$ and $\pi'|Az_j$. The latter is an irreducible representation of Az_j. We have $Pz_j =$ the kernel of $\pi'|Az_j$, which is a primitive ideal of Az_j. Since P is a kernel of an n_k-dimensional irreducible representation, we have, by Lemma 1.4, $n_k = n_j$ i.e. $k = j$. Thus $Pz_j = Pz_k$ is a primitive ideal of Az_k. Next, if P' is a primitive ideal of Az_k it is clear that $P = \{a \in A \mid az_k \in P'\}$ is a primitive ideal of A which is the kernel of n_k-dimensional irreducible representation. Moreover, the fact that $\Omega(A)$ coincides with the space of all maximal ideals in A(cf. Lemma 1.1.) implies that the above correspondence is one-to-one.

LEMMA 1.6. *Let P be a primitive ideal of A which is the kernel of an n_k-dimensional irreducible representation, then we have $P(1-z_k) = A(1-z_k)$.*

PROOF. At first, we assert that $Pz_i = Az_i$ for all $i \neq k$. Suppose Pz_i is a proper closed ideal of Az_i for $i \neq k$. There exists a maximal ideal P' of Az_i containing Pz_i. Set $P_0 = \{a \in A \mid az_i \in P'\}$, then P_0 is an ideal of A and contains P. We have $P_0 = P$ for P is a maximal ideal of A by Lemma 1.1. Thus $Pz_i = P_0 z_i = P'$, that is, Pz_i is a primitive ideal of Az_i. Therefore P is the kernel of an n_i-dimensional irreducible representation which contradicts to $i \neq k$. We have $Pz_i = Az_i$ for all $i \neq k$.

To prove the conclusion of the lemma, it is sufficient to show that $P(1-z_k)$ separates the set of pure states of $A(1-z_k)$ by Kaplansky [11: Theorem 7.2.]. So let φ and ψ be distinct pure states of $A(1-z_k)$ and $\bar{\varphi}$ and $\bar{\psi}$ pure state extensions of φ and ψ to $\tilde{A}(1-z_k)$ respectively. Since $P(\tilde{A}(1-z_k)) = \bigcup_{i \neq k} P(\tilde{A}z_i)$, we have two cases in question.

1. $\overline{\varphi} \in P(\widetilde{A}z_i)$ and $\overline{\psi} \in P(\widetilde{A}z_i)$ for $i \neq j$. Consider the canonical representations $\pi_{\overline{\varphi}}$ and $\pi_{\overline{\psi}}$, then as in the argument of the proof of Lemma 1.5 $\pi_{\overline{\varphi}}$ and $\pi_{\overline{\psi}}$ induce the irreducible representations π_1 and π_2 of Az_i and Az_j. Put $P_1 = \{a \in P(1-z_k) | az_i \in$ the kernel of $\pi_1\}$, $P_2 = \{a \in P(1-z_k) | az_j \in$ the kernel of $\pi_2\}$, then P_1 and P_2 are primitive ideals of $P(1-z_k)$ and, by Lemma 1.4, they are the kernels of n_i and n_k-dimensional irreducible representations of $P(1-z_k)$ respectively. Since $n_i \neq n_j$, P_1 is different from P_2. On the other hand one easily verifies that P is equal to the kernel of $\pi_\varphi | P(1-z_k)$ and P_2 to the kernel of $\pi_\psi | P(1-z_k)$. It follows $\varphi | P(1-z_k) \neq \psi | P(1-z_k)$.

2. $\overline{\varphi}, \overline{\psi} \in P(\widetilde{A}z_i)$. From the assumption there exists an element $a(1-z_k) \in A(1-z_k)$ $(a \in A)$ such that $\varphi(a(1-z_k)) \neq \psi(a(1-z_k))$.

We have

$$\overline{\varphi}(az_i) = \overline{\varphi}(a(1-z_k)) = \varphi(a(1-z_k)) \neq \psi(a(1-z_k)) = \overline{\psi}(az_j).$$

Since $Pz_i = Az_i$, we can find an element of P such as $bz_i = az_j$ and we get

$$\varphi(b(1-z_k)) = \overline{\varphi}(bz_j) = \overline{\varphi}(az_i) \neq \overline{\psi}(az_i) = \overline{\psi}(bz_i) = \psi(b(1-z_k)),$$

i.e. $\quad \varphi | P(1-z_k) \neq \psi | P(1-z_k)$.

This completes the proof.

LEMMA 1.7. *Let $\{P_{n_k,\alpha}\}$ be the family of all primitive ideals in A which are the kernels of n_k-dimensional irreducible representations. Then for every k, $\{P_{n_k,\alpha}\}$ is closed in $\Omega(A)$.*

PROOF. we shall show that $\bigcap_\alpha P_{n_k,\alpha} = A(1-z_k) \cap A$. Take an element $a \in \bigcap_\alpha P_{n_k,\alpha}$ then $az_k \in \bigcap_\alpha P_{n_k,\alpha} z_k$. By Lemma 1.5, $\{P_{n_k,\alpha}z_k\}$ is the family of all primitive ideals in Az_k. Hence $az_k = 0$ i.e. $a \in A(1-z_k) \cap A$. Conversely, if $a \in A(1-z_k) \cap A$ we have $az_k = 0 \in \bigcap_\alpha P_{n_k,\alpha} z_k$ and, by Lemma 1.5, $a \in \bigcap_\alpha P_{n_k,\alpha}$.

Next, suppose that a primitive ideal P contains the intersection of $\{P_{n_k,\alpha}\}$. By the argument at the first part of the proof of Lemma 1.6, $Pz_i = Az_i$ except a number $j \neq i$ and Pz_j is a primitive ideal of Az_j. We assume $j \neq k$. By Lemma 1.6, we have $P(1-z_j) = A(1-z_j)$ so that there exists an element $b \in P$ such as $b(1-z_j) = 1-z_j$, which implies $(1-b)z_j = 1-b$. As $z_j \leqq 1-z_k$, we get $1-b \in A(1-z_k) \cap A = \bigcap_\alpha P_{n_k,\alpha}$. Hence P contains $1-b$ and thus P contains 1, a contradiction. Therefore we have $j = k$, i.e. P is the kernel of an n_k-dimensional irreducible representation. This completes the proof.

THE CONCLUDING PROOF OF THE THEOREM. Let $\{P_{1,\alpha}\}$ be the family of all primitive ideals which are the kernels of one-dimensional representations. Put $R =$ the kernel of $\{P_{1,\alpha}\}^c$, complement of $\{P_{1,\alpha}\}$ in $\Omega(A)$, then the C*-algebra A/R has no one-dimensional representations and $Q(A/R)$ is compact. Hence, by the above discussions, we can see that all irreducible representations of A are finite and there is a fixed upper bound for all these degrees. Denote by $\{P_{n_k,\alpha}\}$ $(n_0=1<n_1<n_2<\cdots\cdots<n_m)$ the family of all primitive ideals in A which are the kernels of n_k-dimensional irreducible representations. We assert that this family is closed in $\Omega(A)$. Since $\{P_{1,\alpha}\}$ is clearly closed in $\Omega(A)$ (cf. [11]), we assume $k \geqq 2$. Then, by Lemma 1.7, the canonical image of $\{P_{n_k,\alpha}\}$ is closed in $\Omega(A/R)$ and since $\Omega(A/R)$ is homeomorphic to $\{P_{1,\alpha}\}^c$, $\{P_{n_k\alpha,}\}$ is closed in $\{P_{1,\alpha}\}^c$ whence in $\Omega(A)$. Put $I = \bigcap_\alpha P_{n_m,\alpha}$ and $J = \bigcap_{i=1}^{m-1}\bigcap_\alpha P_{n_i,\alpha}$. From the definition of I and J it is clear that $I \cap J = \{0\}$. Suppose that $I + J$ is not dense in A, then we can find a maximal ideal P such that $P \supseteqq I + J$. Since P is primitive, P is the kernel of an n_m-dimensional irreducible representation by Lemma 1. 7. On the other hand, it is known that the family $\{P_{n_i,\alpha}\}$ (i = 1, 2 $\ldots,m$-1) is closed in $\Omega(A)$ (cf. [11]), hence P is the kernel of an irreducible representation whose dimension does not exceed n_{m-1}. This is a contradiction. Therefore $I + J$ is dense in A. Hence $I + J = A$, so that one can see that A is isomorphic to the C*-sum of A/I and A/J. Continuing the same argument for A/J, we get desired conclusion: that is, putting $I_i = \bigcap_\alpha P_{n_i,\alpha}\,(i=1,2,\ldots,m)$ we have

$$A \cong A/I_1 \oplus A/I_2 \oplus \cdots\cdots \oplus A/I_m \text{ (C*-sum)}$$

and it is clear, by Lemma 1.7, that each A/I_k is an n_k-dimensionally homogeneous C*-algebra.

To prove the converse of the theorem, we need the following

LEMMA 1. 8. *If A is an n-dimensionally homogeneous C*-algebra, then $Q(A)$ is compact.*

PROOF. Since A is weakly dense in $\tilde{A}$, $\tilde{A}$ satisfies the polynomial identity of the same order as the one which A satisfies (cf. [11]). Hence the dimension of any irreducible representation of $\tilde{A}$ does not exceed n. Therefore $\tilde{A}$ is a W*-algebra of finite type 1. Let x be a maximal ideal of the center of $\tilde{A}$. We denote by P_x the minimal closed ideal in $\tilde{A}$ containing x. Since $\Omega(A)$ is a Hausdorff space by [11: Theorem 4.2]. the first part of the proof of Lemma 12 of [6] shows that $A(x)$ has a faithful irreducible representation where $A(x)$

means the homomorphic image of A by the canonical map $A \to A/P_x$. Moreover Theorem 4 in [6] shows that $\widetilde{A}(x)$ has a faithful irreducible representation. Now, by the hypothesis $A(x)$ is isomorphic to an $n \times n$ full matrix algebra and the first part of this proof shows that $\widetilde{A}(x)$ is isomorphic to a $k \times k$ matrix algebra with $k \leqq n$. Hence $k = n$ and we have $A(x) = \widetilde{A}(x)$.

Take an element $\varphi \in P(A)$, then by [6: Theorem 5] φ can be extended to an element $\tilde{\varphi} \in P(\widetilde{A})$. By Theorem 4 in [11] there exists a maximal ideal x of the center of $\widetilde{A}$ and an element $\tilde{\varphi}' \in P(\widetilde{A}(x))$ such as $\tilde{\varphi} = {}^t\psi_x\ (\tilde{\varphi}')$ where ψ_x means the canonical mapping $\widetilde{A} \to \widetilde{A}/P_x$. We have $\varphi = {}^t\psi_x\ (\tilde{\varphi}' \,|\, A(x))$. As it is easily seen that $Q(A(x))$ is compact the above argument shows that $\varphi \in Q(A)$ i. e. $Q(A)$ is compact.

Using the above lemma, we can prove easily the converse of the theorem. In fact, suppose $A \cong A_1 \oplus A_2 \oplus \cdots \oplus A_m$ where $\{A_i\}$ are all homogeneous C*-algebras, then $P(A) = \bigcup_{i=1}^{m} P(A_i)$ where we identify $P(A_i)$ with its natural embedding in $P(A)$. By Lemma 1.8 we have $P(A_i) = Q(A_i)$, hence $P(A) = Q(A)$ considering with $Q(A) = \bigcup_{i=1}^{m} Q(A_i)$.

Thus the whole proof is completed.

Now, Let us consider the situation of our theorem. One might suppose that the size of the pure state space plays an important rôle in the structure theory of a C*-algebra and, at this point of view, the case treated above is an extreme one and it is the first step to make out this case when the pure state space of a C*-algebra comes into our consideration. The another extreme case is the one where the pure state space becomes the largest one, that is, it coincides with the state space. The next theorem gives the answer for the question which C*-algebras are endowed with this property.

THEOREM 2. *Let A be a C*-algebra, then the necessary and sufficient condition for $P(A) = S(A)$ is that A is a prime C*-algebra without non-zero GCR ideal.*

PROOF. The proof of the sufficiency is essentially included in the proof of Theorem 1 in [6]. In fact, the proof of Theorem 1 in [6] is divided into three steps and the last step is devoted to prove the result associated to the sufficiency of our theorem. Hence it is sufficient to show the necessity of the theorem.

Suppose that $P(A) = S(A)$. We shall show that A is a prime C*-algebra. Let I and J be closed ideals in A sush as $I \cap J = \{0\}$ then one easily verifies that $\widetilde{I} \cap \widetilde{J} = \{0\}$, where $\widetilde{I}$ and $\widetilde{J}$ mean the weak closures of I and J in $\widetilde{A}$. Since $\widetilde{I}$ (resp. $\widetilde{J}$) is a weakly closed ideal of $\widetilde{A}$ we can find a central projection z_1(resp. z_2) such as $\widetilde{I} = \widetilde{A}z_1$ (resp. $\widetilde{J} = \widetilde{A}z_2$). We have $z_1 z_2 = 0$. Hence for an arbitrary element $\psi \in P(\widetilde{A})$, $\psi(z_1)\psi(z_2) = 0$.

By [6: Theorem 5], $P(\widetilde{A})|A = P(A) = S(A)$. If we suppose that $I \neq \{0\}$ and $J \neq \{0\}$, there exists a state φ of A such that $\varphi(I) \neq 0$ and $\varphi(J) \neq 0$. Putting $\bar{\varphi}$ as an extension of φ to an element of $P(\widetilde{A})$, we get $\bar{\varphi}(\widetilde{I}) \neq 0$ and $\bar{\varphi}(\widetilde{J}) \neq 0$, which contradicts to $\varphi(z_1)\varphi(z_2) = 0$. Thus, if $I \cap J = \{0\}$ we have $I = \{0\}$ or $J = \{0\}$; A is a prime C*-algebra.

Next, suppose that A has a non-zero GCR ideal, then A has a non-zero CCR ideal I. Since A becomes a prime C*-algebra, I is isomorphic to the algebra of all completely continuous operators on some Hilbert space (cf. [11 : Lemma 7. 14]). Take an arbitrary state φ of I and denote by $\bar{\varphi}$ the state extension of φ to A. As $\bar{\varphi} \in S(A) = P(A)$ there exists a net of pure states $\{\varphi_\alpha\}$ of A with $\bar{\varphi} = \lim_\alpha \varphi_\alpha$. We have $\lim_\alpha \varphi_\alpha | I = \varphi$. Therefore we may assume that $\varphi_\alpha | I \neq 0$ for all α. We assert that all $\varphi_\alpha | I$ are pure states of I. In fact, by Glimm [6 : Lemma 3], $\{\varphi_\alpha | I\}$ are all states of I and suppose that $\varphi_\alpha | I = \frac{1}{2}(\varphi_1' + \varphi_2')$ for some states φ_1', φ_2' of I. Denoting by φ_1 and φ_2 the state extensions of φ_1' and φ_2' to A we get an extension of $\varphi_\alpha | I$ to A, $\frac{1}{2}(\varphi_1 + \varphi_2)$. However, as shown in the proof of Theorem 2 of Glimm [6] the state extension from a closed ideal to the whole algebra is unique, and we get $\varphi_\alpha = \frac{1}{2}(\varphi_1 + \varphi_2)$ which implies $\varphi_\alpha = \varphi_1 = \varphi_2$. Hence $\varphi_\alpha | I = \varphi_1' = \varphi_2'$. Thus $\{\varphi_\alpha | I\}$ are pure states of I and φ is the limit of the net $\{\varphi_\alpha | I\}$. This contradicts the structure of I noted above. Therefore A has no non-zero GCR ideal. This completes the proof.

2. The structure of homogeneous C*-algebras.

Let A be an n-dimensionally homogeneous C*-algebra with unit. Throughout this section, M_n means an $n \times n$ full matrix algebra and G the group of all *-automorphisms of M_n. If R is a subset of the structure space $\Omega(A)$ of A, we denote by $A(R)$ the quotient algebra by the kernel of R and by $a(R)$ the canonical image of $a \in A$ in $A(R)$. A matrix of M_n is always denoted with indices i,j (or k, l) such as (λ_{ij}), (a_{ij}). Most of the notations and terminologies in the theory of fibre bundles are referred to Steenrod [14]. We use the notations $GL(n, C)$, $U(n)$, $SU(n)$ etc. as usual.

Let G_0 be the group of all automorphisms of M_n. Before going into discussions, we need some considerations on the topologies of G_0 and G. Consider the simple convergence topology on G_0 and G. Since an arbitrary element of G_0 is considered to be a bounded linear operator on the vector space M_n, G_0 is embedded into the full operator algebra $B(M_n)$, which is isomorphic to M_{n^2}, $n^2 \times n^2$ full matrix algebra. Hence G_0 can be embedded into $GL(n^2, C)$ and one verifies easily that this embedding is topologically isomorphic, so that the image of G_0 in $GL(n^2, C)$ is a closed subgroup of $GL(n^2, C)$. Therefore G_0 becomes a topological group (furthermore a Lie group) by the simple convergence topology over M_n. On the other hand, let T_0 be the center of $GL(n, C)$ then it is well known that $GL(n,C)$ is homomorphic to G_0 and the kernel of this homomorphism is T_0. By the straight-forward calculation, we see that this homomorphism is a continuous, hence open homomorphism. Thus G_0 is topologically isomorphic to $GL(n,C)/T_0$ and a similar treatment shows that the group G with the simple convergence topology is topologically isomorphic to the factor group $U(n)/T$, where T denotes the center of $U(n)$. We notice that both G_0 and G are topological transformation groups of M_n.

The following lemma plays the key point of our discussions.

LEMMA. *For any point P of $\Omega(A)$, there exists a neighborhood U of P such as $A(U) \cong C(\overline{U}) \mathbin{\hat{\otimes}_\alpha} M_n$, the C^*-tensor product of $C(\overline{U})$ and M_n, where $\overline{U}$ means the closure of U.*

PROOF. Take a point $P_0 \in \Omega(A)$. Define a continuous function $\gamma(x)$ over $(-\infty, \infty)$ as follows;

$$\gamma\left\{\left(-\infty, -\frac{1}{2}\right] \cup \left\{\frac{1}{2}\right\} \cup \left[\frac{3}{2}, +\infty\right)\right\} = 1,\ \gamma(0) = \gamma(1) = 0,$$

and $\gamma(x)$ is linear on $\left[-\frac{1}{2}, 0\right], \left[0, \frac{1}{2}\right], \left[\frac{1}{2}, 1\right], \left[1, \frac{3}{2}\right]$. Let a be an element of A such as $a(P_0)$ is a non-zero projection, then we have $\gamma(a(P_0)) = \gamma(a)(P_0) = 0$. Denote by U the set of $\left\{P \in \Omega(A);\ \|\gamma(a)(P)\| < \frac{1}{2},\ \|a(P)\| > \frac{3}{4}\right\}$. Since $\Omega(A)$ is a Hausdorff space, $\|a(P)\|$ is a continuous function on $\Omega(A)$ and hence U is a neighborhood of P_0 (cf. [11; Theorem 4. 2 and 4. 1]). Next, choose the function $\delta(x)$ defined by; $\delta\left(\left(-\infty, \frac{1}{4}\right]\right) = 0$, $\delta\left(\left[\frac{3}{4}, +\infty\right)\right) = 1$, $\delta(x)$ is linear on $\left[\frac{1}{4}, \frac{3}{4}\right]$. If $P \in U$, then the spectrum $\sigma(a(P))$ of

$a(P)$ is contained in $\left[-\frac{1}{4},\frac{1}{4}\right]\cup\left[\frac{3}{4},\frac{5}{4}\right]$ but not in $\left[-\frac{1}{4},\frac{1}{4}\right]$. Hence $\delta(a)(P)$ is a non-zero projection in $A(P)$ and $\delta(a)(P_0)=a(P_0)$.

By the assumptions, $A(P)\cong M_n$ for all $P\in\Omega(A)$. Therefore there exist n elements $a_1,a_2,\dots,a_n$ in A such that $a_1(P_0),a_2(P_0),\dots,a_n(P_0)$ are orthogonal minimal projections in $A(P_0)$. Applying the above arguments to a_1 we get a neighborhood U_1 of P_0 such that, putting $e_1=\delta(a_1)$, $e_1(P)$ is always a non-zero projection for $P\in U_1$ and $e_1(P_0)=a_1(P_0)$. Suppose inductively that we have chosen $U_1,\dots,U_m$, $m<n$, to be neighborhoods of P_0 and $e_1,\dots,e_m$ in A such that $\{e_i(P);\ i=1.2,\dots,m\}$ are non-zero orthogonal projections for $P\in\bigcap_{i=1}^{m}U_i$ and $e_i(P_0)=a_i(P_0)$. Applying the first argument to $\left(1-\sum_{i=1}^{m}e_i\right)a_{m+1}\left(1-\sum_{i=1}^{m}e_i\right)$ we can find again a neighborhood U_{m+1} of P_0 such that if we define $e_{m+1}=\delta\left(\left(1-\sum_{i=1}^{m}e_i\right)a_{m+1}\left(1-\sum_{i=1}^{m}e_i\right)\right)$, $e_{m+1}(P)$ is a non-zero projection for any $P\in U_{m+1}$ and it is clear that the system $\{U_1,U_2,\dots,U_m,\ U_{m+1};\ e_1,e_2,\dots,e_m,\ e_{m+1}\}$ also satisfies the inductive assumption. Therefore, we can get a neighborhood U of P_0 and n-elements $e_1,e_2,\dots,e_n$ in A such as $\{e_i(P)\,|\,i=1,2,\dots,n\}$ are non-zero orthogonal minimal projections in $A(P)$ for any $P\in U$.

Since $e_iAe_1(P_0)\neq0$, there exists an element $b_i\in e_iAe_1$ such as $b_i(P_0)\neq0$. We have $b_i^*b_i(P_0)\neq0$. Hence, by the continuity of $\|b_i^*b_i(P)\|$ (cf. [11]), we get a neighborhood U_i of P_0 such as $b_i^*b_i(P)>0$ for any $P\in\overline{U}_i$. In this case it is no loss of generality that we may identify this U_i with the above preceding U_i, so that $b_i^*b_i(P)>0$ for all $P\in\overline{U}$ and $i=1,2,\dots,n$. Notice that the kernel of $\overline{U}$ coincides with the kernel of U. Now, $\{e_i(U);\ i=1,2,\dots,n\}$ are orthogonal abelian projections and $\sum_{i=1}^{n}e_i(U)=1$ in $A(U)$. Since the structure space of $A(U)$ is homeomorphic to $\overline{U}$, one easily verifies that $e_1Ae_1(U)$ may be identified with $C(\overline{U})$, the ring of all continuous functions on $\overline{U}$(cf. [11; Lemma 4.1]). Therefore $b_i^*b_i(P)>0$ $(P\in\overline{U})$ imply that there exists $c_i\in A$ such that $c_i(U)>0$ and $c_ib_i^*b_i(U)=b_i^*b_ic_i(U)=e_1(U)$. Take an element $u_i\in A$ with $u_i(U)=b_i(U)c_i(U)^{1/2}$. We have

$$u_i^*u_i(U)=c_i^{1/2}b_i^*(U)b_i(U)c_i(U)^{1/2}=c_i(U)b_i^*b_i(U)=e_1(U).$$

Besides, as $u_i^*u_i(U)$ is a non-zero projection, $u_iu_i^*(U)$ is a non-zero projection. It follows that $u_iu_i^*(P)\geqq e_i(P)$ for each $P\in U$. Thus $u_iu_i^*(U)=e_i(U)$.

Put $u_{ij}=u_iu_j^*$. By the above discussions, $\{u_{ij}(U);\ i,j=1,2,\dots,n\}$ are the matrix units of $A(U)$, hence $A(U)$ is isomorphic to the C^*-tensor product of $e_1Ae_1(U)$ and M_n. After all, $A(U) \cong C(\overline{U})\hat{\otimes}_\alpha M_n$. This completes the proof.

Now, let $\{U_\alpha\}$ be the family of open sets corresponding to each point of $\Omega(A)$ in the preceding lemma. Denoting by $\{u_{ij}^\alpha\}$ those elements of A which induce the matrix units of $A(U_\alpha)$, we have

$$a(U_\alpha) = \sum_{i,j=1}^{n} a_{ij}^\alpha(U_\alpha)u_{ij}^\alpha(U_\alpha) \qquad \text{for any } a \in A$$

where $\{a_{ij}^\alpha(U)\}$ may be considered to be continuous functions on $\overline{U}_\alpha$. Put $B = \bigcup_{P\in\Omega(A)} A(P)$ and define the map pr from B to $\Omega(A)$ as $pr(a(P)) = P$. Let ϕ_α be the map from $U_\alpha \times M_n$ to $pr^{-1}(U_\alpha)$, defined by

$$\phi_\alpha(P, (\lambda_{ij})) = \sum_{i,j=1}^{n} \lambda_{ij}u_{ij}^\alpha(P).$$

It is clear that ϕ_α is a one-to-one mapping from $U_\alpha \times M_n$ onto $pr^{-1}(U_\alpha)$. If $P \in U_\alpha \cap U_\beta$, we have, for any $a \in A$

$$a(U_\alpha \cap U_\beta) = \sum_{i,j} a_{ij}^\alpha(U_\alpha \cap U_\beta)u_{ij}^\alpha(U_\alpha \cap U_\beta) = \sum_{i,j} a_{ij}^\beta(U_\alpha \cap U_\beta)u_{ij}^\beta(U_\alpha \cap U_\beta).$$

Put $g_{\beta\alpha}(P)[(a_{ij}^\alpha(P))] = (a_{ij}^\beta(P))$, then it is not difficult to conclude that $g_{\beta\alpha}(P)\in G$. Moreover,

$$\phi_\alpha(P, (a_{ij}^\alpha(P))) = \sum_{i,j} a_{ij}^\alpha(P)u_{ij}^\alpha(P) = \sum_{i,j} a_{ij}^\beta(P)u_{ij}^\beta(P) = \phi_\beta(P, (a_{ij}^\beta(P)))$$
$$= \phi_\beta(P, g_{\beta\alpha}(P)[(a_{ij}^\alpha(P))]).$$

Suppose P_σ converges to P in $U_\alpha \cap U_\beta$. Let (λ_{ij}) be an arbitrary element of M_n and $a = \sum_{i,j} \lambda_{ij}u_{ij}^\alpha \in A$, then

$$a(U_\alpha \cap U_\beta) = \sum_{i,j} \lambda_{ij}u_{ij}^\alpha(U_\alpha \cap U_\beta) = \sum_{i,j} a_{ij}^\beta(U_\alpha \cap U_\beta)u_{ij}^\beta(U_\alpha \cap U_\beta).$$

Since $\{a_{ij}^\beta(P)\}$ are continuous functions on U_β, $a_{ij}^\beta(P_\sigma)$ converges to $a_{ij}^\beta(P)$ for each $i, j = 1,2,\dots,n$. Therefore

$$g_{\beta\alpha}(P_\sigma)[(\lambda_{ij})] = (a_{ij}^\beta(P_\sigma)) \longrightarrow (a_{ij}^\beta(P)) = g_{\beta\alpha}(P)[(\lambda_{ij})].$$

Hence $g_{\beta\alpha}(P_\sigma)$ converges to $g_{\beta\alpha}(P)$ in G i. e. $g_{\beta\alpha}$ is a continuous mapping from $U_\alpha \cap U_\beta$ into G.

Let us consider the topology on $pr^{-1}(U_\alpha)$ induced by ϕ_α from $U_\alpha \times M_n$, then one easily verifies that this induces the unique topology on $pr^{-1}(U_\alpha \cap U_\beta)$. Therefore $\{\phi_\alpha\}$ defines a topology on B. In the following we consider the space B endowed with this topology. Then,we have

THEOREM 3. *Let A be an n-dimensionally homogeneous C^*-algebra, then A defines a fibre bundle* $\mathfrak{B}(A) = \{B, pr, \Omega(A), M_n, G\}$.

PROOF. By the definitions of the topology of B and the above arguments the following results are obvious;

(i) pr is a continuous map from B to $\Omega(A)$,

(ii) ϕ_α is a homeomorphism from $U_\alpha \times M_n$ to $pr^{-1}(U_\alpha)$,

(iii) $\phi^{-1}_{\beta,P}\phi_{\alpha,P}$; $M_n \longrightarrow M_n$ $(P \in U_\alpha \cap U_\beta)$ coincides with the operation of an element $g_{\beta\alpha}(P)$ of G and the map $g_{\beta\alpha}: U_\alpha \cap U_\beta \longrightarrow G$ is continuous on $U_\alpha \cap U_\beta$.

Moreover it is clear that $pr\ \phi_\alpha(P, (\lambda_{ij})) = P$ and $pr^{-1}(P)$ is homeomorphic to M_n for each $P \in \Omega(A)$. Therefore $\mathfrak{B} = \{B, pr, \Omega(A), M_n, U_\alpha, \phi_\alpha\}$ is a coordinate bundle.

On the other hand, the above discussion shows that if U and U' are open sets in $\Omega(A)$ such as $A(U) \cong C(\overline{U}) \hat{\otimes}_\alpha M_n$ and $A(U') \cong C(\overline{U}') \hat{\otimes}_\alpha M_n$ where $\overline{U}$ and $\overline{U}'$ mean the closures of U and U', we get, for corresponding functions ϕ and ϕ' to U and U',

$$\phi(P, (\lambda_{ij})) = \phi'(P, g(P)[(\lambda_{ij})]) \qquad \text{for each } P \in U \cap U',$$

where $g(P)$ denotes an element of G corresponding to the couple of (ϕ, ϕ') and $P \in U \cap U'$. Besides, $g(P)$ is a continuous function on $U \cap U'$. This means that the coordinate bundle $\mathfrak{B} = \{B, pr, \Omega(A), M_n, G, U_\alpha, \phi_\alpha\}$ defines uniquely the fibre bundle $\mathfrak{B}(A) = \{B, pr, \Omega(A), M_n, G\}$ independent from the covering $\{U_\alpha\}$ provided that each element of the covering has the above mentioned property.

DEFINITION. *We call this fibre bundle $\mathfrak{B}(A)$ the structure bundle of A.*

Denote by $Y(A)$ the set of all onto $*$-homomorphisms from A to M_n and consider the pointwise convergence topology on $Y(A)$. Define the map $\widetilde{pr}$; $Y(A) \longrightarrow \Omega(A)$ as $\widetilde{pr}(\theta) = \theta^{-1}(0)$ for $\theta \in Y(A)$ and the map $\widetilde{\phi}_\alpha: U_\alpha \times G \longrightarrow \widetilde{pr}^{-1}(U_\alpha)$ such as

$\widetilde{\phi}_\alpha(P, g)(a) = g^{-1}[(a^\alpha_{ij}(P))]$ for $P \in U_\alpha$, $g \in G$ and $a \in A$. Then it is easily checked that $\widetilde{pr}$ is a continuous map and $\widetilde{\phi}_\alpha$ a one-to-one map from $U_\alpha \times G$ onto $\widetilde{pr}^{-1}(U_\alpha)$.

THEOREM 4. $\widetilde{\mathfrak{B}} = \{Y(A), \widetilde{pr}, \Omega(A), G, G, U_\alpha, \widetilde{\phi}_\alpha\}$ *is the associated principal bundle of* $\mathfrak{B} = \{B, pr, \Omega(A), M_n, G, U_\alpha, \phi_\alpha\}$

PROOF. By the definition of $Y(A)$ and $\widetilde{pr}$, it is clear that the fibre over the point $P \in \Omega(A)$ is homeomorphic to G. Suppose that (P_σ, g_σ) converges to (P, g) in $U_\alpha \times G$. Since $a_{ij}^\alpha(P)$ is a continuous function on U_α for each $a \in A$ and $i, j = 1, 2, \ldots, n$, the matrix $(a_{ij}^\alpha(P_\sigma))$ converges to $(a_{ij}^\alpha(P))$. On the other hand g_σ^{-1} converges to g^{-1}. Hence, we get

$$\widetilde{\phi}_\alpha(P_\sigma, g_\sigma)(a) = g_\sigma^{-1}[(a_{ij}^\alpha(P_\sigma))] \to g^{-1}[(a_{ij}^\alpha(P))] = \widetilde{\phi}_\alpha(P, g)(a)$$

because G is a topological transformation group of M_n. Thus $\widetilde{\phi}_\alpha(P, g)$ is a continuous map. Conversely if $\widetilde{\phi}_\alpha(P_\sigma, g_\sigma)$ converges to $\widetilde{\phi}_\alpha(P, g)$ in $\widetilde{pr}^{-1}(U_\alpha)$, then $\widetilde{pr}(\widetilde{\phi}_\alpha(P_\sigma, g_\sigma)) = P_\sigma$ converges to $\widetilde{pr}(\widetilde{\phi}_\alpha(P, g)) = P$. On the other hand, for an arbitrary matrix $(\lambda_{ij}) \in M_n$, put $a = \sum_{i,j} \lambda_{ij} u_{ij}^\alpha$ then $\widetilde{\phi}_\alpha(P_\sigma, g_\sigma)(a) = g_\sigma^{-1}[(\lambda_{ij})]$ and $\widetilde{\phi}_\alpha(P, g)(a) = g^{-1}[(\lambda_{ij})]$. Hence $g_\sigma^{-1}[(\lambda_{ij})]$ converges to $g^{-1}[(\lambda_{ij})]$, that is, g_σ^{-1} converges to g^{-1} in G. Therefore g_σ converges to g and $\widetilde{\phi}_\alpha^{-1}$ is continuous.

At last, for $P \in U_\alpha \cap U_\beta$ we have

$$\widetilde{\phi}_\alpha(P,g)(a) = g^{-1}[(a_{ij}^\alpha(P))] = g^{-1} g_{\beta\alpha}^{-1}(P)[(a_{ij}^\beta(P))]$$
$$= \widetilde{\phi}_\beta(P, g_{\beta\alpha}(P)g)(a).$$

Therefore the coordinate transformation $\widetilde{g}_{\beta\alpha}(P)$ of $\widetilde{\mathfrak{B}}$ coincides with the one of $\mathfrak{B}$ and so $\widetilde{\mathfrak{B}} = \{Y(A), \widetilde{pr}, \Omega(A), M_n, G, G, U_\alpha, \widetilde{\phi}_\alpha\}$ is the associated principal bundle of $\mathfrak{B} = \{B, pr, \Omega(A), M_n, G, U_\alpha, \phi_\alpha\}$.

Consider again the coordinate bundle $\mathfrak{B} = \{B, pr, \Omega(A), M_n, G, U_\alpha, \phi_\alpha\}$ and let A_0 be the space of all cross-sections of $\mathfrak{B}$. Then A_0 becomes an algebra under the pointwise addition and multiplication and natural scalar multiplication. Besides, one easily verifies that A_0 is complete under the norm $\|f\| = \sup_{P \in \Omega(A)} \|f(P)\|$. Define the $*$-operation on A_0 as $f^*(P) = (f(P))^*$ for $f \in A_0$, then it is almost clear that A_0 becomes a C^*-algebra with this $*$-operation. The relation between two C^*-algebras A and A_0 is given in the following

THEOREM 5. *An n-dimensionlly homogeneous C^*-algebra A is $*$-isomorphic to the C^*-algebra A_0 defined by all cross-sections in the structure bundle $\mathfrak{B}(A)$.*

PROOF. By the definition of the topology of B, it is clear that the map: $P \in \Omega(A) \to a(P) \in B$ defines a cross-section f_a in $\mathfrak{B}$ for each $a \in A$. Then the map: $a \in A \to f_a \in A_0$ is an isomorphism from A into A_0. In the following we shall show that this mapping is onto. Take an arbitary element $f \in A_0$ and define the M_n-valued function $a(\theta)$ on $Y(A)$ as follows;

$$\bar{a}(\theta) = g^{-1}[(f_{ij}^\alpha(P))] \text{ if } \theta = \widetilde{\phi}_\alpha(P, g),$$

where

$$f(P) = \sum_{i,j} f_{ij}^{\alpha}(P) u_{ij}^{\alpha}(P).$$

The value of $\bar{a}(\theta)$ is independent of the representation of θ, in fact, if $\theta = \widetilde{\phi_\beta}(P, g')$, we have

$$g'^{-1}[(f_{ij}^{\beta}(P))] = g^{-1}g_{\beta\alpha}^{-1}(P)[(f_{ij}^{\beta}(P)] = g^{-1}[(f_{ij}^{\alpha})].$$

The same argument as in the proof of the continuity of $\widetilde{\phi_\alpha}$ in Theorem 4 shows that $\bar{a}(\theta)$ is a continuous function. Moreover, for $\theta = \widetilde{\phi_\alpha}(P, g')$

$$\bar{a}(g \cdot \theta) = \bar{a}(g \cdot \widetilde{\phi_\alpha}(P,g')) = \bar{a}(\widetilde{\phi_\alpha}(P, g'g^{-1})) = gg'^{-1}[(f_{ij}^{\alpha}(P))]$$
$$= g[\bar{a}(\theta)],$$

where $g\cdot\theta$ means the $*$-homomorphism $g[\theta(a)]$ $(a \in A)$. Therefore, by [10; Theorem 9. 2], there exists a corresponding element $a \in A$ such as $\theta(a) = \bar{a}(\theta)$ and one can easily verify the rest of the proof i.e. $f = f_a$. Thus A is isomorphic to A_0.

With the aid of this theorem, we can prove the following natural correspondence between two $*$-isomorphic n-dimensionally homogeneous C^*-algebras and their structure bundles.

THEOREM 6. *Let A_1 and A_2 be n-dimensionally homogeneous C^*-algebras, then A_1 is $*$-homomorphic to A_2 if and only if there exists a bundle map h from $\mathfrak{B}(A_2)$ to $\mathfrak{B}(A_1)$ such as its induced map $\bar{h}$ from $\Omega(A_2)$ to $\Omega(A_1)$ is one-to-one.*

COROLLARY. *Let A_1 and A_2 be n-dmensionall homogeneous C^*-algebras, then A_1 is $*$-isomorphic to A_2 if and only if $\mathfrak{B}(A_1)$ is equivalent to $\mathfrak{B}(A_2)$.*

We take the slightly different definition from that of Steenrod [14] for the equivalence of coordinate bundles, that is, two coordinate bundles with the same fibre and group are said to be equivalent if there exists a bundle map which induces the homeomorphism between their base spaces. The equivalence of two fibre bundles are understood in the analogous way. We notice that all reults in [14] are not essentially changed under this definition of the equivalence relation.

PROOF OF THEOREM Let π be a $*$-homomorphism from A_1 onto A_2 and $\mathfrak{B}_1 = \{B_1, pr_1, \Omega(A_1), M_n, G, U'_\gamma, \phi'_\gamma\}$ and $\mathfrak{B}_2 = \{B_2, pr_2, \Omega(A_2), M_n, G, U_\alpha, \phi_\alpha\}$ the coordinate bundles belonging to $\mathfrak{B}(A_1)$ and $\mathfrak{B}(A_2)$ respectively. Then, for any $P \in \Omega(A_2)$ we have $\pi^{-1}(P) \in \Omega(A_1)$. Hence π induces the map $\bar{h}$: $\Omega(A_2) \to \Omega(A_1)$ defined by $\bar{h}(P) = \pi^{-1}(P)$ and the map $h: B_2 \to B_1$ defined by

$h(b(P)) = \pi^{-1}(b)(\bar{h}(p))$ for $b \in A_2$. Clearly $\bar{h}$ is continuous and one-to-one. A simple computation shows that $pr_1 h = \bar{h} pr_2$, hence h carries fibres into fibres and induces the map $\bar{h}$. Take an arbitrary element $P \in U_\alpha \cap \bar{h}^{-1}(U'_\gamma)$ and consider the map

$$\bar{g}_{\gamma\alpha}(P)[(\lambda_{ij})] = \phi'^{-1}_{\gamma,\bar{h}(P)} \cdot h \cdot \phi_{\alpha,P}[(\lambda_{ij})] \quad \text{for } (\lambda_{ij}) \in M_n.$$

One may easily verify that this mapping is a $*$-automorphism of M_n, i.e. $\bar{g}_{\gamma\alpha}(P) \in G$. Suppose P_σ converges to P in $U_\alpha \cap \bar{h}^{-1}(U'_\gamma)$. Put $b = \sum_{i,j} \lambda_{ij} u^\alpha_{ij}$ and $\pi^{-1}(b)(\bar{h}(P)) = \sum_{i,j} a^\gamma_{ij}(\bar{h}(P)) v^\gamma_{ij}(\bar{h}(P))$, then the matrix $(a^\gamma_{ij}(\bar{h}(P_\sigma)))$ converges to the matrix $(a^\gamma_{ij}(\bar{h}(P)))$, for $\bar{h}(P_\sigma)$ converges to $\bar{h}(P)$ and $a^\gamma_{ij}(P)$ is a continuous function on $\bar{U}'_\gamma$ for each i, j. Therefore $\bar{g}_{\gamma\alpha}(P_\sigma)[(\lambda_{ij})] = (a^\gamma_{ij}(\bar{h}(P_\sigma)))$ converges to $\bar{g}_{\gamma\alpha}(P)[(\lambda_{ij})] = (a^\gamma_{ij}(\bar{h}(P)))$, which implies that the map $\bar{g}_{\gamma\alpha}(P)$ is a continuous map from $U_\alpha \cap \bar{h}^{-1}(U'_\gamma)$ into G. Hence, by [14: Lemma 2.6] h is a bundle map.

Next, suppose that there exists a bundle map h from the bundle $\mathfrak{B}_1 = \{B_2, pr_2, \Omega(A_2), M_n, G, U_\alpha, \phi_\alpha\}$ to the bundle $\mathfrak{B}_1 = \{B_1, pr_1, \Omega(A_1), M_n, G, U'_\gamma, \phi'_\gamma\}$ which induces the one-to-one continuous mapping $\bar{h}$ from $\Omega(A_2)$ to $\Omega(A_1)$. Since $\Omega(A_2)$ is a compact space and $\Omega(A_1)$ a Hausdorff space, $\bar{h}^{-1}$ is continuous and a slight modification of Lemma 2.7 in [14] shows that h^{-1} is continuous, too.

For $a \in A_1$, put $\pi(a)(P) = h^{-1}(a(\bar{h}(P)))$ $(P \in \Omega(A_2))$, then $\pi(a)(P)$ is a cross-section of $\mathfrak{B}_2$. Hence, by Theorem 5, we can say that $\pi(a)$ belongs to A_2. For $P \in U_\alpha \cap \bar{h}^{-1}(U'_\gamma)$, put $a_1(P) = \sum_{i,j} \lambda_{ij} u^\alpha_{ij}(P)$ and $a_2(P) = \sum_{i,j} \lambda'_{ij} u^\alpha_{ij}(P)$ then

$$\begin{aligned} h(a_1(P) + a_2(P)) &= h\phi_\alpha(P, (\lambda_{ij} + \lambda'_{ij})) = \phi'_\gamma(\bar{h}(P), \bar{g}_{\gamma\alpha}(P)[(\lambda_{ij} + \lambda'_{ij})]) \\ &= \phi'_\gamma(\bar{h}(P), \bar{g}_{\gamma\alpha}(P)[(\lambda_{ij})]) + \phi'_\gamma(\bar{h}(P), \bar{g}_{\gamma\alpha}(P)[(\lambda'_{ij})]) \\ &= h\phi_\alpha(P, (\lambda_{ij})) + h\phi_\alpha(P, (\lambda'_{ij})) = h(a_1(P)) + h(a_2(P)) \end{aligned}$$

Similarly

$$h(a_1(P)a_2(P)) = h(a_1(P))h(a_2(P)), \text{ and}$$

$h(a_1(P)^*) = h(a_1(P))^*$.

Hence π is a $*$-homomorphism from A_1 into A_2.

We shall show that π is an onto map. Let φ and ψ be pure states of A_2 and π_φ, π_ψ their canonical representations. Denote by P_1, P_2 their kernels. Then there exists a pure state φ'(resp. ψ') of $\pi_\varphi(A_2)$ (resp. $\pi_\psi(A_2)$) and a $*$-isomorphism θ_1(resp. θ_2) between $A_2(P_1)$ and $\pi_\varphi(A_2)$ (resp. $A_1(P_2)$ and $\pi_\psi(A_2)$) such that

$$\varphi(a) = \varphi'(\theta_1(a(P_1))) \text{ (resp. } \psi(a) = \psi'(\theta_2(a(P_2))))$$

for all $a \in A_2$. Suppose $\varphi \neq \psi$, then we can find an element $a \in A_2$ such as $\varphi(a) \neq \psi(a)$. Moreover there exists an element $b \in A_1$ such as $b(\bar{h}(P_1)) = h(a(P_1))$ and $b(\bar{h}(P_2)) = h(a(P_2))$ even if $a(P_1) = a(P_2)$ or not. Then we get

$$\varphi(\pi(b)) = \varphi'(\theta_1(\pi(b)(P_1))) = \varphi'[\theta_1(h^{-1}(b(\bar{h}(P_1))))] = \varphi'(\theta_1(a(P_1)))$$
$$= \varphi(a) \neq \psi(a) = \psi'(\theta_2(a(P_2))) = \psi'(\theta_2(\pi(b)(P_2))) = \psi(\pi(b))$$

That is, $\pi(A_1)$ separates the set of pure states of A_2. Hence, by [11 : Theorem 7.2], we have $\pi(A_1) = A_2$.

In the case that A_1 is $*$-isomorphic to A_2, the induced map $\bar{h}$ is a continuous one-to-one map from $\Omega(A_2)$ onto $\Omega(A_1)$, hence $\bar{h}$ is a homeomorphism between the base spaces of the structure bundles. Thus A_1 is $*$-isomorphic to A_2 if and only if $\mathfrak{B}(A_1)$ is equivalent to $\mathfrak{B}(A_2)$.

THEOREM 7. *Let A_1 and A_2 be n-dimensionally homogeneous C^*-algebras. If they are algebraically isomorphic each other, then there exists a $*$-isomorphism between them, that is, A_1 and A_2 are $*$-isomorphic each other.*

PROOF. Let π be an isomorphism from A_1 onto A_2, then π is bicontinuous and we can see that $P \in \Omega(A_1)$ if and only if $\pi(P) \in \Omega(A_2)$. Hence, by the analogous argument as in the proof of the above theorem, one easily verifies that the bundle $\mathfrak{B}(A_1)$ is G_0-equivalent to the bundle $\mathfrak{B}(A_2)$.

Since G_0 is topologically isomorphic to the quotient group of $GL(n, C)$ by its center and G the quotient group of $U(n)$ by its center, a straight-foward calculation using the structure of $GL(n, C)$ and $U(n)$ shows that the homogeneous space G_0/G is a solid space. Therefore, by [14 : Theorem 12.7], $\mathfrak{B}(A_1)$ is G-equivalent to $\mathfrak{B}(A_2)$, which completes the proof.

COROLLARY. *If A_1 is homomorphic to A_2, then A_1 is also $*$-homomorphic to A_2.*

PROOF. Let π be a homomorphism from A_1 onto A_2. As π is continuous, the kernel I of π is a closed two-sided ideal of A_1. Hence I is self-adjoint. Put $A_1' = A_1/I$, then clearly A_1' is an n-homogeneous C^*-algebra and π induces an isomorphism from A_1' onto A_2. By the above theorem A_1' is $*$-isomorphic to A_2, hence A_1 is $*$-homomorphic to A_2.

Next, we shall prove the construction theorem for n-homogeneous C^*-algebras from the bundles $\{B, pr, \Omega, M_n, G\}$ where Ω is an arbitrary compact Hausdorff space.

THEOREM 8. *Let the fibre bundle $\mathfrak{B} = \{B, pr, \Omega, M_n, G\}$ be given where Ω denotes an arbitrary compact Hausdorff space, then there exists, except the relation by $*$-isomorphism, uniquely an n-dimensionally homogeneous C^*-algebra A such that the structure bundle $\mathfrak{B}(A)$ is equivalent to $\mathfrak{B}$.*

PROOF. Let $M_n(\omega)$ be the fibre over the point $\omega \in \Omega$ and $\{U_\alpha\}$, $\{\phi_\alpha\}$ the coordinate neighborhoods and coordinate functions of $\mathfrak{B}$. Without loss of generality, we may assume that $M_n(\omega)$ is an algebra and $M_n(\omega)$ is $*$-isomorphic to M_n by the map $\phi_{\alpha,\omega}^{-1} = p_\alpha$. Let A be the set of all cross-sections in $\mathfrak{B}$. We consider the pointwise addition, multiplication, $*$-operation and natural scalar multiplication in A, then A becomes a $*$-algebra. For $a \in A$, define the norm $\|a\| = \sup_{\omega\in\Omega} \|a(\omega)\|$, then it is almost clear that A becomes a C^*-algebra under this norm structure. We shall show that A is an n-homogeneous C^*-algebra. Take an arbitrary point $\omega \in \Omega$ and put $P_{\omega_0} = \{a \in A \mid a(\omega_0) = 0\}$, then P_{ω_0} is a closed ideal of A. We assert that $A/P_{\omega_0} \cong M_n(\omega_0)$. Suppose $\omega_0 \in U_\alpha$ and consider the map: $a \in A \to a(\omega_0) \in M_n(\omega_0)$, then clearly this mapping is a $*$-homomorphism from A into $M_n(\omega_0)$. Let b be an arbitrary element of $M_n(\omega_0)$. There exists a neighborhood U of ω_0 such as $\overline{U} \subset U_\alpha$ and we can find a continuous function f on Ω such as $f(\omega_0)=1$ and $f(U^c) = 0$ (U^c means the complement of U). We define the function $a(\omega)$ as

$$a(\omega) = \phi_\alpha(\omega, f(\omega)p_\alpha(b)) \qquad \text{for } \omega \in U_\alpha$$
$$= 0 \qquad \text{for } \omega \in U_\alpha^c$$

One easily verifies that $a(\omega)$ is a cross-section of $\mathfrak{B}$, i.e. $a \in A$ and $a(\omega_0) = b$. We have $A/P_{\omega_0} \cong M_n(\omega_0)$.

Next, let Z be the center of A. The above result implies that if $a \in Z$, then $a(\omega) = f(\omega)\cdot 1$ for some $f \in C(\Omega)$ where 1 means the identity of $M_n(\omega)$. Conversely, for any function $f \in C(\Omega)$ the cross-section a_f defined by $a_f(\omega) = f(\omega)\cdot 1$ belongs to Z. Therefore $Z \cong C(\Omega)$ and the spectrum of Z is homeomorphic with Ω.

Now consider an arbitrary irreducible representation π of A, then π induces an irreducible representation of Z having the kernel $\pi^{-1}(0) \cap Z$, which defines a point $\omega_0 \in \Omega$. Take the primitive ideal P_{ω_0} constructed above and let a be an arbitrary element of P_{ω_0}. If ω_σ converges to ω_0 in U_α, $p_\alpha(a(\omega_\sigma))$ converges to $p_\alpha(a(\omega_0))$. Hence for any $\varepsilon > 0$ there exists an index σ_0 such that $\|p_\alpha(a(\omega_\sigma)) - p_\alpha(a(\omega_0))\| < \varepsilon$ for $\sigma \geqq \sigma_0$. We have,

$$\big|\|a(\omega_0)\| - \|a(\omega_\sigma)\|\big| = \big|\|p_\alpha(a(\omega_0))\| - \|p_\alpha(a(\omega_\sigma))\|\big|$$
$$\leqq \|p_\alpha(a(\omega_0)) - p_\alpha(a(\omega_\sigma))\| < \varepsilon,$$

that is, $f(\omega) = \|a(\omega)\| \in C(\Omega)$ and $a_f \in \pi^{-1}(0) \cap Z$. Moreover $a^*a(\omega) \leqq \|a(\omega)\|^2\cdot 1$

implies $a^*a \leqq a_f$. Hence, $a \in \pi^{-1}(0)$ that is $P_{\omega_0} \subset \pi^{-1}(0)$. As P_{ω_0} is a maximal ideal of A, we get $P_{\omega_0} = \pi^{-1}(0)$. Thus any irreducible representation of A is n-dimensional.

Let $\{B', pr', \Omega(A), M_n, G, U'_\gamma, \phi'_\gamma\}$ be the structure bundle of A. For a point $b \in B$, there exists a cross-section $a(\omega)$ such that $a(\omega) = b$ where $pr(b) = \omega$. We define the map $h: B \to B'$ as $h(b) = a(P_\omega)$. This is clearly well-defined and we notice that the restriction of h to $M_n(\omega)$ is a $*$-isomorphism from $M_n(\omega)$ to $A(P_\omega)$. Let $\bar{h}$ be the map defined by $\bar{h}(\omega) = P_\omega$. By the preceeding discussions, $\bar{h}$ is a homeomorphism from Ω to $\Omega(A)$. Moreover we have $pr'h = \bar{h}pr$. Thus h carries fibres into fibres and induces the map $\bar{h}$. Put $\bar{g}_{\gamma\alpha}(\omega) = \phi'^{-1}_{\gamma,\bar{h}(\omega)} h \phi_{\alpha,\omega}$ for $\omega \in U_\alpha \cap \bar{h}^{-1}(U'_\gamma)$. One verifies easily that $\bar{g}_{\gamma\alpha}(\omega) \in G$ and they satisfy;

$$\bar{g}_{\gamma\alpha}(\omega) g_{\alpha\beta}(\omega) = \bar{g}_{\gamma\beta}(\omega), \qquad \omega \in U_\alpha \cap U_\beta \cap \bar{h}^{-1}(U'_\gamma),$$

$$g'_{\delta\gamma}(\bar{h}(\omega)) \bar{g}_{\gamma\alpha}(\omega) = \bar{g}_{\delta\alpha}(\omega), \qquad \omega \in U_\alpha \cap \bar{h}^{-1}(U'_\gamma \cap U'_\delta)$$

where $g_{\alpha\beta}$ and $g'_{\delta\gamma}$ denote the coordinate transformations of $\mathfrak{B}$ and $\mathfrak{B}(A)$, respectively. Take an arbitrary matrix (λ_{ij}) and suppose ω_σ converges to ω_0 in $U_\alpha \cap \bar{h}^{-1}(U'_\gamma)$. Let $h(\phi_\alpha(\omega, (\lambda_{ij})) = \sum_{i,j} b^\gamma_{ij}(\bar{h}(\omega)) u^\gamma_{ij}(\bar{h}(\omega))$. We see that $\bar{h}(\omega_\sigma)$ converges to $\bar{h}(\omega_0)$ and as $b^\gamma_{ij}(P)$ is a continuous function on U'_γ for each i, j,

$$\bar{g}_{\gamma\alpha}(\omega_\sigma)[(\lambda_{ij})] = (b^\gamma_{ij}(\bar{h}(\omega_\sigma))) \to \bar{g}_{\gamma\alpha}(\omega_0)[(\lambda_{ij})] = (b^\gamma_{ij}(\bar{h}(\omega_0))).$$

Therefore the map; $\omega \in U_\alpha \cap \bar{h}^{-1}(U'_\gamma) \to \bar{g}_{\gamma\alpha}(\omega) \in G$ is continuous. It follows from [14: Lemma 2. 6] that h is a bundle map from $\mathfrak{B}$ to $\mathfrak{B}(A)$.

The above theorem offers a new method for the construction of the n-homogeneous C*-algebras, in fact there exist a large number of non-isomorphic n-homogeneous C^*-algebras according to the number of non-equivalent fibre bundles over the compact Hausdorff spaces with fibre M_n and group G.

Now, it is well known that an n-dimensionally homogeneous W^*-algebra is isomorphic to the W^*-tensor product of a commutative W^*-algebra and M_n. One might suspect that the analogous result holds for an arbitrary n-homogeneous C^*-algebra. However this is not the case as we shall show in the following discussions. At first, we have

THEOREM 9. *An n-dimensionally homogeneous C^*-algebra A is isomorphic to the C^*-tensor product of a commutative C^*-algebra and M_n if and only if the structure bundle $\mathfrak{B}(A)$ of A is equivalent to the product bundle.*

PROOF. Combining [14: Corollary 8. 4] with Theorem 4 and the cross-section theorem (cf. [14: p. 36]) one verifies easily that it is sufficient to prove the following result; *an n-dimensionally homogeneous C^*-algebra is isomor-*

phic to the C^*-tensor product of a commutative C^*-algebra and M_n if and only if the principal bundle $\{Y(A), \tilde{pr}, \Omega(A), G, G\}$ admits a cross-section.

Suppose that the bundle $\{Y(A), \tilde{pr}, \Omega(A), G, G\}$ admits a cross-section f. For any element $a \in A$, put $\bar{a}(P) = f(P)(a)$ $(P \in \Omega(A))$. Then, clearly $\bar{a}(P)$ is a M_n-valued continuous function on $\Omega(A)$ and the map : $a \to \bar{a}$ is a $*$-homomorphism from A to $C(\Omega(A), M_n)$, the ring of all M_n-valued continuous functions on $\Omega(A)$. If $\bar{a}(P) = 0$ for all $P \in \Omega(A)$, then a belongs to every kernel of $f(P)$ $(P \in \Omega(A))$ which coincide with $\Omega(A)$ because f is a cross-section. Hence $a = 0$, and A is isomorphically embedded into $C(\Omega(A), M_n)$. Then, by Kaplansky [11: Theorem 3. 4], we see that A is isomorphic to $C(\Omega(A), M_n)$. On the other hand, by Grothendieck [8] and Takesaki [15], $C(\Omega(A), M_n)$ is isomorphic to $C(\Omega(A)) \hat{\otimes}_\alpha M_n$, the C^*-tensor product of $C(\Omega(A))$ and M_n. Thus we have $A \cong C(\Omega(A)) \hat{\otimes}_\alpha M_n$.

Next we assume that $A \cong C \hat{\otimes}_\alpha M_n$ for some commutative C^*-algebra C. We may assume that $C = C(\Omega)$ for a compact space Ω. As we mentioned above, $C(\Omega) \hat{\otimes}_\alpha M_n \cong C(\Omega, M_n)$, so that, by Corollary of Theorem 6, we can set $A = C(\Omega, M_n)$. It follows that the center Z of A is isomorphic to $C(\Omega)$. Take a primitive ideal P of A, then $P \cap Z$ is a primitive ideal of Z. Hence $P \cap Z$ defines a point $\omega_P \in \Omega$. By Kaplansky [10 : Theorem 9.1] the mapping $P \to P \cap Z$ is a homeomorphism between $\Omega(A)$ and $\Omega(Z)$. Therefore the above mapping $P \to \omega_P$ gives a homeomorphism between $\Omega(A)$ and Ω. Moreover, since $P = \{a \in A \mid a(\omega) = 0\}$ is a primitive ideal of A and A is central (cf. [10], [11]) we have

$$P = \{a \in A \mid a(\omega_P) = 0\}.$$

Now each $\omega \in \Omega$ defines an element θ_ω of $Y(A)$ by $\theta_\omega(a) = a(\omega)$ for $a \in A$. Put $f(P) = \theta\omega_P$ for each $P \in \Omega(A)$, then f is a cross-section of $\{Y(A), \tilde{pr}, \Omega(A), G, G\}$. In fact, it is clear that $f(P)$ is a continuous function on $\Omega(A)$ to $Y(A)$ and

$$\tilde{pr}(f(P)) = \text{the kernel of } f(P) = \{a \in A \mid \theta\omega_P(a) = 0\} = \{a \in A \mid a(\omega_P) = 0\}$$
$$= P.$$

This completes the proof.

Theorem 8 shows that the theory of fibre bundles of n-homogeneous C^*-algebras becomes a trivial one if all n-homogeneous C^*-algebras are isomorphic to the C^*-tensor products of some commutative C^*-algebras and M_n. But as we mentioned above, this is not true. We can show the following

THEOREM 10. *For every $n > 1$, there exists an n-dimensionally homogeneous C^*-algebra which is not isomorphic to the C^*-tensor product of a commutative C^*-algebra and M_n.*

PROOF. Considering Theorem 7 and Theorem 8, it sufficies to prove that there exists a principal bundle over a compact space with group G which is not equivalent to a product bundle. For the convenience of discussions, we take the notation G_n for G. Then the group G_n may be considerod to be a closed subgroup of G_{n+1} by the suitable identification. Since G_{n+1} is a Lie group, by the bundle struccture theorem (cf. [14: §7. 4 and 7. 5]), G_n becomes a fibre bundle over G_{n+1}/G_n with fibre G_n and group G_n. Thus we get a principal bundle over the compact space G_{n+1}/G_n with group G_n. We assert that this bundle is not equivalent to the product bundle for $n \geqq 2$. In fact, if this bundle is equivalent to the product bundle we have

$$G_{n+1} \approx G_{n+1}/G_n \times G_n.$$

Hence, by [14: §17. 7], we get

$$\pi_1(G_{n+1}) \cong \pi_1(G_{n+1}/G_n) + \pi_1(G_n).$$

But this relation does not hold for $n \geqq 2$ as shown in the following discussions.

At first, it is known that $U(n) = TSU(n)$. Hence a well-known isomorphism theorem for topological groups shows that $U(n)/T \cong SU(n)/SU(n) \cap T$ (topologically isomorphic) by the canonical correspondence. Therefore we may identify G_n with $SU(n)/SU(n) \cap T$. Let f be the canonical map from $SU(n)$ to G_n, then $\{SU(n), f\}$ is a simply connected convering group of G_n by [2: p.59]. Hence, by [2: p. 54, Proposition 7], the Poincaré group of G_n is isomorphic to $SU(n) \cap T$, which is a cyclic group of order n. Therefore one easily see that the above cited relation does not hold for $n \geqq 2$.

REMARK. Using the homotopy groups of $SU(n)$ we can show another examples of bundles which are not equivalent to the product bundles. However, in this case, the discussions are somewhat complicated so we give here only the brief sketch of these examples.

By the bundle structure theorem used above, $SU(n+1)$ becomes a principal fibre bundle over $SU(n+1)/SU(n)$ with group $SU(n)$. On the other hand we see that G_n is topologically isomorphic to the factor group $SU(n)/SU(n) \cap T$. Hence the above bundle induces a G_n-bundle over the compact space $SU(n+1)$ $/SU(n)$. We can prove that this bundle is not equivalent to the product bundle for $n \geqq 2$. In fact, suppose this bundle is equivalent to the product bundle, then we can show that the original bundle $SU(n+1)$ is $SU(n)$-equivalent to the

$SU(n) \cap T$-bundle and since $SU(n) \cap T$ is a discrete group, the bundle $\{SU(n+1), p, SU(n+1)/SU(n), SU(n), SU(n) \cap T\}$ is equivalent to the product bundle by [14: Corollary of the classification theorem]. We have

$$SU(n+1) \approx SU(n+1)/SU(n) \times SU(n).$$

Therefore we get

$$\pi_{2n}(SU(n+1)) \cong \pi_{2n}(SU(n+1)/SU(n)) + \pi_{2n}(SU(n)).$$

Now, by the theorem of Bott, we have $\pi_{2n}(SU(n+1)) = 0$ and by the theorem of Borel-Hirzebruch $\pi_{2n}(SU(n))$ is a cyclic group of order $n!$, which implies a contradiction (cf. [1], [16]).

This completes the proof.

COROLLARY. *Let A_1 and A_2 be n-dimensionally homogeneous C^*-algebras, then the isomorphic relation between their centers does not necessarily imply the isomorphic relation between A_1 and A_2*

The proof is trivial once we consider the C^*-tensor product $C(\Omega(A)) \hat{\otimes}_\alpha M_n$ for the structure space $\Omega(A)$ of a C^*-algebra A which is not itomorphic to the C^*-tensor product of a commutative C^*-algebra and M_n.

At last, we shall study the relation between the unitary group U_A of A and the group of $*$-automorphisms G_A of A leaving the center elementwise fixed. Since $U(n)/T \cong SU(n)/SU(n) \cap T$ by the canonical correspondence, we may identify G with $SU(n)/SU(n) \cap T$. Since $SU(n) \cap T$ has a local cross-section in $SU(n)$(cf. [14: § 7.4]) there exists a neighborhood V of the unit of $G(=SU(n)/SU(n) \cap T)$ and a continuous mapping f of V into $SU(n)$ such that $f(g)$ induces the (inner) $*$-automorphism g for each $g \in V$. It follows T has a local cross-section f in $V(n)$ whose values belong to $SU(n)$. From [14: Corollary of Theorem 7.4], $U(n)$ is considered to be a principal bundle over $G(=U(n)/T)$ with group T. The structure of this bundle is the following:

Let V and f be a neighborhood of the unit of G and a local cross-section over V mentioned above, then the family $\{\sigma \cdot V \mid \sigma \in SU(n)\}$ becomes an open covering of G which is the system of coordinate neighborhoods of G. We define the local cross-section f_σ on $\sigma \cdot V = V_\sigma$ by $f_\sigma(g) = \sigma f(\sigma^{-1}g)$ for $g \in V_\sigma$. For each $(g, \lambda) \in V_\sigma \times T$, define the function $\phi_\sigma(g, \lambda) = \lambda f_\sigma(g)$. We get the coordinate transformation $g_{\sigma\tau}(g) = (f_\sigma(g))^{-1} f_\tau(g)$ for $g \in V_\sigma \cap V_\tau$ which belongs to $SU(n) \cap T$. Hence $U(n)$ is a fibre bundle over G with fibre T and group $SU(n) \cap T$. As it was shown in the proof of Theorem 10, the Poincaré group of G is a cyclic group of order n. It follows that the Poincaré group of $G \times T$ is isomorphic to the product group of order n and the additive group of all integers, the Poincaré group of T. On the other hand, it is known that the Poincaré group of $U(n)$ is isomorphic to the additive group of all integers. Therefore the above bundle is not equivalent to a product bundle.

Now let $G(P)$ be the group of all $*$-automorphisms of $A(P)$, then we have $G(P) \cong G$. We consider on $G(P)$ the simple convergence topology on $A(P)$, then the preceding isomorphism relation becomes topological. Put $B^G = \bigcup_{P \in \Omega(A)} G(P)$. For every $b \in B^G$, there exists a unique point $P \in \Omega(A)$ such as $b \in G(P)$. We define $p_G(b) = P$. Let U_α be a coordinate neighborhood of the structure bundle of A. We define the coordinate function ψ_α of $U_\alpha \times G$ onto $p_G^{-1}(U_\alpha)$ by

$$\psi_\alpha(P, g)[a] = \sum_{i,j} \lambda_{ij}(g[p_\alpha(a)])u_{ij}^\alpha(P)$$

for each $a \in A(P)$ where $\lambda_{ij}(g[p_\alpha(a)])$ means the (i, j)-component of the matrix $g[p_\alpha(a)]$ and $p_\alpha = \phi_{\alpha,P}^{-1}$. Take an arbitrary element $P \in U_\alpha \cap U_\beta$, then we have, for each $a \in A$

$$a(P) = \sum_{i,j} a_{ij}^\alpha(P)u_{ij}^\alpha(P) = \sum_{i,j} a_{ij}^\beta(P)u_{ij}^\beta(P)$$
$$= \phi_\alpha(P, (a_{ij}^\alpha(P))) = \phi_\beta(P, (a_{ij}^\beta(P)))$$

and

$$(a_{ij}^\beta(P)) = g_{\beta\alpha}(P)[(a_{ij}^\beta(P))].$$

Hence we get

$$\psi_\alpha(P, g)[a(P)] = \sum_{i,j} \lambda_{ij}(g[a_{kl}^\alpha(P))])u_{ij}^\alpha(P)$$
$$= \sum_{i,j} \lambda_{ij}(g_{\beta\alpha}(P)g[(a_{kl}^\alpha(P))])u_{ij}^\beta(P)$$
$$= \sum_{i,j} \lambda_{ij}(g_{\beta\alpha}(P)gg_{\beta\alpha}^{-1}(P)[(a_{kl}^\beta(P))])u_{ij}^\beta(P)$$
$$= \psi_\beta(P, g_{\beta\alpha}(P)gg_{\beta\alpha}^{-1}(P))[a(P)],$$

that is,

$$\psi_{\beta\alpha}(P)(g) = \psi_{\beta,P}^{-1}\psi_{\alpha,P}(g) = g_{\beta\alpha}(P)gg_{\beta\alpha}^{-1}(P) = g_{\beta\alpha}(P)\cdot g.$$

We introduce the topology on B^G by the family of the mapping $\{\psi_\alpha\}$. Then the above arguments show that B^G is a fibre bundle over $\Omega(A)$. The fibre of this bundle is G and the group is also G, but acting on the fibre as inner automorphisms.

Set $B^U = \bigcup_{P \in \Omega(A)} U(P)$ where $U(P)$ denotes the unitary group of $A(P)$. Since

$U(n)$ is an invariant subspace of M_n under G, B^U becomes a fibre bundle over $\Omega(A)$ with fibre $U(n)$ and group G as a subbundle of B. For each $u \in U(P)$, we define a $*$-automorphism $g_u \in G(P)$ of $A(P)$ by $g_u[a] = uau^{-1}$ for $a \in A(P)$. Set the mapping $\nu: B^U \to B^G$ by $\nu(u) = g_u$. Denote by η the natural mapping: $U(n) \to G$. Then we have, for $a \in A(P)$,

$$\begin{aligned}\nu(u)[a] &= uau^{-1} \\ &= \sum_{i,j} \lambda_{ij}(p_\alpha(u)p_\alpha(a)p_\alpha(u)^{-1})u_{ij}^\alpha(P) \\ &= \psi_\alpha(P, \eta p_\alpha(u))[a]\end{aligned}$$

where $pr(u) = P \in U_\alpha$. Hence, by [14: Theorem 9.6], we get the following result:

B^U is a fibre bundle over B^G relative to the projection ν. The fibre of the bundle is one-dimensional torus group T and the group is $SU(n) \cap T$ acting on the fibre by left multiplications.

The restriction of the structure group from T to $SU(n) \cap T$ follows from the following observations.

Set

$$W_{\alpha\sigma} = \psi_\alpha(U_\alpha \times V_\sigma) \qquad \text{and } v_{\alpha\sigma}(\psi_\alpha(P, g), \lambda) = \phi_\alpha(P, \lambda f_\sigma(g))$$

for each $(P, g, \lambda) \in U_\alpha \times V_\sigma \times T$, where V_σ and $f_\sigma(g)$ mean the notations used in the first paragraph of our discussions. As it was shown in the proof of the above cited theorem, $\{W_{\alpha\sigma}\}$ and $\{v_{\alpha\sigma}\}$ are the coordinate neighborhoods and the coordinate functions. The coordinate transformations are

$$\gamma_{\beta\tau,\alpha\sigma}(b) = f_\tau(g_{\beta\alpha}(P)gg_{\beta\alpha}^{-1}(P))^{-1}g_{\beta\alpha}(P)[f_\sigma(g)]$$
$$\text{for } b = \psi_\alpha(P, g) \in W_{\alpha\sigma} \cap W_{\beta\tau}$$

which are easily seen to belong to $SU(n) \cap T$.

From Theorem 5, one can see that there exists a one-to-one correspondence between U_A and the family of all cross-section in B^U. That is, for each element $u \in U_A$ the mapping: $P \to u(P) \in B^U$ defines a cross-section over $\Omega(A)$ and conversely, for each cross-section f over $\Omega(A)$, there exists an element $u \in U_A$ such as $u(P) = f(P)$.

Consider a $*$-automorphism θ leaving Z elementwise fixed We have $\theta(P) \cap Z = P \cap Z$ for each primitive ideal P of A and as A is central this implies $\theta(P) = P$ i.e. P is invariant by θ. Hence θ induces a $*$-automorphism $\theta(P)$ on $A(P)$ and one sees without difficulty that the mapping $P \to \theta(P)$ defines a cross-section of B^G over $\Omega(A)$. Conversely, suppose $f(P)$ is a cross-section of B^G. A straight-forward calculation shows that $f(P)[a(P)]$ defines a cross-section of

B for each $a \in A$. Hence there exists an element $\bar{a} \in A$ such as $\bar{a}(P) = f(P)[a(P)]$ by Theorem 5. Define the mapping θ by $\theta(a) = \bar{a}$, then one verifies easily that θ is a $*$-automorphism of A leaving Z elementwise fixed and $\theta(P) = f(P)$.

Put $\Omega^\theta = \{\theta(P) : P \in \Omega(A)\} \subset B^G$. For each element $u \in U_A$, we define the inner automorphism $\theta_u(a) = uau^{-1}$. Then the mapping $\theta_u(P) \rightarrow u(P)$ is a cross-section of the relative $(SU(n) \cap T,\ SU(n) \cap T)$-bundle B^U over the base space $(B^G, \Omega^{\theta u})$. Conversely, let θ be a $*$-automorphim of A leaving Z elementwise fixed. If there exists a cross-section f in the relative bundle B^U over Ω^θ, then f defines a cross-section f' of the bundle B^U over $\Omega(A)$ by $f'(P) = f(\theta(P))$. Hence we can find an element $u \in U_A$ such as $u(P) = f'(P)$. We have $\theta_u(P) = \theta(P)$ for all $P \in \Omega(A)$. Thus θ is an inner automorphism of A. We have

THEOREM 11. *B^U is a fibre bundle over B^G relative to the map ν. The fibre of this bundle is T and the group is $SU(n) \cap T$ acting on the fibre by left multiplications. A $*$-automorphism θ of A leaving the center elementwise fixed is inner if and only if there exists a cross-section in the relative $(SU(n) \cap T,\ SU(n) \cap T)$-bundle B^U over the base (B^G, Ω^θ).*

COROLLARY. *If $\Omega(A)$ is arcwise-connected, arcwise locally connected and simply connected, then every $*$-automorphism of A leaving the center elementwise fixed is inner.*

PROOF. As it is clear that Ω^θ is homeomorphic with $\Omega(A)$ for each $*$-automorphism θ, Ω^θ satisfies the same conditions as $\Omega(A)$ does. Hence, by [14 : §13. 9], the relative bundle B^U over Ω^θ is equivalent to the product bundle. Therefore this bundle always has a cross-section.

REMARK. Assume $\Omega = G$ and $A = C(\Omega) \hat{\otimes}_\alpha \mathrm{M}_n$, then by Theorem 9, $B \approx G \times M_n$, hence $B^U \approx G \times U(n)$. Moreover we get $B^G \approx G \times G$. A is considered to be the ring of all M_n-valued continuous functions on G.

Define the $*$-automorphism θ of A by

$$\theta(a)(g) = g[a(g)] \text{ for each } a \in A \text{ and } g \in G.$$

We have $\Omega^\theta = \{(g, g) | g \in G\}$. We shall show that this automorphism is not inner. In fact, suppose there exists a cross-section f in B^U over Ω^θ, then we have

$f(g, g) = (g, \sigma(g)) \in B^U$. Since $f(g.g)$ induces the automorphism $\theta(g)$, $\sigma(g)$ induces the automorphism g for every $g \in G$. Hence $\sigma(g)$ is a cross-section of the bundle $U(n)$ over G. But the first part of our discussions in this section shows that $U(n)$ is not a trivial bundle over G, which is a contradiction (cf. [14 : §8. 31]). Thus there exists an outer $*$-automorphism in A.

3. Concluding remarks Though we assumed the unit in a C^*-algebra throughout this paper, the assumption is not essential in our discussions of §2 except Theorem 11. Some minor modifications and usual transitions for lack of the unit such as taking the word *"vanishing at infinity"* for *"all"* make us enable to prove the analogous results for an n-homogeneous C^*-algebra without unit, so we omit the details.

ADDED IN PROOF. After writing this paper, Glimm's new paper has appeared "Type I C^*-algebras, Ann. Math., 73(1961), 572-612" in which he treats the analogous problem to our Theorem 1 without the assumption of the unit.

He informed us kindly that Fell's unpublished paper, "The structure of algebras of operator fields", treated the same problems as our §2 and his results overlap with ours in many points, though we do not know which parts of this paper overlap with his. All results of this paper were derived independently of those of the above paper.

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