ALGEBRAIC DERIVATIONS IN THE FIELD OF MIKUSIŃSKI'S OPERATORS

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Let $\mathfrak{C}$ be the set of complex-valued functions in R^1, continuous in $0 \leqq t < +\infty$ and vanishing identically in $t < 0$. Under pointwise addition and convolution as multiplication, $\mathfrak{C}$ forms a commutative algebra over the complex numbers which contains no zero divisor. Elements of the quotient field $\mathfrak{D}$ of the ring $\mathfrak{C}$ are the operators of Mikusiński. Every function $f(t)$ in $\mathfrak{C}$ is regarded as an element of $\mathfrak{D}$ which is denoted by $\{f\}$. Recently, Mikusiński [3] [4] has discussed some properties of an algebraic derivation D in $\mathfrak{D}$ which is defined by $Du = \{-tu(t)\}$ for any $u \in \mathfrak{C}$. In any algebra A, an algebraic derivation F is defined as a linear mapping of A into some algebra B, which contains A as a subalgebra, such that $F(ab) = F(a)b + aF(b)$ for any $a,b \in A$. Here we shall study algebraic derivations in the field $\mathfrak{D}$ which are continuous in some sense specified below. Since any derivation in $\mathfrak{D}$ is completely determined by its effects on the ring $\mathfrak{C}$, we have only to consider algebraic derivations from $\mathfrak{C}$ into $\mathfrak{D}$. Notations are the same as in Mikusiński [3], unless otherwise stated.

Clearly, $\mathfrak{C}$ is a locally convex space with respect to the topology of uniform convergence on compact sets in R^1. Moreover, it is a locally multiplicatively-convex F-algebra in the sense of Michael [2]. A sequence $\{a_n : n = 1, 2,\dots\}$ in $\mathfrak{D}$ is said to be convergent to an element $a \in \mathfrak{D}$ if there exists an element $b \in \mathfrak{D}$ such that ba_n and ba are contained in $\mathfrak{C}$ and $ba_n \to ba$ with respect to the topology of $\mathfrak{C}$. Then any sequence has at most one limit. The pseudo-topology of $\mathfrak{D}$ thus defined is used in what follows. It is known that there is no locally convex Hausdorff topology in $\mathfrak{D}$ which induces this notion of convergence for sequences (cf. [1], [3]).

THEOREM 1. *A linear mapping F of $\mathfrak{C}$ into $\mathfrak{D}$ is a continuous derivation if and only if there exists an element $a \in \mathfrak{D}$ such that $F = aD$, i.e. $F(u) = a \cdot D(u)$ for any $u \in \mathfrak{C}$, where $D(u) = \{-tu(t)\}$.*

PROOF. Let F be a continuous derivation of $\mathfrak{C}$ into $\mathfrak{D}$. Since $\{t^k\} = k!\, l^{k+1} (k = 0, 1, 2,\dots)$, any polynomial $\{f(t)\} = \left\{\sum_{k=0}^{n} \alpha_k t^k\right\}$ is expressed as

$\sum_{k=0}^{n} k!\,\alpha_k l^{k+1}$, where l is the Heaviside's unit function, i.e. $l(t) = 0$ for $t < 0, = 1$ for $t \geqq 0$. Here we mean $\{f(t)\} = \{f(t)l(t)\}$ for any polynomial $f(t)$. Applying F, we have

$$F(f) = \sum_{k=1}^{n} k!\,\alpha_k F(l^{k+1}) = \sum_{k=0}^{n} (k+1)!\,\alpha_k l^k \cdot F(l)$$

$$= (\alpha_0 + \sum_{k=1}^{n} (k+1)k\alpha_k \{t^{k-1}\}) \cdot F(l)$$

$$= s^2 \{tf(t)\} \cdot F(l), \text{ where } s = l^{-1}.$$

Putting $a = - s^2 \cdot F(l)$, we have $F(f) = a \cdot D(f)$ for any polynomial $\{f\}$ in $\mathfrak{C}$.

For any $u \in \mathfrak{C}$, there exists, by a theorem of Weierstrass, a sequence of polynomials $\{f_n\}$ which tends to u in $\mathfrak{C}$. By the continuity of $F, F(f_n)$ tends to $F(u)$ in $\mathfrak{D}$. On the other hand, it is clear that the sequence $F(f_n) = a \cdot \{- tf_n(t)\}$ tends to $a \cdot \{- tu(t)\}$ in $\mathfrak{D}$. Since the limit is unique, we have thus $F(u) = a \cdot \{- tu(t)\} = a \cdot D(u)$.

It is an easy matter to show that $F = aD$ is a continuous derivation of $\mathfrak{C}$ into $\mathfrak{D}$ for any $a \in \mathfrak{D}$. Q. E. D.

THEOREM 2. *Let F be a continuous derivation of $\mathfrak{C}$ into $\mathfrak{D}$. If F brings $\mathfrak{C}$ into itself, then the element a, stated in Theorem* 1, *represents a measure μ on R^1 with carrier in $0 \leqq t < +\infty$ and*

$$(1) \qquad F(u) = \mu * D(u) = - \int_0^\infty (t - \tau)u(t - \tau)d\mu(\tau)$$

for all $u \in \mathfrak{C}$. Thus F is a continuous endomorphism of the linear space $\mathfrak{C}$.

PROOF. By assumption, $F(l)$ is a function in $\mathfrak{C}$ which we denote by $h(t)$. It is shown in the proof of Theorem 1 that $F(u) = s^2 \cdot \{tu(t)\}\{h(t)\}$ for any $u \in \mathfrak{C}$. Put $T = - (d^2/dt^2)h(t)$. Then T is a distribution with carrier in $0 \leqq t < +\infty$. For any $u \in \mathfrak{C}$, $T*(- tu(t))$ is a well-defined distribution whose carrier is contained in $0 \leqq t < +\infty$. Let g be a non-zero twice continuously differentiable function with compact carrier in $0 \leqq t < +\infty$. Then we have $F(u) \cdot \{g\} = s^2 \{tu(t)\} \cdot \{h(t)\} \cdot \{g(t)\} = \{tu(t)\} \cdot \{h(t)\} \cdot \{g''(t)\} = (tu) * h * g''$. On the other hand, it is clear that $(- tu)*T*g = (tu)*h*g''$. Since $F(u)$ is a function in $\mathfrak{C}$, $F(u) \cdot \{g\} = F(u)*g$ and therefore $F(u)*g = (- tu)*T*g$. By [5; Chap. VI, Th. XIV], we have $F(u) = T*(- tu(t))$ almost everywhere. This also shows that $T*(- tu(t))$ is a locally bounded function for any $u \in \mathfrak{C}$.

Now we shall show that T is a measure. Let φ be any continuous function in R^1 with compact carrier. Then, for some real k, $\psi(t) = \varphi(t - k)$ is in $\mathfrak{C}$ and $\psi(t) = 0$ in a neighborhood of $t = 0$. Thus there is a continuous function $u \in \mathfrak{C}$ such that $\psi(t) = - tu(t)$ and therefore $T*\psi(t) = T*(- tu(t))$ is locally bounded. Since $\psi(t) = \varphi(t - k) = \tau_k \varphi = \delta_{(k)} * \varphi$, we have $T*\psi = T*(\delta_{(k)}*\varphi)$

$= \delta_{(k)} * (T * \varphi) = \tau_k (T * \varphi)$. It follows readily that $T * \varphi$ is a locally bounded function in R^1. As φ is arbitrary, T is a measure by a theorem of Schwartz [5; pp.48-49].

Hence, denoting the measure by μ, we get $F(u) = \mu * (-tu) = \mu * D(u)$, where the equalities hold not only almost everywhere but everywhere. It is now easy to see that $F(u_n)$ tend to $F(u)$ in $\mathfrak{C}$ whenever u_n tend to u in $\mathfrak{C}$. The theorem is thus established.

It is also obvious that, for measure μ with carrier in $0 \leqq t < +\infty$, the mapping F defined by (1) is a continuous derivation of $\mathfrak{C}$ into itself. The Mikusinski's derivation D is characterized by the fact that it is the only continuous derivation of $\mathfrak{C}$ into $\mathfrak{O}$ satisfying $D(s) = 1$. The measure corresponding to D by Theorem 2 is the Dirac's measure δ.

Since any derivation $F = aD$, $a \in \mathfrak{O}$, is extendable to the whole field $\mathfrak{O}$ in a unique fashion and since the element a corresponding to F is determined uniquely, we have the following

THEOREM 3. *An algebraic derivation F of $\mathfrak{O}$ into itself induces a continuous mapping of $\mathfrak{C}$ into $\mathfrak{O}$, if and only if it is of the form aD where a is an element in $\mathfrak{O}$ which is determined uniquely by F.*

Finally, it is noted that the continuity hypothesis or the like cannot be removed from our theorems. To see this, we shall make some remarks on arbitrary (not assumed to be continuous) derivations in $\mathfrak{O}$. Let P be the algebra of polynomials in l and L the quotient field of P. Then the field $\mathfrak{O}$ of operators is a transcendental extension of L. In other words, if we let L_0 the largest subfield of $\mathfrak{O}$ which is algebraic over L, then $L_0 \neq \mathfrak{O}$ or, equivalently, $\mathfrak{C} \not\subset L_0$. Suppose on the contrary that $\mathfrak{C} \subset L_0$, then, for any $x \in \mathfrak{C}$, there exist a finite number of elements $a_0, a_1, \ldots, a_n \in L$ such that $a_0 x^n + a_1 x^{n-1} + \ldots + a_n = 0$. We may assume that this equation is irreducible in L and the coefficients a_i are contained in P. In this case, $a_n \neq 0$. We must have such an equation for $x = \{e^t\}$. By a simple calculation, we get $x^m = \{t^{m-1} e^t / (m-1)!\}$ for $m = 1, 2, \ldots$. It follows from this and the fact that $a_0 x^n + a_1 x^{n-1} + \ldots + a_{n-1} x = -a_n \neq 0$ that $a_0 x^n + \ldots + a_{n-1} x = o(t^k e^t)$ for some k when t tends to $+\infty$. Since a_n is a polynomial, we have arrived at a contradiction.

Since L_0 is a separable extension of L, any derivation F of P into $\mathfrak{O}$ has, as is well known, the unique extension to L_0 which is again denoted by F. But $\mathfrak{O}$ is transcendental over L_0 and therefore there are infinitely many extensions F from L_0 to $\mathfrak{O}$ unless we impose any restriction on F.

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