ON LINEARLY INDEPENDENT ALMOST COMPLEX STRUCTURES IN A DIFFERENTIABLE MANIFOLD

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Introduction. Let M be a difierentiable manifold. If M admits k almost complex structures $\Phi_1, \Phi_2, \ldots \Phi_k$ ($\Phi_1^2 = \Phi_2^2 = \cdots = \Phi_k^2 = -E$)[1], then we say that they are *linearly independent with respect to real constant coefficients* if $\alpha_1\Phi_1 + \alpha_2\Phi_2 + \ldots + \alpha_k\Phi_k = 0$[2] ($\alpha_1, \alpha_2, \ldots \alpha_k$: real constants) always implies $\alpha_1 = \alpha_2 = \ldots = \alpha_k = 0$. If a differentiable manifold M admits k almost complex structures linearly independent in real constant coeffiecients and any other almost complex structure (if it exists) is given by a linear combination (with respect to real constant coefficients) of the k independent almost complex structures, then we say that the *dimension of the set of almost complex structures* in the sense of linear combination in real constant coefficients is equal to k. We remark that a linear combination of almost complex structures does not in general give an almost complex structure, but the number k has an invariant meaning.

The purpose of this paper is to clearify the structure of the manifold M whose dimension of the set of almost complex structures in the above sense is equal to 2 or 3 and put in order all such manifolds.

1. Case where the dimension of almost complex structures is equal to 2. With respect to almost complex structures in a differentiable manifold M we have already known the following remarkable structures[3]:

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1) $E(=\delta^i_j; i, j = 1, \ldots, m)$ denotes the unit matrix of degree m, where m is the dimension of M. We omit eventually to indicate the degree of a unit matrix E or the dimension of the manifold M.

2) This equation is written in tensor forms such that $\alpha_1 \overset{(1)}{\phi}{}_i{}^j + \alpha_2 \overset{(2)}{\phi}{}_j{}^i + \ldots + \alpha_k \overset{(k)}{\phi}{}_j{}^i = 0$, where $\Phi = (\overset{(1)}{\phi}{}_j{}^i), \ldots, \Phi_k = (\overset{(k)}{\phi}{}_j{}^i)$ $(i, j = 1, \ldots, m)$. In this paper, since we only deal with tensor fields of type (1,1), we do not use tensor calculations as far as possible, making use of matrix-operations. The meanings of Φ_1^2 or $\Phi_1\Phi_2$ is of course $\Phi_1^2 = (\overset{(1)}{\phi}{}_j{}^a \overset{(1)}{\phi}{}_i{}^a)$ or $\Phi_1\Phi_2 = (\overset{(1)}{\phi}{}_{ja} \overset{(2)}{\phi}{}_i{}^a)$, and so on. In general, capital letters Φ, Ψ, T, A, B, $C, \ldots$ will denote matrices given by the components of respective tensor fields.

3) With respect to almost quaternions structure, see for instance, C. Ehresmann [1]; P. Libermann [2]; M. Obata [3], [4]; C. J. Hsu [7]; H. Wakakuwa [5], [6]. The structural group of the tangent bundle of such a manifold is reducible to $Sp(n)$, the unitary symplectic group.

The almost complex-product structure ot the first kind is the same as Ltbermann's

(I) *Almost quaternion structure* (Φ, Ψ, T): $\Phi^2=\Psi^2=\mathrm{T}^2=-E$; $\Phi\Psi=-\Psi\Phi=\mathrm{T}$,

$$\Psi\mathrm{T}=-\mathrm{T}\Psi=\Phi,\ \mathrm{T}\Phi=-\Phi\mathrm{T}=\Psi.$$

(II) *Almost complex-product structure of the first kind* (Φ, Ψ, T):

$\Phi^2=-E,\ \Psi^2=\mathrm{T}^2=E;\ \Phi\Psi=-\Psi\Phi=\mathrm{T},\ \Psi\mathrm{T}=-\mathrm{T}\Psi=-\Phi,\ \mathrm{T}\Phi=-\Phi\mathrm{T}=\Psi.$

(III) *Almost complex-product structure of the second kind* (Φ, Ψ, T):

$$\Phi^2=\Psi^2=-E,\ \mathrm{T}^2=E;\ \Phi\Psi=\Phi\Psi=-\mathrm{T},\ \Psi\mathrm{T}=\mathrm{T}\Psi=\Phi,\ \mathrm{T}\Phi=\Phi\mathrm{T}=\Psi.$$

Now, we set forth the following new structure:

(IV) structure (Φ, Ψ, T): $\Phi^2=\Psi^2=\mathrm{T}^2=-E$;

$$\Phi\Psi+\Psi\Phi=2E\ (\Phi\neq-\Psi),\ \mathrm{T}=E-\Phi\Psi-\Phi\ (=\Psi\Phi-E-\Phi).$$

We can easily verify that the three almost complex structures Φ, Ψ and T in (IV) satisfy

$$(1.1)\qquad \Phi\Psi+\Psi\Phi=2E(\Phi\neq-\Psi),\ \Phi\mathrm{T}+\mathrm{T}\Phi=2E,\ \Psi\mathrm{T}+\mathrm{T}\Psi=-2E.$$

The above three structures are characterized by the following simple forms, for the existence of the remaining structure T necessarily follows by means of Φ and Ψ.

(I′) $\qquad \Phi^2=\Psi^2=-E;\ \Phi\Psi=-\Psi\Phi,$

(II′) $\qquad \Phi^2=-E,\ \Psi^2=E;\ \Phi\Psi=-\Psi\Phi,$

(III′) $\qquad \Phi^2=\Psi^2=-E;\ \Phi\Psi=\Psi\Phi,$

(IV′) $\qquad \Phi^2=\Psi^2=-E;\Phi\Psi+\Psi\Phi=2E\ (\Phi\neq-\Psi).$

The verifications are easy for the cases (I′), (II′) and (III′) by putting $\Phi\Psi=-\Psi\Phi=\mathrm{T}$ or $\Phi\Psi=\Psi\Phi=\tau$. We only prove for the case (IV′). If there exist two almost complex structure Φ and Ψ satisfying the above relation (IV′), then we have $E-\Phi\Psi-\Phi=\Psi\Phi-E-\Phi$. If we put these common tensor field by T then we see that

$$\mathrm{T}^2=(E-\Phi\Psi-\Phi)(\Psi\Phi-E-\Phi)=-E,$$

making use of (IV′).

We will state several lemmas with respect to the above four structures (I), (II), (III), (IV).

LEMMA 1.1. *The three almost complex structures* Φ, Ψ *and* T *in the structure* (I) *or* (IV) *are linearly independent with respect to real constant*

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quaternion structure of the second kind".

The Ψ and T in (II) and the T in (III) are almost *product* structures. When we speak of almost product structures, we omit of course the trivial almost product structure $\pm E(=\pm\delta^i_j)$.

coefficients.

PROOF. For the structure (I), the linear independence of Φ, Ψ and T easily follows from the quaternic relations. We will prove the lemma for the structure (IV).

In the structure (IV), assume that there is a linear relation such that $\alpha\Phi + \beta\Psi + \gamma\mathrm{T} = 0$.

Taking account of the equations obtained by multiplying Φ from the left and from the right to the above equation and making use of (1. 1), we get $\alpha = \beta + \gamma$. And since $\mathrm{T} = E - \Phi\Psi - \Phi$, we have $(\beta + \gamma)\Phi + \beta\Psi + \gamma(E - \Phi\Psi - \Phi) = 0$ or

$$\beta(\Phi + \Psi) + \gamma(E - \Phi\Psi) = 0$$

By multiplying Φ from the left to the last equation, we get

$$-\beta(E - \Phi\Psi) + \gamma(\Phi + \Psi) = 0.$$

From the last two equations, we have $(\beta^2 + \gamma^2)(\Phi + \Psi) = 0$. Since $\Phi \neq -\Psi$, we get $\beta = \gamma = 0$ and hence $\alpha = 0$. Q. E. D.

The two almost complex structures Φ and Ψ in (III) are linealy independent with respect to real constant coefficients, since $\mathrm{T} \neq \pm E$.

The structure (II) admits apparently only one almost complex structure but we can state the following lemma.

LEMMA 1. 2. *The structure* (II) *admits two almost complex structures linearly independent with respect to real constant coefficients.*

PROOF. If we put $\Phi_1 = \alpha\Phi + \beta\Psi$, where α and β are non zero real constants satisfying $\alpha^2 - \beta^2 = 1$, then we see that Φ_1 is an almost complex structure linearly independent with Φ. Q. E. D.

Summing up the Lemmas 1. 1 and 1. 2, we can state

LEMMA 1. 3. *In the structures* (I) *and* (IV), *there are three almost complex structures linearly independent with respect to real constant coefficients. In the structures* (II) *and* (III), *there are two almost complex structures linearly independent with respect to real constant coefficients.*

LEMMA. 1. 4. *Let M be a differentiable manifold. And assume that M admits two linearly independent* (*with respect to real constant coefficients*) *almost complex structures $A = (a_j^i)$, $B = (b_j^i)$ satisfying*

$$AB + BA = cE \ (c: \text{ real constant}).$$

Then we can necessarily find in M one of the three structures (I), (II), (IV).

PPOOF. If $c = 0$, we can find in M the structure (I) (see (I′)). If $|c| < 2$, we put

$$\Phi = A,\quad \Psi = \left(\frac{c}{2}A + B\right) \Big/ \sqrt{1 - \frac{c^2}{4}}\ .$$

Then we see that $\Phi^2 = -E$ and $\Psi^2 = -E$, $\Phi\Psi + \Psi\Phi = 0$. Hence we can also find the structure (I) (see (I′)).

If $|c| > 2$, we put

$$\Phi = A,\quad \Psi = \left(\frac{c}{2}A + B\right) \Big/ \sqrt{\frac{c^2}{4} - 1}\ .$$

Then, we see that $\Phi^2 = -E$ and $\Psi^2 = E$, $\Phi\Psi + \Psi\Phi = 0$. Hence, we can find the structure (II) (see (II′)).

Lastly, if $|c| = 2$, we can assume without any loss of generality that $c = 2$. For, if $c = -2$, we consider $-B$ instead of B. Hence A and B satisfy $AB + BA = 2E\,(A \neq - B)$ and by putting $\Phi = A$, $\Psi = B$ we can find the structure (IV) (see (IV′)).

LEMMA 1.5. *Let $A = (a_j^i)$, $B = (b_j^i)$ be two almost complex structures linearly independent with respect to real constant coefficients in a differentiable manifold M. If*

$$\alpha A + \beta B \qquad (\alpha, \beta:\ \textit{real constants};\ \alpha\beta \neq 0)$$

gives again an almost complex structure, then we can find in M one of the three structures (I), (II), (IV).

PEOOF. Since $\alpha A + \beta B$ $(\alpha\beta \neq 0)$ gives an almost complex structure, we have $(\alpha A + \beta B)^2 = -E$ or $AB + BA = \dfrac{\alpha^2 + \beta^2 - 1}{\alpha\beta} E$. Hence, by virtue of the above Lemma 1.4, we can find in M one of the structures (I), (II), (IV).

Q.E.D.

LEMMA 1.6. *Let $A = (a_j^i)$, $B = (b_j^i)$ be two almost complex structures linearly independent with respect to real constant coefficients in a differentiable manifold M. If A and B satisfy*

$$ABA = \alpha A + \beta B \ (\alpha, \beta:\ \textit{real constants};\ \alpha^2 + \beta^2 \neq 0),$$

then we can find in M one of the four structures (I), (II), (III), (IV).

PROOF. If $\beta = 0$ $(\alpha \neq 0)$, we have $ABA = \alpha A$, from which we get

$B = -\alpha A$. But this contradicts to the assumption that A and B are linearly independent. Hence we see that $\beta \neq 0$. If $\alpha = 0\,(\beta \neq 0)$, we have $ABA = \beta B$ and we see that $ABA = \pm B$. If $ABA = B$, we have $AB + BA = 0$. Hence we can find the structure (I). If $ABA = -B$, we have $AB = BA$. Hence we can find the structure (III) (see (III')).

Lastly, if $\alpha\beta \neq 0$, we see that M admits one of the steuctures (I),(II),(IV) by virtue of Lemma 1.5. Q.E.D.

THEOREM. *Let M be a differentiable manifold. If the dimension of almost complex structures of M in the sense of linear combination in real constant coefficients is equal to* 2 (see the definition in the Introduction), *then M admits the structure* (II) *or the structure* (III).

PROOF. By virtue of the definition, there are two almost complex structures $A = (a_j^i)$, $B = (b_j^i)$ linearly independent with respect to real constant coefficients and any other almost complex structure is a linear combination of A and B with respect to real constant coefficients. Since ABA is an almost complex structure, we can write

$$ABA = \alpha A + \beta B \qquad (\alpha, \beta : \text{real};\ \alpha^2 + \beta^2 \neq 0).$$

Hence, by virtue of Lemma 1.6, we can find in M one of the structures (I),(II),(III),(IV). But, the structures (I) and (IV) admits three almost complex structures linearly independent with respect to real constant coefficients (Lemma 1.3), which contradicts to the assumption. Therefore, we find the structure (II) or (III) in M. Q.E.D.

COROLLARY 1.1. *Let M be a differentiable manifold which does not admit any almost product structure. Then the dimension of the set of almost complex structures in the sense of linear combination in real constant coefficients is* $= 0, 1$ *or* $\geqq 3$.

PROOF. If the dimension of the set of almost complex structures is just equal to 2, then we find in M the structure (II) or (III) by virtue of the Theorem. But these structures admits one or two almost product structures contradictorily to the assumption.

2. The structure (IV). Since the structures (I), (II) and (III) are already treated ([1]~[7]), we state here several remarks on the new structure (IV).

LEMMA 2.1. *The structure* (IV) *is equivalent to*

(IV)$_1$ *strurture* (Φ, Σ): $\Phi^2 = -E$, $\Sigma^2 = 0$[4] $(\Sigma \neq 0)$, $\Phi\Sigma + \Sigma\Phi = 0$.

4) R.S. Clark and M.R. Bruckheimer ([8]) have treated a tensor field J in a $2n$-dimensional manifold M_{2n} such that $J^2 = 0$ and rank $J = 2n$ all over the M_{2n}.

PROOF. If we are given the structure (IV) in M, we put $\Sigma = \Phi + \Psi$. Then we can verify that $\Sigma^2 = 0$, $\Phi\Sigma + \Sigma\Phi = 0$ by virtue of the relations in (IV). Conversely, if we are given the structure $(IV)_1$ in the present lemma, then we can find the structure (IV) by putting $\Psi = \Sigma - \Phi$, $T = E - \Phi\Psi - \Phi$.

In the structure (IV) or $(IV)_1$, if $\Sigma = \Phi + \Psi$ is of constant rank $r(< \dim M)$ over the whole manifold M, then we say that (IV) or $(IV)_1$ is of requlr type. If otherwise, we say that (IV) or $(IV)_1$ is of singular type.

LEMMA 2. 2. *If the structure* (IV) *or* $(IV)_1$ *in* M *is of regular type, then we can find in* M *an almost product structure.*

PROOF. By the definition, $\Sigma = \Phi + \Psi$ is of constant rank r all over the M. The characteristic roots of Σ are all zero and there corresponds a subspace of dimension $s = \dim M - r$. These define a distribution D of dimension s throughout the M. Since there exists always a distribution D' complementary to D, M admits an almost product structure. Q. E. D.

We will here introduce the normal forms of the structure (IV) or $(IV)_1$ at the tangent space of a point P of M. Since the structures (IV) and $(IV)_1$ are equivalent (Lemma 2. 1), we consider only the structure $(IV)_1$:

$$\Phi^2 = -E, \ \Sigma^2 = 0, \ \Phi\Sigma + \Sigma\Phi = 0. \tag{2.1}$$

The dimension of M is even: $\dim M = 2n$, since M admits an almost complex stlucture Φ. It is well known that at the tangent space of P we can choose a complex frame such that the almost complex structure Φ takes the form

$$\Phi = \begin{pmatrix} iE_n & 0 \\ 0 & -iE_n \end{pmatrix}, \tag{2.2}$$

where E_n denotes the unit matrix of degree n. In this case, the last equation of (2. 1) implies that Σ is necessarily of the form $\Sigma = \begin{pmatrix} 0 & A \\ A' & 0 \end{pmatrix}$, where A and A' are $(n \times n)$-matrices. And since Σ has a real representation together with Φ, it must be self-adjoint: $A' = \overline{A}$, that is

$$\Sigma = \begin{pmatrix} 0 & A \\ \overline{A} & 0 \end{pmatrix}. \tag{2.3}$$

From the second equation of (2. 1), we have $A\overline{A} = 0$, and the $(n \times n)$-matrix A is degenerate. Hence there exists at least a non zero vector z such that $Az = 0$. Hereby the vector z should be in general complex. Let $(z_1, z_2, \cdots, z_n)$ be the components of z and consider a regular $(n \times n)$-matrix of the form

$$X=\begin{pmatrix} z_1 & \\ z_2 & \\ \vdots & * \\ \vdots & \\ z_n & \end{pmatrix},$$

where the elements in the part $*$ are arbitrary except the condition that X is regular. Then, we can see that $X^{-1}A\overline{X}$ is of the form

$$X^{-1}A\overline{X}=\left(\begin{array}{c|c} 0 & \\ \vdots & * \\ 0 & \end{array}\right), \tag{2.4}$$

taking account of $Az=0$. We write this $X^{-1}A\overline{X}$ anew by A. We can see that there exists a complex $(n\times n)$-matrix Y such that (see Appendix)

$$Y^{-1}A\overline{Y}=\begin{pmatrix} 0 & e_1 & & & 0 \\ \vdots & \ddots & \ddots & & \\ \vdots & & \ddots & \ddots & \\ \vdots & & & \ddots & e_{n-1} \\ 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix}, \tag{2.5}$$

where $e_1,\dots,e_{n-1}$ are equal to 0 or 1. But since $A\overline{A}=0$ and $Y^{-1}A\overline{Y}$ is real, the square of the last matrix is 0. Hence we see that at least one of the neighboring e_i and e_{i+1} $(i=1,\dots,n-2)$ must be zero. If we consider a trans-formation by a matrix of the form $\begin{pmatrix} Y & 0 \\ 0 & \overline{Y}\end{pmatrix}$ for the Y satisfying (2.5), then we obtain the final form of Σ such that

$$\Sigma=\left(\begin{array}{c|c} 0 & \begin{matrix} 0 & e_1 & & & 0 \\ \vdots & \ddots & \ddots & & \\ \vdots & & \ddots & \ddots & \\ \vdots & & & \ddots & e_{n-1} \\ 0 & \cdots & \cdots & \cdots & 0 \end{matrix} \\ \hline \begin{matrix} 0 & e_1 & & & 0 \\ \vdots & \ddots & \ddots & & \\ \vdots & & \ddots & \ddots & \\ \vdots & & & \ddots & e_{n-1} \\ 0 & \cdots & \cdots & \cdots & 0 \end{matrix} & 0 \end{array}\right),$$

where $e_1,\dots,e_{n-1}$ are equal to 0 or 1 and at least one of the neighboring e_i and $e_{i\ 1}$ $(i=1,\dots,n-2)$ is equal to zero. Hence we know that the rank of Σ is even and it is $\leqq n$. And furthermore, we can transform Σ to the form

$$\Sigma = \left(\begin{array}{cc|cc} \multicolumn{2}{c|}{\multirow{2}{*}{0}} & 0 & 0 \\ & & E_p & 0 \\ \hline 0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ E_p & 0 & & \end{array}\right),$$

where E_p is a unit matrrix of degree $p\left(\leqq \dfrac{n}{2}\right)$.

It is easily seen that under the above normalization of Σ the form (2.2) of Φ is unchanged. Consequently, with respect to a complex frame, the normal forms of the structure $(\mathrm{IV})_1$ are given by

$$\Phi = \begin{pmatrix} iE_n & 0 \\ 0 & -iE_n \end{pmatrix}, \qquad \Sigma = \left(\begin{array}{cc|cc} \multicolumn{2}{c|}{\multirow{2}{*}{0}} & 0 & 0 \\ & & E_p & 0 \\ \hline 0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ E_p & 0 & & \end{array}\right),$$

where E_n and E_p are unit matrices of degree n and $p\left(\leqq \dfrac{n}{2}\right)$ respectively.

Making use of a matrix

$$(2.6) \qquad I = \frac{1}{\sqrt{2}}\begin{pmatrix} E_n & iE_n \\ E_n & -iE_n \end{pmatrix}, \quad \left(I^{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} E_n & E_n \\ -iE_n & iE_n \end{pmatrix}\right),$$

where E_n is the unit matrix of degree n, we have $I^{-1}\Phi I$ and $I^{-1}\Sigma I$ which give the normal forms of Φ and Σ with respect to a real frame.
These are as follows:

$$\Phi = \left(\begin{array}{c|c} 0 & -E_n \\ \hline E_n & 0 \end{array}\right), \qquad \Sigma = \left(\begin{array}{cc|cc} 0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ E_p & 0 & & \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{0}} & 0 & 0 \\ & & -E_p & 0 \end{array}\right),$$

where E_n and E_p are unit matrices of degree n and $p\left(\leqq \dfrac{n}{2}\right)$ respectively.

3. Two new structures. We introduce in this section, two new structures and their normal norms.

(V) *structure* (Φ, Ψ, T): $\Phi^2 = \Psi^2 = \mathrm{T}^2 = -E$;

$$\Phi\Psi = \Psi\Phi, \ \Psi\mathrm{T} = \mathrm{T}\Psi, \ \mathrm{T}\Phi = \Phi\mathrm{T},$$

where Φ, Ψ and T are linearly independent with respect to real constant coefficients.

If M admits such a structure (V), the dimension of M is of course even: $M = M_{2n}$. The normal forms of this structure at the tangent space of a point P of M_{2n} is introduced as follows.

With respect to a suitable complex frame, one of the Φ, Ψ and T, for instance Φ takes the form: $\Phi = \begin{pmatrix} iE_n & 0 \\ 0 & -iE_n \end{pmatrix}$. And in this case, by virtue of the relations $\Phi\Psi = \Psi\Phi$ and $\mathrm{T}\Phi = \Phi\mathrm{T}$, we see that Ψ and T take the forms such that $\Psi = \begin{pmatrix} P & 0 \\ 0 & P' \end{pmatrix}, \mathrm{T} = \begin{pmatrix} Q & 0 \\ 0 & Q' \end{pmatrix}$, where P, Q, P' and Q' are $(n \times n)$-matrices. Since Ψ and T have their real representation together with Φ we must have $P' = \overline{P}, Q' = \overline{Q}$, that is,

$$\Psi = \begin{pmatrix} P & 0 \\ 0 & \overline{P} \end{pmatrix}, \qquad \mathrm{T} = \begin{pmatrix} Q & 0 \\ 0 & \overline{Q} \end{pmatrix}.$$

Furthermore, we see that $P^2 = \overline{P}^2 = -E_n$, $Q^2 = \overline{Q}^2 = -E_n$, since Ψ and T are almost complex structures. Hence the characteristic roots of P and Q ($\overline{P}$ and $\overline{Q}$) are $\pm i$. Making use of a suitable complex $(n \times n)$-matrix X, we can get

$$X^{-1}PX = \left(\begin{array}{c|c} \begin{matrix} i & e_1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & e_{r-1} \\ 0 & & & i \end{matrix} & 0 \\ \hline 0 & \begin{matrix} -i & e'_1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & e'_{-1} \\ 0 & & & -i \end{matrix} \end{array}\right), \quad (r + s = n),$$

where $e_1, \cdots e_{r-1}$, $e'_1, \cdots, e'_{s-1}$ are equal to 1 or 0. But we can easily see that $e_1, \cdots, e_{1-1}$, $e'_1, \cdots, e'_{s-1}$ all vanish by virtue of $P^2 = -E_n$. Therefore, making use of a $(2n \times 2n)$-matrix $\mathfrak{X}$ such that $\mathfrak{X} = \begin{pmatrix} X & 0 \\ 0 & \overline{X} \end{pmatrix}$, we get

$$\mathfrak{X}^{-1}\Psi\mathfrak{X} = \begin{pmatrix} P & 0 \\ 0 & \overline{P} \end{pmatrix}, \quad P = \begin{pmatrix} iE_r & 0 \\ 0 & -iE_s \end{pmatrix}, \qquad (r + s = n),$$

where E_r and E_s are unit matrices of degree r and s respectively. Under such a transformation, the form of Φ is unchanged: $\mathfrak{X}^{-1}\Phi\mathfrak{X} = \Phi$. Now, in this case, by virtue of $\Psi\mathrm{T} = \mathrm{T}\Psi$, we have

$$\mathfrak{X}^{-1}\mathrm{T}\mathfrak{X} = \begin{pmatrix} Q & 0 \\ 0 & \overline{Q} \end{pmatrix}, \quad Q = \begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix}$$

where Q_1 and Q_2 are matrices of degree r and s respectively. Let Y be a complex $(n \times n)$-matrix such that $Y = \begin{pmatrix} Y_1 & 0 \\ 0 & Y_2 \end{pmatrix}$, where the degrees of Y_1 and Y_2 are r and s respectively. If we choose Y_1 and Y_2 suitably, then we have

$$Y^{-1}QY = \begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix},$$

$$Q_1 = \begin{pmatrix} iE_{r_1} & 0 \\ 0 & -iE_{r_2} \end{pmatrix}, \ (r_1 + r_2 = r); \quad Q_2 = \begin{pmatrix} iE_{s_1} & 0 \\ 0 & -iE_{s_2} \end{pmatrix}, \ (s_1 + s_2 = s)$$

where E_{r_1}, E_{r_2}, E_{s_1} and E_{s_2} are unit matrices of degree r_1, r_2, s_1 and s_2 respectively.

If we consider a $(2n \times 2n)$-matrix $\mathfrak{Y}$ such that $\mathfrak{Y} = \begin{pmatrix} Y & 0 \\ 0 & \overline{Y} \end{pmatrix}$, then we see that

$$\mathfrak{Y}^{-1}\mathrm{T}\mathfrak{Y} = \begin{pmatrix} Q & 0 \\ 0 & \overline{Q} \end{pmatrix}, \quad Q = \begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix},$$

Q_1 and Q_2 being given by the preceding diagonal forms. Under such a transformation, the already obtained forms of Φ and Ψ are unchanged.

Consequently, with respect to a complex frame, the normal forms of the structure (V) are given by

$$\begin{cases} \Phi = \begin{pmatrix} iE_n & 0 \\ 0 & -iE_n \end{pmatrix}, \\ \Psi = \begin{pmatrix} P & 0 \\ 0 & \overline{P} \end{pmatrix}, \quad P = \begin{pmatrix} iE_r & 0 \\ 0 & -iE_s \end{pmatrix} \quad (r + s = n), \\ \mathrm{T} = \begin{pmatrix} Q & 0 \\ 0 & \overline{Q} \end{pmatrix}, \quad Q = \left(\begin{array}{cc|cc} iE_{r_1} & 0 & & \\ 0 & -iE_{r_2} & \multicolumn{2}{c}{0} \\ \hline & & iE_{s_1} & 0 \\ \multicolumn{2}{c|}{0} & 0 & -iE_{s_2} \end{array}\right) \quad (r_1 + r_2 = r, \ s_1 + s_2 = s). \end{cases}$$

Making use of a matrix I of (2.6), we obtain their normal forms $I^{-1}\Phi I$, $I^{-1}\Psi I$

and $I^{-1}\mathrm{T}I$ with respect to a real frame. These are as follows.

$$\begin{cases} \Phi = \begin{pmatrix} 0 & -E_n \\ E_n & 0 \end{pmatrix}, \\ \Psi = \begin{pmatrix} 0 & -P_1 \\ P_1 & 0 \end{pmatrix}, \ P_1 = \begin{pmatrix} E_r & 0 \\ 0 & -E_s \end{pmatrix}, \ (r+s=n), \\ \mathrm{T} = \begin{pmatrix} 0 & Q_1 \\ Q_1 & 0 \end{pmatrix}, \ Q_1 = \left(\begin{array}{cc|cc} E_{r_1} & 0 & & \\ 0 & -E_{r_2} & \multicolumn{2}{c}{0} \\ \hline & & E_{s_1} & 0 \\ \multicolumn{2}{c|}{0} & 0 & -E_{s_2} \end{array}\right), \ (r_1+r_2=r, \ s_1+s_2=s), \end{cases}$$

where the indices of the unit matrices E's denote their degrees.

Now, the other new structure is given as follows:

(VI) *structuree* (Φ, Ψ, T): $\Phi^2 = \Psi^2 = \mathrm{T}^2 = -E$;

$$\Phi\Psi = \mathrm{T}\Phi, \ \Psi\Phi\Psi = \alpha\Phi + \alpha\Psi + \mathrm{T}, \ (\alpha: \text{real constant}; \ \neq -1),$$

Φ, Ψ, T being linearly independent in real constant coefficients.

The normal forms of this structure are introduced as follows. Of course the dimension of the manifold M is even: $M = M_{2n}$. First we consider the case $\alpha \neq -1, 3$. With respect to a suitable complex frame, we can take normal form of Φ such that $\Phi = \begin{pmatrix} iE_n & 0 \\ 0 & -iE_n \end{pmatrix}$. And by virtue of the self-adjointness of Ψ and T and the relation $\Phi\Psi = \mathrm{T}\Phi$, we have

$$\Psi = \begin{pmatrix} P & Q \\ \overline{Q} & \overline{P} \end{pmatrix}, \quad \mathrm{T} = \begin{pmatrix} P & -Q \\ -\overline{Q} & \overline{P} \end{pmatrix},$$

where P and Q are of degree n.

Since Ψ and T are almost complex structures, we get

$$(3.1) \qquad P^2 + Q\overline{Q} = -E_n, \ PQ + Q\overline{P} = 0;$$

furthermore the last relation of (VI) implies that

$$(3.2) \qquad -P^2 + Q\overline{Q} = i(\alpha+1)P - \alpha E_n, \ -PQ + Q\overline{P} = i(\alpha-1)Q.$$

From the first equation of (3.1) and (3.2) we have

$$(3.3) \qquad P^2 = -\frac{i(\alpha+1)}{2}P + \frac{\alpha-1}{2}E_n,$$

and from the last equations of (3.1) and (3.2) we have

$$(3.4) \qquad PQ = -QP = -\frac{i(\alpha-1)}{2}Q.$$

By virtue of (3.3) we can see that the characteristic roots of P is $-i$ and $-i\dfrac{\alpha-1}{2}$. As in the preceding cases, we see that for a suitable complex $(n\times n)$-matrix X, the new form $X^{-1}PX$ of P is given by[5)]

$$P=\begin{array}{c}\begin{array}{cc}\overbrace{\hphantom{-i\quad e_1\quad \ddots\quad 0}}^{r}\qquad & \overbrace{\hphantom{-i\tfrac{\alpha-1}{2}\quad e'_1\quad 0}}^{s}\end{array}\\ \left(\begin{array}{cccc|cccc} -i & e_1 & & 0 & & & & \\ & \ddots & \ddots & & & 0 & & \\ & & \ddots & e_{r-1} & & & & \\ 0 & & & -i & & & & \\ \hline & & & & -i\frac{\alpha-1}{2} & e'_1 & & 0 \\ & 0 & & & & \ddots & \ddots & \\ & & & & & & \ddots & e'_{s-1} \\ & & & & 0 & & & -i\frac{\alpha-1}{2} \end{array}\right)\end{array}\begin{array}{l}\\ \left.\vphantom{\begin{matrix}a\\a\\a\\a\end{matrix}}\right\}r\\ \left.\vphantom{\begin{matrix}a\\a\\a\\a\end{matrix}}\right\}s\end{array},\quad (r+s=n),$$

where $e_1,\dots e_{r-1}$, $e'_1,\dots,e'_{s-1}$ are equal to 1 or 0. But, since $\alpha\neq 3$ we know that these are all zero by virtue of (3. 3), that is the new form of P is given by

$$P=\begin{array}{c}\begin{array}{cc}\overbrace{\hphantom{-i\quad \ddots\quad 0}}^{r}\qquad & \overbrace{\hphantom{-i\tfrac{\alpha-1}{2}\quad 0}}^{s}\end{array}\\ \left(\begin{array}{ccc|ccc} -i & & 0 & & & \\ & \ddots & & & 0 & \\ 0 & & -i & & & \\ \hline & & & -i\frac{\alpha-1}{2} & & 0 \\ & 0 & & & \ddots & \\ & & & 0 & & -i\frac{\alpha-1}{2} \end{array}\right)\end{array}\begin{array}{l}\\ \left.\vphantom{\begin{matrix}a\\a\\a\end{matrix}}\right\}r\\ \left.\vphantom{\begin{matrix}a\\a\\a\end{matrix}}\right\}s\end{array},\quad (r+s=n).$$

Now, if we put

---

5) It is allowable to consider a transformat'on of Ψ, T by a matrix of the from $\begin{pmatrix} X & 0 \\ 0 & \overline{X}\end{pmatrix}$, since it has a real meaning. Hereby the form of Φ is unchanged.

$$Q = \begin{array}{cc} \overbrace{\quad}^{r} & \overbrace{\quad}^{s} \end{array} \left(\begin{array}{c|c} Q_1 & Q_2 \\ \hline Q_3 & Q_4 \end{array} \right) \begin{array}{l} \}r \\ \}s \end{array},$$

then by virtue of the equations of (3. 4) we know that $Q_1 = Q_2 = Q_3 = 0$, that is,

$$Q = \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & Q_4 \end{array} \right) \begin{array}{l} \}r \\ \}s \end{array}, \quad (r + s = n).$$

From the first equations of (3. 1), we see that $Q_4\overline{Q}_4 = \sigma E_s$, where E_s is a unit matrix of degree s and $\sigma = (\alpha + 1)(\alpha - 3)/4 \neq 0$. If $\sigma > 0$, we get $\frac{Q_4}{\sqrt{\sigma}} \cdot \frac{\overline{Q}_4}{\sqrt{\sigma}} = E_s$, and we can take a complex $(s \times s)$-matrix Y_s such that $Y_s^{-1} \frac{Q_4}{\sqrt{\sigma}} \overline{Y}_s = E_s$ ([4], Lemma 1 and 2′) or $Y_s^{-1}Q_4\overline{Y}_s = \sqrt{\sigma}\, E_s$. If $\sigma < 0$, we get $\frac{Q_4}{\sqrt{-\sigma}} \cdot \frac{\overline{Q}_4}{\sqrt{-\sigma}} = - E_s$ and we can take a complex $(s \times s)$-matrix Y_s' such that $Y_s'^{-1} \frac{Q_4}{\sqrt{-\sigma}} \overline{Y}_s' = J_{s/2}$, $J_{s/2} = \begin{pmatrix} 0 & -E_{s/2} \\ E_{s/2} & 0 \end{pmatrix}$ ([4], ibid.) or $Y_s'^{-1}Q_4\overline{Y}_s' = \sqrt{-\sigma}\, J_{s/2}$. In this case, s must be even. In each case, we put

$$Y = \begin{pmatrix} E_r & 0 \\ 0 & Y_s \end{pmatrix} \text{ or } Y' = \begin{pmatrix} E_r & 0 \\ 0 & Y_s' \end{pmatrix},$$

where E_r is a unit matrix of degree r. Then making use of a matrix $\mathfrak{Y} = \begin{pmatrix} Y & 0 \\ 0 & \overline{Y} \end{pmatrix}$ or $\mathfrak{Y}' = \begin{pmatrix} Y' & 0 \\ 0 & \overline{Y}' \end{pmatrix}$, we get the normal forms $\mathfrak{Y}^{-1}\Psi\mathfrak{Y}$, $\mathfrak{Y}^{-1}\mathrm{T}\mathfrak{Y}$ (or $\mathfrak{Y}'^{-1}\Psi\mathfrak{Y}'$, $\mathfrak{Y}'^{-1}\mathrm{T}\mathfrak{Y}'$) of Ψ and T, whereas the form of Φ is unchanged. These are as follows.

$$\Phi = \begin{pmatrix} iE_n & 0 \\ 0 & -iE_n \end{pmatrix}, \Psi = \begin{pmatrix} P & Q \\ \overline{Q} & \overline{P} \end{pmatrix}, \tau = \begin{pmatrix} P & -Q \\ -\overline{Q} & \overline{P} \end{pmatrix},$$

where

$$
P = \begin{array}{c} \overbrace{\hphantom{-i \;\cdots\; -i}}^{r} \;\; \overbrace{\hphantom{-i\frac{\alpha-1}{2} \;\cdots\; -i\frac{\alpha-1}{2}}}^{s} \\ \left(\begin{array}{ccc|ccc} -i & & 0 & & & \\ & \ddots & & & 0 & \\ 0 & & -i & & & \\ \hline & & & -i\frac{\alpha-1}{2} & & 0 \\ & 0 & & & \ddots & \\ & & & 0 & & -i\frac{\alpha-1}{2} \end{array}\right) \end{array} \begin{array}{l} \}\, r \\ \\ \}\, i \end{array}, \quad Q = \begin{array}{c} \overbrace{\hphantom{0}}^{r} \;\; \overbrace{\hphantom{Q_4}}^{s} \\ \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & Q_4 \end{array}\right) \end{array} \begin{array}{l} \}\, r \\ \}\, s \end{array},
$$

$$
Q_4 = \sqrt{\sigma}\, E_s (\sigma > 0) \text{ or } \sqrt{-\sigma} \times \begin{pmatrix} 0 & -E_{s/2} \\ E_{s/2} & 0 \end{pmatrix} (\sigma < 0).
$$

With respect to a real frame, these normal forms are given by

$$
\Phi = \begin{pmatrix} 0 & -E_n \\ E_n & 0 \end{pmatrix}, \quad \Psi = \begin{pmatrix} Q' & P' \\ -P' & -Q' \end{pmatrix}, \quad T = \begin{pmatrix} -Q' & P' \\ -P' & Q' \end{pmatrix},
$$

where

$$
P' = \begin{array}{c} \overbrace{\hphantom{1 \;\cdots\; 1}}^{r} \;\; \overbrace{\hphantom{\frac{\alpha-1}{2} \;\cdots\; \frac{\alpha-1}{2}}}^{s} \\ \left(\begin{array}{ccc|ccc} 1 & & 0 & & & \\ & \ddots & & & 0 & \\ 0 & & 1 & & & \\ \hline & & & \frac{\alpha-1}{2} & & 0 \\ & 0 & & & \ddots & \\ & & & 0 & & \frac{\alpha-1}{2} \end{array}\right) \end{array} \begin{array}{l} \}\, r \\ \\ \}\, s \end{array}, \quad Q' = \begin{array}{c} \overbrace{\hphantom{0}}^{r} \;\; \overbrace{\hphantom{Q'_4}}^{s} \\ \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & Q'_4 \end{array}\right) \end{array} \begin{array}{l} \}\, r \\ \}\, s \end{array},
$$

$$
Q'_4 = \begin{cases} \sqrt{\sigma}\, E_s & (\sigma > 0) \\ \text{or} & \\ \sqrt{-\sigma} \begin{pmatrix} 0 & -E_{s/2} \\ E_{s/2} & 0 \end{pmatrix} & (\sigma < 0). \end{cases}
$$

If $\alpha = -1$ in the present structure, then it gives rise the structure (IV). For, in such a case, we have $\Psi\Phi\Psi = -\Phi - \Psi + T$, from which we get $-\Phi\Psi = -\Psi\Phi + E + \Psi T$ and $-\Psi\Phi = -\Phi\Psi + E + T\Psi$. Hence $\Psi T + T\Psi =$

$-2E$ holds true and Ψ, $-\mathrm{T}$ give rise the structure (IV).

In the case $\alpha = 3$, the relation (3.1) also holds true and (3.3), (3.4) become

$$P^2 = -2iP + E_n;\ PQ = -Q\overline{P} = -iQ, \tag{3.5}$$

respectively. We know that the characteristic roots of P are all $-i$ and we can take Φ, Ψ, T such that

$$\Phi = \begin{pmatrix} iE_n & 0 \\ 0 & -iE_n \end{pmatrix},\quad \Psi = \begin{pmatrix} P & Q \\ \overline{Q} & \overline{P} \end{pmatrix},\quad \mathrm{T} = \begin{pmatrix} P & \overline{Q} \\ -\overline{Q} & \overline{P} \end{pmatrix},$$

where

$$P = \begin{pmatrix} -i & e_1 & & & 0 \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & & & \ddots & e_{n-1} \\ 0 & & & & -i \end{pmatrix},\quad (e_1,\dots,e_{n-1} = 0 \text{ or } 1;\ \text{all of them are not zero}).$$

From the first relation of (3.5), we see that at least one of the neighboring e_i and $e_{i+1}(i = 1,\dots,n-2)$ is zero. An example of normal forms of this structure is set forth as follows:

$$P = \begin{pmatrix} -i & 1 & & & 0 \\ & \ddots & 0 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 0 \\ 0 & & & & -i \end{pmatrix},\quad Q = \begin{pmatrix} 0 & 0 & \sqrt{2} & \\ \vdots & 0 & \vdots & \\ \vdots & -\sqrt{2}\,i & \vdots & \\ \vdots & 0 & \vdots & 0 \\ \vdots & \vdots & \vdots & \\ 0 & 0 & 0 & \end{pmatrix},$$

all relations of (3.1) and (3.5) are satisfied and hence all properties of (VI) hold true.

We remark that in case of $\alpha = 3$, $\Sigma = 2\Phi + \Psi + \mathrm{T}$ is a tensor field such that $\Sigma^2 = 0$ which is verified making use of the relations in (VI) ($\alpha = 3$). And this tensor field Σ also satisfies $\Phi\Sigma = \Sigma\Phi$.

In the case $\alpha \neq -1$, there does not exist a relation such that $\Phi\Psi + \Psi\Phi = \rho E$. It is verified by taking account of the forms of P.

4. The last new structure. The last structure is set forth as follows: (VII) *structure* (Φ, Ψ, T): $\Phi^2 = \Psi^2 = \mathrm{T}^2 = -E$;

$$\begin{cases} \Psi\mathrm{T} + \mathrm{T}\Psi = 2fE, \\ \mathrm{T}\Phi + \Phi\mathrm{T} = 2gE, \\ \Phi\Psi + \Psi\Phi = 2hE \end{cases} \quad (f, g, h:\ \text{real constants};\ |f|, |g|, |h| > 1),$$

Φ, Ψ, T being linearly independent in real constant coefficients.

The normal forms of this structure are introduced as follows. The dimension of the manifold M is even: $M = M_{2n}$. First we take Φ such that $\Phi = \begin{pmatrix} iE_n & 0 \\ 0 & -iE_n \end{pmatrix}$ and we put $\Psi = \begin{pmatrix} P_1 & P_2 \\ \overline{P_2} & \overline{P_1} \end{pmatrix}$, $\mathrm{T} = \begin{pmatrix} Q_1 & Q_2 \\ \overline{Q_2} & \overline{Q_1} \end{pmatrix}$, taking account of the self-adjointness of Ψ and T. From the last two relations of (VII), we see that $P_1 = -ihE_n$, $Q = -igE_n$, that is,

$$\Psi = \begin{pmatrix} -ihE_n & P_2 \\ \overline{P_2} & ihE_n \end{pmatrix}, \quad \mathrm{T} = \begin{pmatrix} -igE_n & Q_2 \\ \overline{Q_2} & igE_n \end{pmatrix}.$$

Since Ψ and T are almost complex structures, we have

$$P_2\overline{P_2} = (h^2 - 1)E_n, \quad Q_2\overline{Q_2} = (g^2 - 1)E_n, \tag{4.1}$$

where $h^2 - 1 > 0$, $g^2 - 1 > 0$. Now, the first equation of (4.1) is written such that $\dfrac{P_2}{\sqrt{h^2-1}} \cdot \dfrac{\overline{P_2}}{\sqrt{h^2-1}} = E_n$. We can take a complex $(n \times n)$-matrix X such that $X^{-1}\dfrac{P_2}{\sqrt{h^2-1}} \cdot \overline{X} = E_n$ or $X^{-1}P_2\overline{X} = \sqrt{h^2-1}\,E_n$ ([4], Lemma 1 and 2′).

Hence if we consider a transformation by a matrix $\mathfrak{X} = \begin{pmatrix} X & 0 \\ 0 & \overline{X} \end{pmatrix}$ (see the footnote 5)), the new forms $\mathfrak{X}^{-1}\Psi\mathfrak{X}$, $\mathfrak{X}^{-1}\mathrm{T}\mathfrak{X}$ of Ψ and T are as follows:

$$\Psi = \begin{pmatrix} -ihE_n & \sqrt{h^2-1}\,E_n \\ \sqrt{h^2-1}\,E_n & ihE_n \end{pmatrix}, \quad \mathrm{T} = \begin{pmatrix} -igE_n & Q_2 \\ \overline{Q_2} & igE_n \end{pmatrix},$$

whereas the form of Φ is unchanged. By virtue of the relation $\Psi\mathrm{T} + \mathrm{T}\Psi = 2fE_n$, we get

$$Q_2 + \overline{Q_2} = \frac{2(f + gh)}{\sqrt{h^2-1}} E_n. \tag{4.2}$$

Now we decompose the matrix Q_2 such that $Q_2 = R + iS$, where R and S are real $(n \times n)$-matrices. Then from (4.2) we see that R is a diagonal form: $R = \dfrac{(f + gh)}{\sqrt{h^2-1}} E_n$. And from the second equation of (4.1) we have

$$R^2 + S^2 = (g^2 - 1)E_n, \quad RS = SR, \tag{4.3}$$

the second equation necessarily holds true since R is proportional to E_n. Taking account of the first equation of (4.3) and $R = \dfrac{(f + gh)}{\sqrt{h^2-1}} E_n$, we have

$S^2 = \dfrac{\Delta}{h_2 - 1} E_n$, where $\Delta = \begin{vmatrix} 1 & -h & -g \\ -h & 1 & -f \\ -g & -f & 1 \end{vmatrix}$. If $\Delta > 0$, we can choose a real matrix Y such that $Y^{-1}SY=\sqrt{\dfrac{\Delta}{h-1}}\,E_n$ and if $\Delta<0$ we can also choose a real matrix Y' such that $Y'^{-1}SY'=\sqrt{\dfrac{-\Delta}{h^2-1}}\,J_{n/2}$, $J_{n/2}=\begin{pmatrix}0 & -E_{n/2}\\ E_{n/2} & 0\end{pmatrix}$. In the last case n must be even. If $\Delta=0$, we can choose a real matrix Y'' such that $Y''^{-1}SY''=\left(\begin{array}{c|c}0 & 0\\ \hline E_p & 0\end{array}\right)$, where E_p is a unit matrix of degree $p\left(\leqq\dfrac{n}{2}\right)$ (see §2, the normalization of Σ such that $\Sigma^2=0$). Under a transfomation by a real $(2n\times 2n)$-matrix $\mathfrak{Y}=\begin{pmatrix}Y & 0\\ 0 & Y\end{pmatrix}$, $\mathfrak{Y}'=\begin{pmatrix}Y' & 0\\ 0 & Y'\end{pmatrix}$ or $\mathfrak{Y}''=\begin{pmatrix}Y'' & 0\\ 0 & Y''\end{pmatrix}$, the already obtained normal forms of Φ and Ψ are unchanged.

Consequently, with respect to a suitable complex frame, the normal forms of the present structure are as follows, where we have put $\Delta=\begin{vmatrix}1 & -h & -g\\ -h & 1 & -f\\ -g & -f & 1\end{vmatrix}$.

(i) structure (VII) of the first type ($\Delta>0$):

$$\Phi=\begin{pmatrix}iE_n & 0\\ 0 & -iE_n\end{pmatrix},\ \Psi=\begin{pmatrix}-ihE_n & \sqrt{h^2-1}\,E_n\\ \sqrt{h^2-1}E_n & ihE_n\end{pmatrix},\ \mathrm{T}=\begin{pmatrix}-igE_n & Q\\ Q & igQ_n\end{pmatrix},$$

where

$$Q=\frac{(f+gh)}{\sqrt{h^2-1}}\,E_n+i\,\frac{\sqrt{\Delta}}{\sqrt{h^2-1}}\,E_n.$$

(ii) structure (VII) of the second type ($\Delta<0$):

$$\Phi=\begin{pmatrix}iE_n & 0\\ 0 & -iE_n\end{pmatrix},\ \Psi=\begin{pmatrix}-ihE_n & \sqrt{h^2-1}\,E_n\\ \sqrt{h^2-1}\,E_n & ihE_n\end{pmatrix},\ \mathrm{T}=\begin{pmatrix}-igE_n & Q\\ Q & igE_n\end{pmatrix},$$

where

$$Q=\frac{(f+gh)}{\sqrt{h^2-1}}\,E_n+i\,\frac{\sqrt{-\Delta}}{\sqrt{h^2-1}}\,J_{n/2},\ J_{n/2}=\begin{pmatrix}0 & -E_{n/2}\\ E_{n/2} & 0\end{pmatrix},$$

(n: even).

(iii) strucutre (VII) of the third type ($\Delta=0$):

$$\Phi=\begin{pmatrix}iE_n & 0\\ 0 & -iE_n\end{pmatrix},\ \Psi=\begin{pmatrix}-ihE_n & \sqrt{h^2-1}\,E_n\\ \sqrt{h^2-1}\,E_n & ihE_n\end{pmatrix},\ \mathrm{T}=\begin{pmatrix}-igE_n & Q\\ \overline{Q} & igE_n\end{pmatrix},$$

where

$$Q=\frac{(f+gh)}{\sqrt{h^2-1}}E_n+iK_n,\ K_n=\left(\begin{array}{c|c}0 & 0\\ \hline E_p & 0\end{array}\right)\ \left(p\leqq\frac{n}{2}\right)$$

With respect to a real frame these respective normal forms are:

(i) $(\Delta>0)$: $\Phi=\begin{pmatrix}0 & -E_n\\ E_n & 0\end{pmatrix},\ \Psi=\begin{pmatrix}\sqrt{h^2-1}\,E_n & hE_n\\ -hE_n & -\sqrt{h^2-1}\,E_n\end{pmatrix},$

$$T=\left(\begin{array}{c|c}\dfrac{f+gh}{\sqrt{h^2-1}}E_n & \left(g+\dfrac{\sqrt{\Delta}}{\sqrt{h^2-1}}\right)E_n\\ \hline \left(-g+\dfrac{\sqrt{\Delta}}{\sqrt{h^2-1}}\right)E_n & -\dfrac{f+gh}{\sqrt{h^2-1}}E_n\end{array}\right),$$

(ii) $(\Delta<0)$: $\Phi=\begin{pmatrix}0 & -E_n\\ E_n & 0\end{pmatrix},\ \Psi=\begin{pmatrix}\sqrt{h^2-1}\,E_n & hE_n\\ -hE_n & -\sqrt{h^2-1}\,E_n\end{pmatrix},$

$$T=\left(\begin{array}{c|c}\dfrac{f+gh}{\sqrt{h^2-1}}E_n & gE_n+\dfrac{\sqrt{-\Delta}}{\sqrt{h^2-1}}J_{n/2}\\ \hline -gE_n+\dfrac{\sqrt{-\Delta}}{\sqrt{h^2-1}}J_{n/2} & -\dfrac{f+gh}{\sqrt{h^2-1}}E_n\end{array}\right)$$

$$\left(J_{n/2}=\left(\begin{array}{c|c}0 & -E_{n/2}\\ \hline E_{n/2} & 0\end{array}\right),\ n\text{: even}\right),$$

(iii) $(\Delta=0)$: $\Phi=\begin{pmatrix}0 & -E_n\\ E_n & 0\end{pmatrix},\ \Psi=\begin{pmatrix}\sqrt{h^2-1}\,E_n & hE_n\\ -hE_n & -\sqrt{h^2-1}\,E_n\end{pmatrix},$

$$T=\left(\begin{array}{c|c}\dfrac{f+gh}{\sqrt{h^2-1}}E_n & gE_n+K_n\\ \hline -gE_n+K_n & -\dfrac{f+gh}{\sqrt{h^2-1}}E_n\end{array}\right)$$

$$\left(K_n=\left(\begin{array}{c|c}0 & 0\\ \hline E_p & 0\end{array}\right),\ p\leqq\frac{n}{2}\right).$$

If a differentiable manifold M admits the structure (VII), we can find in M almost product structures. For, since $|2f|>2$, $|2g|>2$, $|2h|>2$, M

admits the complex-product structures (II) (see the proof of Lemma 1.4), which contain almost product structures.

5. "Distributions" of almost complex structures. Let M be a differentiable manifold and suppose that the dimension of the set of almost complex structures linearly independent in real constants is equal to k. Then there are just k almost complex structures $\Phi_1,\dots,\Phi_k$ linearly independent in real constant coefficients and any other almost complex structure (if it exists) is given by a linear combination of them with respect to real constant coefficients. Let $x_1,\dots,x_k$ be k real numbers such that $x_1\Phi_1+\dots+x_k\Phi_k$ gives again an almost complex structure in M. The set of all such real numbers $(x_1,\dots,x_k)$ gives a subset in a k-dimensional affine space R^k. We call the subset in R^k the *"distribution"* of almost complex structures of $(\Phi_1,\dots,\Phi_k)$ and call $\Phi_1,\dots,\Phi_k$ the bases of the distribution.

We can consider the other bases $\Phi_1' = \sum_i a_1^i\Phi_i,\dots,\Phi_k' = \sum_i a_k^i\Phi_i$, where a's are real constants and there are some relations among them to make Φ''s be almost complex structures and $\det|a_i^j| \neq 0$. Hence the "distribution" has a meaning in affine geometry with some affine transformations leaving invariant the origin.

In this section, we will consider the "distribution" of almost complex structures of (Φ,Ψ,T) of the structures (I), (IV), (V), (VI) and (VII) (or briefly "distributions" of almost complex structures of the structures (I)$\sim$(VII)) in the above sense. They are subsets in a 3-dimensional affine space R^3.

1°. The "distribution" of almost complex structures of the structure (I). Let (Φ,Ψ,T) give the structure (I) (almost quaternion structure), then the necessary and sufficient condition that a linear combination $x\Phi+y\Psi+z\mathrm{T}$ $(x,y,z:$ real constants) is again an almost complex structure is that $x^2+y^2+z^2=1$. Hence the distribution of almost complex structures of the structure (I) a sphere with respect to a natural metric in R^3 (bases Φ,Ψ,T). With respect is to general bases $\Phi'=a_1\Phi+a_2\Psi+a_3\mathrm{T}$, $\Psi'=b_1\Phi+b_2\Psi+b_3\mathrm{T}$, $\mathrm{T}'=c_1\Phi+c_2\Psi+c_3\mathrm{T}$, $(a_1,\dots,c_3$ are real constants; the sum of squares of a's, b's, or c's is equal to 1; Φ', Ψ', T' are independent) the distribution is an *ellipsoid.*

2°. The "distribution" of almost complex structures of the structure (IV). If (Φ,Ψ,T) gives the structure (IV), then the necessary and sufficient consition that a linear combination $x\Phi+y\Psi+z\mathrm{T}$ $(x,y,z$: real constants) is again an almost complex structure is that $x-y-z=\pm 1$. Hence the distribution of almost complex structures of the structure (IV) is *two parallel planes* in R^3.

3°. The "distribution" of almost complex structures of the structure (V).

Let (Φ,Ψ,T) give the structure (V). At first we remark that if we put $\Phi\Psi(=\Psi\Phi) = T_1$, $\Psi T(= T\Psi) = \Phi_1$, $T\Phi(= \Phi T) = \Psi_1$, then Φ_1, Ψ_1, T_1 give almost product structures and they satisfy $\Phi_1\Psi_1 = \Psi_1\Phi_1 = - T_1$, $\Psi_1 T_1 = T_1\Psi_1 = - \Phi_1$, $T_1\Phi_1 = \Phi_1 T_1 = - \Psi_1$. These almost product structures are not identical with $\pm E$, because Φ, Ψ and T are linearly independent. Suppose that $x\Phi + y\Psi + zT$ (x, y, z: real constants) gives an almost complex structure. Then, from $(x\Phi + y\Psi + zT)^2 = -E$, we have $2yz\Phi_1 + 2zx\Psi_1 + 2xyT_1 = (x^2 + y^2 + z^2 - 1)E$ or $2zx\Psi_1 + 2xyT_1 = (x^2 + y^2 + z^2 - 1)E - 2yz\Phi_1$. If we consider the squares of both sides and take account of $\Phi_1 \neq \pm E$, we have $4z^2x^2 + 4x^2y^2 = (x^2 + y^2 + z^2 - 1)^2 + 4y^2z^2$ and $2x^2yz = yz(x^2 + y^2 + z^2 - 1)$. The equations obtained by cyclically changing x, y, z in the last two equations are also hold true. If one of the x, y, z is equal to zero, then the other one is also equal to zero, which is the trivial case. If $x \neq 0$, $y \neq 0$, $z \neq 0$, then we have $x = \pm 1, y = \pm 1$, $z = \pm 1$. This is the necessary condition. In some cases, these values of x, y, z make $x\Phi + y\Psi + zT$ be an almost complex structure and in other cases, they does not (for instance, in the normal forms in §3, if $s_2 = 0$ such a case occurs and if $r_1r_2 \neq 0$, $s_1s_2 \neq 0$ such a case does not cocur.) But in any cases, the distribution of almost complex structures of the structure (V) is *isolated*.

4°. The "distribution" of almost complex structures of the structure (VI). Let (Φ, Ψ, T) be the structure (VI) and we remark that $\Phi\Psi + \Psi\Phi = \Phi T + T\Phi$ which is obtained from $\Phi\Psi = T\Phi\,(\Psi\Phi = \Phi T)$.

Now, from the relation $\Psi\Phi\Psi = \alpha\Phi + \alpha\Psi + T\,(\alpha \neq -1)$, we have

$$(\alpha + 1)(\Phi\Psi + \Psi\Phi) + (\Psi T + T\Psi) - 2\alpha E = 0. \tag{5.1}$$

On the other hand we consider in R^3 a curve C such that

$$C: \frac{x(y + z)}{\alpha + 1} = \frac{yz}{1} = \frac{x^2 + y^2 + z^2 - 1}{2\alpha}, \tag{5.2}$$

then we can easily see that C is a plane curve defined by

$$x(y + z) - (\alpha + 1)yz = 0, \quad x - y - z = \pm 1. \tag{5.3}$$

C is composed of two branches which are conics on the plane $x - y - z = 1$ and $x - y - z = - 1$ respectively. Let (x, y, z) be on C and we put

$$\Sigma = x\Phi + y\Psi + zT.$$

Then, we see that Σ gives an almost complex structure for each $(x, y, z) \in C$. For, by virtue of $\Phi\Psi + \Psi\Phi = \Phi T + T\Phi$ and (5.1), (5.2) we have $\Sigma^2 = - E$.

Assume that there exists an almost complex structure Σ' of the form

$x'\Phi+y'\Psi+z'\mathrm{T}$, where (x',y',z') is not on C. Then, from $\Sigma'^2=-E$ we have

(5.4) $x'(y'+z')(\Phi\Psi+\Psi\Phi)+y'z'(\Psi\mathrm{T}+\mathrm{T}\Psi)-(x'^2+y'^2+z'^2-1)E=0,$

where (x',y',z') does not satisfy (5.2). Taking account of (5.1) and (5.4), the case $\dfrac{x'(y'+z')}{\alpha+1.}=\dfrac{y'z'}{1}\neq\dfrac{x'^2+y'^2+z'^2-1}{2\alpha}$ can not occur, hence $\dfrac{x'(y'+z')}{\alpha+1}\neq\dfrac{y'z'}{1}$. And in this case, by virtue of (5.1) and (5.4) we have $\Phi\Psi+\Psi\Phi=\rho E$, where ρ is a real constant (depending on x',y',z'). But since the structure (VI) does not satisty this relation (see the remark at the end of §3), the distribution is the curve C itself.

That is, the distribution of almost complex structures of the structure (VI) is a *curve C whose two branches are conics on two parallel planes* in R^3.

5°. The "distribution" of almost complex structures of the structure (VII) (of the first, the second and the third type). Let (Φ,Ψ,T) be the structure (VII) (of the first, the second or the third type). If $x\Phi+y\Psi+z\mathrm{T}$ (x,y,z: real constants) gives an almost complex structure, we have by virtue of (VII),

$$x^2+y^2+z^2-2fyz-2gzx-2hxy=1\ \ (|f|,|g|,|h|>1).$$

Now, we put $\Delta=\begin{vmatrix} 1 & -h & -g \\ -h & 1 & -f \\ -g & -f & 1 \end{vmatrix}$. Then, according to $\Delta>0$ (the first type (i)) or $\Delta<0$ (the second type (ii)) the distribution is a *hyperboloid of two or one sheets* respectively. If $\Delta=0$ (the third type (iii)), the distribution is a *hyperbolic cylinder.*

The "distributions" of the structures (I), (IV), (V), (VI) and (VII) are different in the sense of affine geometry with affine transformations leaving invariant the origin, we can easily get the following lemma.

LEMMA 5.1. *The five structures* (I), (IV), (V), (VI) *and* (VII) *are linearly independnent in real constant coefficients. That is, any one of them can not be obtained from the other one by a linear combination with respect to real constants.*

6. Case where the dimension of the set of almost complex structures is equal to 3.

LEMMA 6.1. *Let $A=(a_j^i)$, $B=(b_j^i)$ and $C=(c_j^i)$ be three almost complex structures in a differentiable manifold M. Assume that A,B,C satisfy*

(6.1) $$AB+BA=\rho E,\ AC+CA=\rho' E,\ BC=CB,$$

where ρ, ρ' are real constants such that $|\rho|, |\rho'| > 2$. Then, A, B and C are linearly dependent in real constant coefficients.

PROOF. We will prove by normalizations of A, B and C. The dimension of M is even: $M = M_{2n}$. At the tangent space of P, we choose a complex frame such that A takes the form $A = \begin{pmatrix} iE_n & 0 \\ 0 & -iE_n \end{pmatrix}$. And in this case, by virtue of the first two equations of (6.1), we see that B and C are of the form

$$B = \begin{pmatrix} -i\frac{\rho}{2}E_n & B_1 \\ \overline{B_1} & i\frac{\rho}{2}E_n \end{pmatrix}, \quad C = \begin{pmatrix} -i\frac{\rho'}{2}E_n & C_1 \\ \overline{C_1} & i\frac{\rho'}{2}E_n \end{pmatrix}.$$

Since B and C are almost complex structures, we have

$$B_1\overline{B_1} = \left(\frac{\rho^2}{4} - 1\right) E_n, \quad C_1\overline{C_1} = \left(\frac{\rho'^2}{4} - 1\right) E_n, \tag{6.2}$$

where $\frac{\rho^2}{4} - 1 > 0, \frac{\rho'^2}{4} - 1 > 0$, because $|\rho|, |\rho'| > 2$. As in the normalization of the structure (VII), we can choose a unitary matrix X such that $X^{-1}B_1\overline{X} = \rho_1 E_n$, where $\rho_1 = \sqrt{\frac{\rho^2}{4} - 1}$. And under a transformation by the matrix $\mathfrak{X} = \begin{pmatrix} X & 0 \\ 0 & \overline{X} \end{pmatrix}$, the form of A is unchanged and the form of B becomes

$$B = \begin{pmatrix} -i\frac{\rho}{2}E_n & \rho_1 E_n \\ \rho_1 E_n & i\frac{\rho}{2}E_n \end{pmatrix}, \quad \left(\rho_1 = \sqrt{\frac{\rho_2}{4} - 1}\right).$$

The third equation of (6.1) implies that $C_1 = \rho_2 E_n$, where $\rho_2 = \rho_1\rho'/\rho$. Hence, the normal forms of A, B and C are as follows

$$A = \begin{pmatrix} iE_n & 0 \\ 0 & -iE_n \end{pmatrix}, \quad B = \begin{pmatrix} -i\frac{\rho}{2}E_n & \rho_1 E_n \\ \rho_1 E_n & i\frac{\rho}{2}E_n \end{pmatrix}, \quad C = \begin{pmatrix} -i\frac{\rho'}{2}E_n & \rho_2 E_n \\ \rho_2 E_n & i\frac{\rho'}{2}E_n \end{pmatrix},$$

where $\rho_1 = \sqrt{\frac{\rho_2}{4} - 1}$, $\rho_2 = \rho_1\rho'/\rho$. Accordingly there exists a linear relation in real constant c efficients such that

$$\left(\frac{\rho\rho_2}{2} - \frac{\rho'\rho_1}{2}\right) A + \rho_2 B - \rho_1 C = 0, \quad (\rho_1 \neq 0, \ \rho_2 \neq 0).$$

Since P is an arbitrary point of M_{2n}, this equation is a tensor equation all over the M_{2n}. Q. E. D.

LEMMA 6. 2. *Let $A = (a_j^i)$, $B = (b_j^i)$ be two almost complex structures linearly independent in real constant coefficients in a differentiable manifold M. And suppose that A, B satisfy*

$$(6.2) \qquad ABA = \lambda A + \mu B,$$

where λ, μ are real constants. Then, we get $\mu \neq 0$, and if $\lambda \neq 0$ we have $\mu = 1$.

PROOF. If $\mu = 0$, we see that $\lambda = \pm 1$ and in this case $ABA = \pm A$ or $A = \pm B$. But this contradicts to the assumption. Hence $\mu \neq 0$. Suppose that $\lambda \neq 0$. Since ABA is an almost complex structure, we have $\lambda\mu(AB + BA) = (\lambda^2 + \mu^2 - 1)E$. On the other hand, by multiplying A from the left and from the right to (6. 2), we get $-BA = -\lambda E + \mu AB$ and $-AB = -\lambda E + \mu BA$ respectively. Hence we have $(\mu + 1)(AB + BA) = 2\lambda E$. Consequently we see that $\mu = 1$ or $\mu + 1 = \pm \lambda$ by a simple calculation. If $\mu + 1 = \pm \lambda (\neq 0)$, we get $AB + BA = \pm 2E$ from which we have $ABA = \pm 2A + B$. Taking account of (6. 2) we get $\lambda A + \mu B = \pm 2A + B$ and hence $\mu = 1$, because A and B are linearly independent. In any cases, we know that $\mu = 1$. Q. E. D.

LEMMA 6. 3. *Suppose that the dimension of almost complex structures of a differentiable manifold M in the sense of linear combination in real constant coefficients is equal to* 3 (see the definition in the Introduction). *Let $A = (a_j^i)$, $B = (b_j^i)$, $C = (c_j^i)$ be three almost complex structures linearly independent in real constant coefficients satisfying*

$$(6.3) \qquad ABA = \lambda_1 A + \mu_1 B,$$

$$(6.4) \qquad CAC = \lambda_2 C + \mu_2 A,$$

$$(6.5) \qquad BCB = \lambda_3 B + \mu_3 C,$$

where $\lambda_1, \mu_1, \lambda_2, \mu_2, \lambda_3, \mu_3$ are real constants and $\lambda_1^2 + \mu_1^2 \neq 0$, $\lambda_2^2 + \mu_2^2 \neq 0$, $\lambda_3^2 + \mu_3^2 \neq 0$.

Then only one of following cases can occur:

1° $\lambda_1 = \lambda_2 = \lambda_3 = 0$; $\mu_1 = \mu_2 = \mu_3 = 1$,

2° $\lambda_1 = \lambda_2 = \lambda_3 = 0$; $\mu_1 = \mu_2 = \mu_3 = -1$,

3° $0 < |\lambda_1| < 2$, $\lambda_2, \lambda_3 = 0$[6]; $\mu_1 = \mu_2 = \mu_3 = 1$,

4° $0 < |\lambda_1| < 2$, $0 < |\lambda_2| < 2$, $\lambda_3 = 0$[6]; $\mu_1 = \mu_2 = \mu_3 = 1$,

5° $|\lambda_1| = |\lambda_2| = |\lambda_3| = 2$, $\lambda_1\lambda_2\lambda_3 < 0$; $\mu_1 = \mu_2 = \mu_3 = 1$,

6° $0 < |\lambda_1|, |\lambda_2|, |\lambda_3| < 2$; $\mu_1 = \mu_2 = \mu_3 = 1$,

6) Of course the cases obtained by cyclically changing λ_1, λ_2, λ_3 are contained.

7° $|\lambda_1|, |\lambda_2|, |\lambda_3| > 2;\ \mu_1 = \mu_2 = \mu_3 = 1$.

In any case, we can find one of the structures (I), (IV), (V), (VII).

PROOF. At first as in the Lemma 6.2, any one of μ_1, μ_2, μ_3 can not vanish. Because, if for instance $\mu_1 = 0$, then A and B are linearly dependent which contradicts to the assumption.

We classify several cases following to the values of $\lambda_1, \lambda_2, \lambda_3$ and consider their possibilities.

Case (i) $\lambda_1 = \lambda_2 = \lambda_3 = 0$. In this case we see that $\mu_1 = \pm 1$, $\mu_2 \pm 1$, $\mu_3 = \pm 1$. If all of the μ_1, μ_2, μ_3 are equal to 1, we have $ABA = B$, $CAC = A$, $BCB = C$ or $AB + BA = 0$, $AC + CA = 0$, $BC + CB = 0$. And A, B, C give rise the structure (I) (case 1°). The case $\mu_1 = 1, \mu_2 = \mu_3 = -1$ or the case $\mu_1 = \mu_2 = 1$, $\mu_3 = -1$ can not occur. For instance assume that $\mu_1 = 1$, $\mu_2 = \mu_3 = -1$. Then we have $AB + BA = 0$ and $AC = CA$. A and B give rise the structure (I) by putting $\Phi = A, \Psi = B, \mathrm{T} = \Phi\Psi\ (= -\Psi\Phi)$. Any other almost complex structure for instance C is a linear combination of these Φ, Ψ, T, i.e., $C = \alpha\Phi + \beta\Psi + \gamma\mathrm{T}\ (\alpha^2 + \beta^2 + \gamma^2 = 1)$. Since $\Phi C (= AC = CA) = C\Phi$, we must have $\beta\mathrm{T} - \gamma\Psi = 0$ taking account of the relations in (I). But since Ψ and T are linearly independent, we get $\beta = \gamma = 0$ and hence $C = \alpha\Phi = \pm\Phi = \pm A$. This contradicts to the assumption that A and C are linearly independent. It is similar for the case $\mu_1 = \mu_2 = 1$, $\mu_3 = -1$.

If all of μ_1, μ_2, μ_3 are equal to -1, we have $AB = BA$, $AC = CA$, $BC = CB$. And these A, B, C give rise the structure (V) (case 2°).

Case (ii) $\lambda_1 \neq 0$, $\lambda_2 = \lambda_3 = 0$. In this case, $\mu_2 = \pm 1, \mu_3 = \pm 1$.

If $\mu_2 = \mu_3 = 1$, we have $AC + CA = 0$, $BC + CB = 0$. We put $\Phi = A$, $\Psi = C$, $\mathrm{T} = \Phi\Psi\ (= -\Psi\Phi)$, then Φ, Ψ, T give rise the almost quaternion structure (I). The almost complex structure B is given as a linear combination of Φ, Ψ and T: $B = \alpha\Phi + \beta\Psi + \gamma\mathrm{T}$. From $BC + CB = 0$, we get $\beta = 0$, i.e., $B = \alpha\Phi + \gamma\mathrm{T} (\alpha^2 + \gamma^2 = 1, \gamma \neq 0)$ taking account of the quaternic relations among Φ, Ψ and T. Now, consider the relation $ABA = \lambda_1 A + \mu_1 B$. By virtue of Lemma 6.2, we know that $\mu_1 = 1$ since $\lambda_1 \neq 0$. Substituting $A = \Phi, B = \alpha\Phi + \gamma\mathrm{T}\ (\alpha^2 + \gamma^2 = 1,\ \gamma \neq 0)$, we get $\lambda_1 + 2\alpha = 0$ since Φ and T are linearly independent. Conversely, we can verify that the present case can occur if $\mu_1 = 1$, $|\lambda_1| = |-2\alpha| < 2$, where Φ, Ψ, T give give the structure (I) (case 3°).

If $\mu_2 = 1, \mu_3 = -1$, we have $AC + CA = 0$, $BC = CB$. As in the above case, $\Phi = A$, $\Psi = C$, $\mathrm{T} = \Phi\Psi (= -\Psi\Phi)$ give rise the structure (I) and $B = \alpha\Phi + \beta\Psi + \gamma\mathrm{T}\ (\alpha^2 + \beta^2 + \gamma^2 = 1)$. From the relation $BC = CB$ and from the quaternic relations among Φ, Ψ, T, we have $\alpha = \gamma = 0$ since Φ and T are linearly independent. Hence $B = \pm\Psi = \pm C$, which contradicts to the assumption that B and C are linearly independent. That is, the case $\mu_2 = 1$, $\mu_3 = -1$ (also the case $\mu_2 = -1$, $\mu_3 = 1$) can not occur.

The case $\mu_2 = \mu_3 = -1$ also can not occur, which is proved as follows. If $\mu_2 = \mu_3 = -1$, we have $AC = CA$, $BC = CB$ and since $\lambda_1 \neq 0$ we know that $\mu_1 = 1$ (Lemma 6.2). Hence, we have from (6.3), $AB + BA = \lambda_1 E$. If $|\lambda_1| < 2$, we obtain the structure (I) by putting $\Phi = A$, $\Psi = \rho A + \sigma B$, $T = \Phi\Psi (= -\Psi\Phi)$, where ρ, σ are suitable real constants (see the proof of Lemma 1.4). The almost complex structure C is a linear combination of Φ, Ψ, T: $C = \alpha\Phi + \beta\Psi + \gamma T$. By virtue of $AC = CA$, we get $\beta = \gamma = 0$ as in the preceding cases. Hence $C = \alpha\Phi = \pm\Phi = \pm A$, which contradicts to the assumption. If $|\lambda_1| > 2$, we obtain the structure (II) by putting $\Phi = A$, $\Psi = \rho A + \sigma B$ (ρ, σ: suitable real constants) $T = \Phi\Psi (= -\Psi\Phi)$ (see also the proof of Lemma 1.4), where Ψ and T are almost product structures. We remark the relation $\Phi\Psi = -\Psi\Phi$ and $\Phi C (= AC = CA) = C\Phi$. Since $AC = CA$, $BC = CB$, we have $\Psi C = C\Psi$. If we put $\Psi C = C\Psi = \Psi'$, then Ψ' is an almost complex structure and $\Psi' C = C\Psi'$. We can easily verify that $\Phi\Psi' = \Phi(\Psi C) = -\Psi\Phi C = -\Psi C\Phi = -\Psi'\Phi$ or $\Phi\Psi' + \Psi'\Phi = 0$. Hence $\Phi, \Psi', T' = \Phi\Psi' (= -\Psi'\Phi)$ give rise the structure (I). In this case $C = \alpha\Phi + \beta\Psi' + \gamma T'$ and from the relation $\Psi' C = C\Psi'$ we get $C = \pm\Psi'$. Therefore $\Psi = \pm E$, but this is a contradiction. Lastly, consider the case $|\lambda_1| = 2$. In this case we have $AB + BA = \pm 2E$. We can assume that $AB + BA = 2E$ without any loss of generality. Because, if otherwise we consider $-A$ instead of A, the form of (6.3), (6.4), (6.5) are however unchanged. If we put $\Phi = A$, $\Psi = \pm B$, $T = E - \Phi\Psi - \Phi$, we obtain the structure (IV) and $C = \alpha\Phi + \beta\Psi + \gamma T$. From the relation $AC = CA$ we have $\beta\Phi\Psi + \gamma\Phi T = \beta\Psi\Phi + \gamma T\Phi$. Since $\Phi\Psi + \Psi\Phi = 2E$, $\Phi T + T\Phi = 2E$, we get $\beta(\Phi\Psi - E) + \gamma(\Phi T - E) = 0$. By multiplying Φ from the left and taking account of the linear independence of Φ, Ψ, T, We get $\beta = \gamma = 0$. Hence $C = \alpha\Phi = \pm A$, which is also a contradiction. We remark that in the proofs for the cases $|\lambda_1| \leqq 2$, we have used only one of the relations $AC = CA$, $BC = CB$.

Case (iii) $\lambda_1 \neq 0$, $\lambda_2 \neq 0$, $\lambda_3 = 0$. In this case, $\mu_3 = \pm 1$. Since $\lambda_1 \neq 0$, $\lambda_2 \neq 0$, we have $\mu_1 = 1$, $\mu_2 = 1$ and $AB + BA = \lambda_1 E$, $AC + CA = \lambda_2 E$ (Lemma 6.2). If $\mu_3 = 1$, then $\Phi = B$, $\Psi = C$, $T = \Phi\Psi (= -\Psi\Phi)$ give rise the structure (I). And A is a linear combination of Φ, Ψ, T: $A = \alpha\Phi + \beta\Psi + \gamma T$ ($\alpha^2 + \beta^2 + \gamma^2 = 1$). From the relations $AB + BA = \lambda_1 E$, $AC + CA = \lambda_2 E$, we see that $\lambda_1 + 2\alpha = 0$, $\lambda_2 + 2\alpha = 0$. Hence $|\lambda_1| < 2$, $|\lambda_2| < 2$. Therefore the present case can occur if and only if $0 < |\lambda_1| < 2$, $0 < |\lambda_2| < 2$; $\mu_1 = \mu_2 = \mu_3 = 1$, where Φ, Ψ, T give the structure (I) (case 4°). Consider the case $\mu_3 = -1$, where we see that $BC = CB$. the relations $AB + BA = \lambda_1 E$, $AC + CA = \lambda_2 E$ also hold true. But the case $|\lambda_1| \leqq 2$ or $|\lambda_2| \leqq 2$ can not occur, which is verified as in the case (ii) (see the remark at the end of case (ii)). The case $|\lambda_1| > 2$ or $|\lambda_2| > 2$ also can not occur, for, in this case the structures A, B, C are not linearly independent by virtue of Lemma 6.1.

Case (iv) $\lambda_1 \neq 0$, $\lambda_2 \neq 0$, $\lambda_3 \neq 0$. By virtue of Lemma 6.2, we know that $\mu_1=\mu_2=\mu_3=1$. In this case we have $AB+BA=\lambda_1 E$, $AC+CA=\lambda_2 E$, $BC+CB=\lambda_3 E$. If any one of $|\lambda_1|, |\lambda_2|, |\lambda_3|$ is equal to 2, then the case 5° occurs, where the structure (IV) is admitted. For instance assume that $|\lambda_1|=2$. Without any loss of generality we can take $\lambda_1=2$, if otherwise we consider $-A$ instead of A ($-\lambda_1, -\lambda_2$ instead of λ_1, λ_2). If we put $\Phi=A$, $\Psi=B$, $\mathrm{T}=E-\Phi\Psi-\Phi$, we get the structure (IV). The almost complex structure C is given by a linear combination of Φ, Ψ, T: $C=\alpha\Phi+\beta\Psi+\gamma\mathrm{T}$, where $\alpha-\beta-\gamma=\pm 1$ (see §5). Hence we can easily see that $AC+CA=\pm 2E$, $BC+CB=\mp 2E$. This case can occur if and only if $|\lambda_1|=|\lambda_2|=|\lambda_3|=2$ and $\lambda_1\lambda_2\lambda_3<0$ (case 5°). If any one of $|\lambda_1|, |\lambda_2|, |\lambda_3|$ is <2, then the remaining two are also <2. For instance assume that $|\lambda_1|<2$. Then we can find the structure (I) by putting $\Phi=A$, $\Psi=\rho A+\sigma B$ (ρ, σ: suitable real constants), $\mathrm{T}=\Psi\Phi(=-\Psi\Phi)$. Hereby $C=\alpha\Phi+\beta\Psi+\gamma\mathrm{T}$ ($\alpha^2+\beta^2+\gamma^2=1$). From the relations $AC+CA=\lambda_2 E$, we can easily get $2\alpha+\lambda_2=0$ or $|\lambda_2|=|-2\alpha|<2$. Similarly we have $|\lambda_3|<2$. Hence if $0<|\lambda_1|, |\lambda_2|, |\lambda_3|<2$; $\mu_1=\mu_2=\mu_3=1$, the present case can occur, where the structure (I) is admitted (case 6°).

Lastly if any one of $|\lambda_1|, |\lambda_2|, |\lambda_3|$ is >2, then the remaining two are also >2. For instance assume that $|\lambda_1|>2$. If $|\lambda_2|\leqq 2$, we must have $|\lambda_1|\leqq 2$ by virtue of the above considerations. But this is a contradiction. Hence if $|\lambda_1|, |\lambda_2|, |\lambda_3|>2$; $\mu_1=\mu_2=\mu_3=1$, the present case occurs (case 7°) and we can find the structure (VII). Q. E. D.

THEOREM 6. 1. *Let M be a differentiable manifold. If the dimension of the set of almost complex structures of M in the sense of linear combination in real constant coefficients is just equal to* 3 (see the definition in the Introduction), *then M admits one of the structures* (I), (IV), (V), (VI), (VII).

PROOF. According to the definition, M admits three and just three almost complex structures A, B, C linearly independent with respect to real constant coefficients and any other almost complex structure is given by a linear combination of the three in real constant coefficients. Now, if all of the three almost complex structures ABA, CAC, BCB are linearly dependent with A, B; C, A; B, C respectively, then the proof is complete by virtue of Lemma 6. 3.

We cosider the other case, for instance, ABA is linearly independent with A and B. Put

$$C' = -ABA, \tag{6.6}$$

then A, B and C' are linearly independent and other almost complex structure is expressed by a linear combination of A, B and C' in real constant coefficients. We remark that

$$(6.7) \qquad AB = C'A, \ BA = AC'.$$

Now, since BAB is also an almost complex structure, we can write

$$(6.8) \qquad BAB = \alpha A + \beta B + \gamma C',$$

where α, β and γ are real constants all of which are not zero.

If $\gamma = 0$ in (6.8), then from $BAB = \alpha A + \beta B$ we have $ABA = \alpha B + \beta A$, i.e., $C' = -\alpha B - \beta A$. But this is a contradiction since A, B, C' are linearly independent. Therefore from now on, we consider $\gamma \neq 0$ in (6.8). Without any loss of generality, we can assume that $\gamma > 0$. For, if $\gamma < 0$, we put $A' = -A$. Then the equation (6.6) is unchanged for A', and (6.8) becomes $BA'B = \alpha A + \beta' B - \gamma C'$, where $\beta' = -\beta$. This is no other than the case of $\gamma > 0$ in (6.8).

By multiplying A from the left and B from the right to (6.8), we get $-ABA = -\alpha B - \beta A + \gamma AC'B$. Since $-ABA = C'$ and $AC'B = BAB$ by virtue of (6.7), we have

$$(6.9) \qquad BAB = \frac{\beta}{\gamma} A + \frac{\alpha}{\gamma} B + \frac{1}{\gamma} C'.$$

From (6.8) and (6.9), we have $\alpha = \dfrac{\beta}{\gamma}$, $\beta = \dfrac{\alpha}{\gamma}$, $\gamma^2 = 1$, and hence $\alpha = \beta$, $\gamma = 1$, since A, B, C' are linearly independent and $\gamma > 0$. In this case (6.8) becomes

$$(6.10) \qquad BAB = \alpha A + \alpha B + C'.$$

If $\alpha = -1$ in (6.10), we can easily get $BC' + C'B = -2E$. Hence $\Phi = B$, $\Psi = -C'$ gives the structure (IV) (see the later Remark).

If $\alpha \neq -1$ in (6.10), we can find the structure (VI) taking account of (6.7), (6.10) and the linear independence of A, B, C'. Q.E.D.

REMARK. In the case $\alpha = -1$, if we put $A = \dfrac{1}{2}\Phi - \dfrac{1}{2}\Psi$, then A is an almost complex structure lying on the distribution of the structure (IV). This A satisfies (6.7) and (6.10).

APPENDIX

The ordinary normalization of a complex $(n \times n)$-matrix A is such that

$$A \rightarrow \begin{pmatrix} \lambda_1 & e_1 & & & 0 \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & & & \ddots & e_{n-1} \\ 0 & & & & \lambda_n \end{pmatrix},$$

where $\lambda_1,\cdots,\lambda_n$ are characteristic roots of A and $e_i = 0$ or 1 $(i = 1,\cdots,n-1)$. But the normalization in §2 (p. 7) is such that $A \to Y^{-1}A\overline{Y}$. In this normalization the property $A\overline{A} = 0$ plays an important role. The trasformations considered in this Appendix are always such that $A \to Y^{-1}A\overline{Y}$. As has been already shown, we can transform A such that $A \to X^{-1}A\overline{X} = \begin{pmatrix} 0 & \\ \vdots & * \\ 0 & \end{pmatrix}$.

Next, we can find a matrix Z such that

$$Z^{-1}A\overline{Z} = \left(\begin{array}{c|c} 0 & e_1 \ 0 \cdots 0 \\ \hline 0 & \\ \vdots & A' \\ 0 & \end{array}\right), \quad (e_1 = 0 \text{ or } 1).$$

Such a Z is composed of the matrices of the following types:

$$(1) \quad \begin{array}{c} \\ \\ \\ i \\ \\ \\ j \\ \\ \end{array} \begin{array}{c} \begin{array}{ccccc} & & i & & j \end{array} \\ \begin{pmatrix} 1 & & \vdots & & \vdots & & \\ & \ddots & \vdots & 0 & \vdots & 0 & \\ & & 1 \vdots & & \vdots & & \\ \cdots & \cdots & 0 & \cdots & 1 & \cdots & \cdots \\ & & \vdots & 1 & \vdots & & \\ & 0 & \vdots & \ddots & \vdots & 0 & \\ & & \vdots & 1 & \vdots & & \\ \cdots & \cdots & 1 & \cdots & 0 & \cdots & \cdots \\ & & \vdots & & \vdots & 1 & \\ & 0 & \vdots & 0 & \vdots & \ddots & \\ & & \vdots & & \vdots & & 1 \end{pmatrix} \end{array}, \quad E + \alpha E_{ij}, \quad \begin{pmatrix} 1 & & & & \\ \beta & & 0 & & \\ & 1 & & & \\ & & \ddots & & \\ & 0 & & \ddots & \\ & & & & 1 \end{pmatrix},$$

$(\alpha, \beta:$ complex)

where E is the unit matrix of degree n and $E_{ij} = \delta_{ij}$ (Kronecker's delta). We write the $Z^{-1}A\overline{Z}$ anew by A. If $e_1 = 0$, we can advance to the next step since A is a direct sum. To consider the case $e_1 = 1$, we put

$$A' = \begin{pmatrix} a_{22} & \cdots\cdots & a_{2n} \\ a_{32} & \cdots\cdots & a_{3n} \\ \vdots & & \vdots \\ a_{n2} & \cdots\cdots & a_{nn} \end{pmatrix}.$$

If at least one of $a_{32},\cdots a_{n2}$ is not zero, we can transform A such that

$$A = \left(\begin{array}{c|ccc} 0 & 1 & 0 \cdots\cdots & 0 \\ \hline 0 & a_{22} & & \\ \vdots & \lambda & & \\ \vdots & 0 & * & \\ \vdots & \vdots & & \\ 0 & 0 & & \end{array}\right),$$

which is done by the matrices of the former two types of (1) $(i, j \geqq 3)$. Furhtermore, if we consider a transformation by a matrix $E + \dfrac{1}{\lambda} E_{13}$ we can take A of the form $\left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & * \end{array}\right)$. In this case we can also advance to the next step.

If all of $a_{32}, \cdots, a_{n2}$ are equal to zero, we see that

$$A = \left(\begin{array}{c|ccc} 0 & 1 & 0 \cdots\cdots & 0 \\ \hline 0 & a_{22} & a_{23} \cdots & a_{2n} \\ \vdots & 0 & & \\ \vdots & \vdots & * & \\ 0 & 0 & & \end{array}\right),$$

and since $A\overline{A} = 0$, we have $a_{22} = a_{23} = \ldots = a_{2n} = 0$. That is, A is transformed into the form

$$A = \left(\begin{array}{cc|c} 0 & 1 & \\ 0 & 0 & 0 \\ \hline & & \\ 0 & & * \end{array}\right).$$

In any cases, we can reduce to the normalization of a matrix of degree $n-1$ or $n-2$. And we get the final form of A:

$$A = \begin{pmatrix} 0 & e_1 & & & 0 \\ \vdots & & \ddots & & \\ \vdots & & & \ddots & \\ \vdots & & & & e_{n-1} \\ 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix}.$$

We remark that at least one of the neighboring e_i and $e_{i+1} (i = 1, \cdots, n-2)$

is zero.

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