ON INFINITESIMAL CONFORMAL AND PROJECTIVE TRANSFORMATIONS OF COMPACT K-SPACES

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In previous papers [3], [4][1], we discussed infinitesimal automorphisms of compact O^*-spaces and conformally flat K-spaces. In this paper we shall deal with infinitesimal conformal and projective transformations of compact K-spaces under a certain condition. We have known that a generalized Chern 2-form $\hat{K}$ can be defined in almost-Hermitian spaces. After preparing notations, definitions and identities in §§1, 2, we shall give in §3 a necessary and sufficient condition that the form $\hat{K}$ to be harmonic. A non-flat K-space of constant curvature, if $n \neq 6$, can not admit an infinitesimal automorphism [4]. In connection with this fact we shall know a more precise result that there does not exist a non-flat K-space of constant curvature if $n \neq 6$. In §4 we shall prove that the scalar curvature of a K-space with vanishing $\hat{K}$ is a non-negative constant. In §5 we shall obtain a lemma which is useful in the last section. In the last section §6, we shall prove that in such a compact space an infinitesimal conformal (or projective) transformation is necessarily an isometry if $n \neq 6$.

1. Preliminaries.[2] Let M be an n dimensional almost-Hermitian space, then its almost-complex structure $\varphi_i^{\ h}$ and the positive definite Riemannian metric g_{ji} satisfy by definition

$$\varphi_i^{\ r}\varphi_r^{\ h} = -\delta_i^h, \tag{1.1}$$

$$g_{ri}\varphi_j^{\ r} = -g_{jr}\varphi_i^{\ r}. \tag{1.2}$$

From (1.2) it follows that tensors $\varphi_{ji} = \varphi_j^{\ r}g_{ri}$ and $\varphi^{ji} = \varphi_r^{\ i}g^{rj}$ are skew-symmetric.

Let $\left\{ \begin{matrix} h \\ ji \end{matrix} \right\}$, $R_{kji}^{\ \ \ h}$ and $R_{ji} = R_{rji}^{\ \ \ r}$ be Christoffel symbols, Riemannian curvature tensor and Ricci tensor respectively. We shall denote by ∇_j the operator of covariant derivative with respect to $\left\{ \begin{matrix} h \\ ji \end{matrix} \right\}$.

It is well known that with respect to the canonical connection defined by

1) The number in brackets refers to Bibliography at the end of the paper.
2) As to notations we follow S. Tachibana [2], [3] and [4].

$$\Gamma_{ji}^h = \left\{ {h \atop ji} \right\} - (1/2)\,\varphi_r{}^h \nabla_j \varphi_i{}^r$$

tensors g_{ji} and $\varphi_i{}^h$ are covariantly constant. If we denote by $K_{lji}{}^h$ the curvature tensor of Γ_{ji}^h and put

$$\hat{K}_{ji} = 2\,K_{jir}{}^t \varphi_t{}^r, \qquad \hat{R}_{ji} = 2\,R_{jir}{}^t \varphi_t{}^r,$$

then we have [3]

$$\hat{K}_{ji} = \hat{R}_{ji} - \varphi_s{}^t \nabla_j \varphi_r{}^s \nabla_i \varphi_t{}^r.$$

The differential form $\hat{K} = \hat{K}_{ji}\, dx^j \wedge dx^i$ is closed [1], [3]. This form $\hat{K}$ is a generalization of Chern 2-form of Hermitian spaces.

If we put $R^*_{ji} = (1/2)\,\varphi^{tr} R_{trsi} \varphi_j{}^s$, then we have

$$\hat{R}_{ji} = 4\varphi_j{}^r R^*_{ri}$$

and hence we have

$$\hat{K}_{ji} = 4\varphi_j{}^r R^*_{ri} - \varphi_s{}^t \nabla_j \varphi_r{}^s \nabla_i \varphi_t{}^r. \tag{1.3}$$

2. K-spaces. In the rest of this paper we shall only concern ourselves with K-spaces. A K-space[3)] is by definition an almost-Hermitian space satisfying

$$\nabla_j \varphi_{ih} + \nabla_i \varphi_{jh} = 0. \tag{2.1}$$

It is easily seen that the following equations hold good,

$$\nabla_r \varphi_i{}^r = 0, \tag{2.2}$$

$$\varphi_j{}^r \nabla_r \varphi_i{}^h = - \varphi_r{}^h \nabla_j \varphi_i{}^r. \tag{2.3}$$

From these equations it follows that tensors $\nabla_k \varphi_{ji}$ and $\varphi_k{}^r \nabla_r \varphi_{ji}$ are skew-symmetric. Since we have $T^{kji} R_{kjih} = 0$ for any skew-symmetric tensor T^{kji}, we get

LEMMA 2.1.[4)]

$$\nabla^k \varphi^{ji} R_{kjih} = 0, \qquad \varphi_r{}^k \nabla^r \varphi^{ji} R_{kjih} = 0.$$

For the sake of convenience in following sections we shall prepare well known identities. First by Bianchi's identity we have

$$\nabla^j R_{jitr} = \nabla_r R_{ti} - \nabla_t R_{ri}, \tag{2.4}$$

from which by transvection with g^{it} we get

$$\nabla_r R = 2\nabla^j R_{jr}. \tag{2.5}$$

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3) S. Tachibana [2].

4) $\nabla^r = g^{rl}\nabla_l$.

Next applying Ricci's identity to φ_{ih} we have

(2. 6) $$\nabla_k\nabla_j\varphi_{ih} - \nabla_j\nabla_k\varphi_{ih} = - R_{kji}{}^r\varphi_{rh} - R_{kjh}{}^r\varphi_{ir}.$$

From (2. 6) we can get the following identities [2],

(2. 7) $$\nabla^r\nabla_r\varphi_{ih} = -(1/2)\varphi^{tr}R_{trih} + R_{ir}\varphi_h{}^r,$$

(2. 8) $$R_{jr}\varphi_i{}^r = - R_{ri}\varphi_j{}^r,$$

(2. 9) $$R^*_{ji} = R^*_{ij},$$

(2. 10) $$R^*_{jr}\varphi_i{}^r = - R^*_{ri}\varphi_j{}^r,$$

(2. 11) $$R_{ji} - R^*_{ji} = \nabla_j\varphi_{rs}\nabla_i\varphi^{rs},$$

(2. 12) $$R - R^* = \nabla_j\varphi_{rs}\nabla^j\varphi^{rs} \geqq 0,$$

where $R^* = g^{ji}R^*_{ji}$.

If we transvect (2. 6) with φ^{kj}, then we have

$$2\varphi^{kj}\nabla_k\nabla_j\varphi_{ih} = - \varphi^{kj}(R_{kji}{}^r\varphi_{rh} + R_{kjh}{}^r\varphi_{ir})$$
$$= - 2(R^*_{hi} - R^*_{ih}),$$

from which and (2. 9) it follows

(2. 13) $$\varphi^{kj}\nabla_k\nabla_j\varphi_{ih} = 0.$$

If we transvect (2.6) with $\nabla^k\varphi^{ji}$, then we have

$$2\nabla^k\varphi^{ji}\nabla_k\nabla_j\varphi_{ih} = - \nabla^k\varphi^{ji}(R_{kji}{}^r\varphi_{rh} + R_{kjh}{}^r\varphi_{ir}),$$

from which and Lemma 2.1 we get

(2. 14) $$\nabla^k\varphi^{ji}\nabla_k\nabla_j\varphi_{ih} = 0.$$

3. The form $\hat{K}$. In a K-space we have from (1.3)

$$\hat{K}_{ji} = \varphi_j{}^r(4R^*_{ri} + \nabla_r\varphi_s{}^t\nabla_i\varphi_t{}^s).$$

Substituting (2. 11) into the last equation we get

(3. 1) $$\hat{K}_{ji} = \varphi_j{}^r(5R^*_{ri} - R_{ri}).$$

In the first place we shall ask for the condition the closed form $\hat{K}$ to be harmonic. From (3. 1) it holds

$$\nabla^j\hat{K}_{ji} = \varphi^{jr}(5\nabla_jR^*_{ri} - \nabla_jR_{ri}).$$

On the other hand

$$2\varphi^{jr}\nabla_jR^*_{ri} = \varphi^{jr}\nabla_j(\varphi^{ts}R_{tsui}\varphi_r{}^u)$$
$$= \varphi^{jr}(\nabla_j\varphi^{ts}R_{tsui}\varphi_r{}^u + \varphi^{ts}\nabla_jR_{tsui}\varphi_r{}^u + \varphi^{ts}R_{tsui}\nabla_j\varphi_r{}^u).$$

In this equation the first term of right hand side vanishes by virtue of Lemma 2.1 and so is the last term because of (1.1) and (2.2). On taking account of (2.2), (2.4) and (2.8), as we have

$$2\varphi^{jr}\nabla_j R^*_{ri} = -\varphi^{ts}\nabla^u R_{tsui} = 2\varphi^{st}\nabla_s R_{ti} = -\varphi_i{}^t\nabla_t R,$$

we obtain the equation

$$\nabla^j \hat{K}_{ji} = -2\varphi_i{}^r\nabla_r R. \tag{3.2}$$

Thus we have

THEOREM 3.1[5]. *A necessary and sufficient condition the form $\hat{K}$ of a K-space to be harmonic is that the scalar curvature R is constant.*

We notice that the form $\hat{K}$ may be trivial.

Now let our K-space be a space of constant curvature, then we have

$$R_{ji} = (R/n)g_{ji}, \qquad R^*_{ji} = (R/n(n-1))g_{ji}$$

and

$$\hat{K}_{ji} = -\frac{n-6}{n(n-1)}R\varphi_{ji}.$$

Since $\hat{K}$ is closed, we have $\hat{K} = 0$ and hence $n = 6$ or $R = 0$. Thus we have

THEOREM 3.2. *If $n > 1$ and $n \neq 6$, then there does not exist a non-flat K-space of constant curvature.*

4. K-spaces with vanishing $\hat{K}$. If the form $\hat{K}$ vanishes, then we have of constant scalar curvature by virtue of (3.2). In this case we shall prove more precisely the following

THEOREM 4.1. *The scalar curvature R of a K-space with vanishing $\hat{K}$ is a non-negative constant.*

LEMMA 4.1. *In any K-space, $R - R^*$ is a non-negative constant.*

PROOF. From (2.12) it is sufficient to prove the equation $\nabla^j\varphi^{rs}\nabla_k\nabla_j\varphi_{rs}=0$. On the other hand we have

$$\nabla^j\varphi^{rs}\nabla_k\nabla_j\varphi_{rs} = \nabla^j\varphi^{rs}(\nabla_j\nabla_k\varphi_{rs} - R_{kjr}{}^t\varphi_{ts} - R_{kjs}{}^t\varphi_{rt}) = 0$$

by virtue of (2.14) and Lemma 2.1. q. e. d.

From (3.1) we have obviously

LEMMA 4.2. *In order that the form $\hat{K}$ vanishes it is necessary and suf-*

5) For a Kählerian space, see K. Yano [5] p.235.

ficient that

$$5R^*_{ji} = R_{ji}. \tag{4.1}$$

PROOF OF THEOREM 4.1. From (4.1) we have $5R^* = R$. From Lemma 4.1 we have $R^* = R - a^2$, where a is a constant. Hence $R = (5/4)a^2$. q. e. d.

5. A lemma. Consider a vector field v^h in a K-space and we put $N(v)_h = \varphi_h{}^t\nabla_t\varphi_{rs}\nabla^r v^s$. If we operate ∇^h to the equation then we have

$$\nabla^h N(v)_h = \varphi^{ht}\nabla_h\nabla_t\varphi_{rs}\nabla^r v^s + \varphi_h{}^t\nabla_t\varphi_{rs}\nabla^h\nabla^r v^s.$$

The first term of right hand side vanishes because of (2.13) and we have

$$\begin{aligned} 2\varphi_h{}^t\nabla_t\varphi_{rs}\nabla^h\nabla^r v^s &= -2\varphi_t{}^h\nabla^t\varphi^{rs}\nabla_h\nabla_r v_s \\ &= -\varphi_t{}^h\nabla^t\varphi^{rs}(\nabla_h\nabla_r v_s - \nabla_r\nabla_h v_s) = 0 \end{aligned}$$

by virtue of Ricci's identity and Lemma 2.1. Thus we get

LEMMA. 5.1. *For any vector field v^h in a K-space it holds that* $\nabla^h N(v)_h = 0$.

Next we shall give identities concerning a vector field which is useful in the next section.

For a vector field v^h, as we have

$$\nabla_i\nabla_r\nabla^r v^i = \nabla_r\nabla_i\nabla^r v^i = \nabla^r(\nabla_r\nabla_i v^i + R_{irt}{}^i v^t),$$

we obtain

$$\nabla_i(\nabla^r\nabla_r v^i) = \nabla^r\nabla_r\nabla^i v_i + \nabla^r(R_{ri}v^i). \tag{5.1}$$

Let us denote by $\underset{v}{\pounds}$ the operator of Lie derivative with respect to a vector field v^h. Then the identity

$$\underset{v}{\pounds}\left\{ {h \atop ji} \right\} = \nabla_j\nabla_i v^h + R_{rji}{}^h v^r \tag{5.2}$$

is well known. Transvecting (5.2) with $\varphi_l{}^j\varphi_h{}^i$ we get

$$\varphi_l{}^j\varphi_h{}^i\underset{v}{\pounds}\left\{ {h \atop ji} \right\} = \varphi_l{}^j\nabla_j g + N(v)_l + 2R^*_{lr}v^r, \tag{5.3}$$

where we have put $g = \varphi_h{}^i\nabla_i v^h$.

6. Infinitesimal transformations of compact K-spaces with vanishing $\hat{K}$. Consider a compact K-space with vanishing $\hat{K}$, then we have $5R^*_{ji} = R_{ji}$ by virture of Lemma 4.2. Hence for a vector field v^h we have

$$5\nabla^j(R^*_{ji}v^i) = \nabla^j(R_{ji}v^i). \tag{6.1}$$

Now let v^h be an infinitesimal conformal transformation, then by definition there exists a scalar function ρ satisfying $\underset{v}{\pounds}\, g_{ji} = 2\rho g_{ji}$ and as is well known the equation

$$\underset{v}{\pounds}\left\{ {h \atop ji} \right\} = \rho_j\delta_i^h + \rho_i\delta_j^h - \rho^h g_{ji}, \qquad \rho_i \equiv \nabla_i\rho, \tag{6.2}$$

holds good. From (6. 2) we have

$$\rho_i = (1/n)\nabla_i f, \qquad f \equiv \nabla_i v^i,$$

and

$$\nabla^r\nabla_r v^h + R_r{}^h v^r = (2-n)\rho^h. \tag{6.3}$$

On the other hand transvecting (6. 2) with $\varphi_l{}^j\varphi_h{}^i$ we get

$$\varphi_l{}^j\nabla_j g + N(v)_l + 2R^*_{lr}v^r = -2\rho_l$$

by virtue of (5. 3). If we operate ∇^l to the last equation and take account of Lemma 5. 1, then we obtain

$$\nabla^l(R^*_{lr}v^r) = -\nabla^l\rho_l.$$

Hence by virtue of (6. 1) we have $\nabla^l(R_{lr}v^r) = -5\nabla^l\rho_l$.

Next operating ∇_h to (6. 3) and taking account of (5. 1) and the last equation we have $\nabla^i\nabla_i f - 10\nabla^i\rho_i = (2-n)\nabla^i\rho_i$, from which we get

$$\frac{2(n-6)}{n}\nabla^i\nabla_i f = 0.$$

If we assume $n \neq 6$, then we have $\nabla^i\nabla_i f = 0$. Since the space in consideration is compact, we obtain $f \equiv \nabla_i v^i = 0$. Thus

THEOREM 6. 1. *In an $n\,(\neq 6)$ dimensional compact K-space with vanishing $\hat{K}$, an infinitesiaml conformal transformation is necessarily an isometry.*

In the same way we can deal with an infinitesimal projective transformation. Let v^h be such a transformation, then by definition there exists a vector field ρ_i such that

$$\underset{v}{\pounds}\left\{ {h \atop ji} \right\} = \rho_j\delta_i^h + \rho_i\delta_j^h.$$

From this equation we have

$$\nabla^r\nabla_r v^h + R_r{}^h v^r = 2\rho^h,$$

$$\rho_i = (1/(n+1))\nabla_i f, \qquad f \equiv \nabla_r v^r.$$

If $\hat{K} \equiv 0$, then we can get

$$\frac{n-6}{n+1}\nabla^r\nabla_r f = 0.$$

Thus we have

THEOREM 6.2. *In an* $n\,(\neq 6)$ *dimensional compact K-space with vanishing* $\hat{K}$, *an infinitesimal projective transformation is necessarily an isometry.*

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