

ON CHARACTERISTIC CLASSES DEFINED BY CONNECTIONS

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1. Introduction In his lecture notes [2], Chern proves that the image of the Weil homomorphism of a differentiable principal fibre bundle is the characteristic algebra of the bundle. His proof, based on the theory of invariant integrals of E. Cartan, is rather computational. We shall give here an alternative proof using a theorem of Leray.

To state our result more explicitly, we have to fix our notations. By $(P, P/G, G)$ we denote a (differentiable) principal fibre bundle with total space P , base space P/G and structure group G . For every closed subgroup H of G , the principal bundle $(P, P/H, H)$ is defined in a natural way.

By $I^k(G)$ we denote the space of $\text{ad} \cdot G$ -invariant homogeneous polynomial functions of degree k defined on the Lie algebra \mathfrak{g} of G . In the same manner as every quadratic form can be identified with a symmetric bilinear form, every element of $I^k(G)$ can be identified with a symmetric multilinear map $\mathfrak{g} \times \dots \times \mathfrak{g} \rightarrow R$ of degree k invariant by $\text{ad} \cdot G$ in the following sense:

$$f(\text{ad} \cdot s(X_1), \dots, \text{ad} \cdot s(X_k)) = f(X_1, \dots, X_k), \quad f \in I^k(G), \quad X_i \in \mathfrak{g}, \quad s \in G.$$

Set

$$I(G) = \sum_{k=0}^{\infty} I^k(G).$$

Then $I(G)$ is an algebra.

Let ω be a connection form in $(P, P/G, G)$ and Ω its curvature form. For every $f \in I^k(G)$, there exists a (unique) closed differential form \bar{f} of degree $2k$ defined on P/G such that

$$\pi^*(\bar{f}) = f(\Omega, \dots, \Omega),$$

where π is the projection $P \rightarrow P/G$. Denote by $w(f)$ the cohomology class defined by \bar{f} . The algebra homomorphism $w: I(G) \rightarrow H^*(P/G; R)$ is independent of the connection chosen (see [2]) and is called the Weil homomorphism.

Our main result is the following

THEOREM A. *Let G be a compact Lie group (not necessarily connected), T its maximal torus, $N = N_G(T)$ the normalizer of T in G . Together with a*

principal fibre bundle $(P, P/G, G)$, consider principal bundles $(P, P/N, N)$ and $(P, P/T, T)$. Then

(1) The diagram

$$\begin{array}{ccccc} I(G) & \longrightarrow & I(N) & \longrightarrow & I(T) \\ \downarrow & & \downarrow & & \downarrow \\ H^*(P/G; R) & \longrightarrow & H^*(P/N; R) & \longrightarrow & H^*(P/T; R) \end{array}$$

is commutative, where the vertical maps are Weil homomorphisms, the upper horizontal arrows denote restriction maps and the lower horizontal arrows are homomorphisms induced from the natural projections $P/T \rightarrow P/N \rightarrow P/G$.

(2) Both $I(G) \rightarrow I(N)$ and $H^*(P/G; R) \rightarrow H^*(P/N; R)$ are isomorphisms (onto).

(3) Both $I(N) \rightarrow I(T)$ and $H^*(P/N; R) \rightarrow H^*(P/T; R)$ are monomorphisms and, in both cases, the images consist of exactly those elements which are invariant by N/T .

As a consequence of Theorem A, we shall obtain

THEOREM B. *Let $(P, P/G, G)$ be a principal fibre bundle. If G is compact, then the image of the Weil homomorphism $w: I(G) \rightarrow H^*(P/G; R)$ is the characteristic algebra of the bundle.*

The isomorphism $H^*(P/G; R) \rightarrow H^*(P/N; R)$ is due to Leray ([4] p.103, see also [1] p.194). The monomorphism $H^*(P/N; R) \rightarrow H^*(P/T; R)$ and the statement concerning its image are trivial consequences of the fact that the Weyl group N/T of G is finite. For the proof of Theorem B, it is sufficient to know that $H^*(P/G; R) \rightarrow H^*(P/N; R)$ is a monomorphism when P is an n -universal bundle for G where n is the dimension of P/G . This can be proved by spectral sequences using the fact that $H^i(G/T; R) = 0$ for i odd (see [1] p.194).

2. Proof of Theorem A. We shall first prove that the restriction map $I(G) \rightarrow I(T)$ is a monomorphism and its image consists of elements invariant by N . Note that, since T is abelian, $I(T)$ is the algebra of all polynomial functions defined on the Lie algebra \mathfrak{t} of T . Let $f \in I^k(G)$. It is evident that the restriction of f to \mathfrak{t} is invariant by N . Assume that

$$f(X, \dots, X) = 0 \text{ for all } X \in \mathfrak{t}.$$

Since any element of \mathfrak{g} can be mapped into \mathfrak{t} by $\text{ad} \cdot s$ for a suitable $s \in G$,

$$f(X, \dots, X) = 0 \text{ for all } X \in \mathfrak{g}.$$

Let f' be any element of $I(N)$, i.e., any element of $I(T)$ invariant by N . We can define an element f of $I(G)$ by the following equation:

$$f(X, \dots, X) = f'(\text{ad} \cdot s(X), \dots, \text{ad} \cdot s(X)) \quad X \in \mathfrak{g},$$

where s is an element of G such that $\text{ad}\cdot s(X)$ is in \mathfrak{t} . Since f' is invariant by N , f is well defined by the above equation. This completes the proof of our assertion.

To complete the proof of Theorem A, it remains to show the commutativity of the diagram. We shall reduce the proof to the commutativity of several simple diagrams.

PROPOSITION 1. *Let $(P, P/G, G)$ and $(E, E/G, G)$ be two principal fibre bundles with same structure group G . Suppose there exists a bundle map $P \rightarrow E$. Then the diagram*

$$\begin{array}{ccc} & I(G) & \\ & \swarrow \quad \searrow & \\ H^*(E/G; R) & \rightarrow & H^*(P/G; R) \end{array}$$

is commutative, where the horizontal arrow is induced by the bundle map and the other two arrows are Weil homomorphisms.

PROOF. Choose an arbitrary connection on E and consider the induced connection on P by the bundle map $P \rightarrow E$. Construct Weil homomorphisms using these two connections. Then the commutativity of the diagram holds already at the cochain level. QED.

PROPOSITION 2. *Let $(Q, Q/H, H)$ be a subbundle of $(P, P/G, G)$ so that $Q/H = P/G$. Set $M = Q/H = P/G$. Then the diagram*

$$\begin{array}{ccc} I(G) & \longrightarrow & I(H) \\ & \swarrow \quad \searrow & \\ & H^*(M; R) & \end{array}$$

is commutative, where the horizontal arrow is the restriction map and the other two arrows are Weil homomorphisms.

PROOF. From an arbitrarily chosen connection in Q , construct the Weil homomorphism $I(H) \rightarrow H^*(M; R)$. Considering the connection of Q as a connection of P , construct the Weil homomorphism $I(G) \rightarrow H^*(M; R)$. Then the commutativity holds already at the cochain level. Q. E. D.

PROPOSITION 3. *Let H be a closed subgroup of G . Consider bundles $(P, P/G, G)$ and $(P, P/H, H)$. Then the diagram*

$$\begin{array}{ccc} I(G) & \longrightarrow & I(H) \\ \downarrow & & \downarrow \\ H^*(P/G; R) & \longrightarrow & H^*(P/H; R) \end{array}$$

is commutative, where the lower horizontal arrow is induced from the natural projection $P/H \rightarrow P/G$.

PROOF. Let $(E, E/G, G)$ be the bundle induced from $(P, P/G, G)$ by the map $P/H \rightarrow P/G$ so that $E/G = P/H$. Then $(P, P/H, H)$ is a subbundle of $(E, E/G, G)$ in a natural way; the imbedding $P \rightarrow E$ is defined by $P \rightarrow P \times P/H \rightarrow E$, where $P \rightarrow P \times P/H$ is given by $u \rightarrow (u, uH)$ for $u \in P$ and $P \times P/H \rightarrow E$ is given by the very definition of E . Consider the diagram

$$\begin{array}{ccc} I(G) & \longrightarrow & I(H) \\ \downarrow & & \downarrow \\ H^*(P/G; R) & \longrightarrow & H^*(P/H; R) \end{array}$$

where the diagonal arrow is the Weil homomorphism of $(E, E/G, G)$. The other maps in the diagram are self explanatory. The lower left triangle is commutative by Proposition 1 and the upper right triangle is commutative by Proposition 2. QED.

The commutativity of the diagram in Theorem A is an immediate consequence of Proposition 3.

3. Proof of Theorem B. Let n be the dimension of P/G . Let $(E, E/G, G)$ be an n -universal bundle for G so that the first n homotopy groups of E are trivial. There exists a bundle map $P \rightarrow E$. The image of the induced homomorphism $H^*(E/G; R) \rightarrow H^*(P/G; R)$ is by definition the characteristic algebra of $(P, P/G, G)$. Because of Proposition 1, it suffices to prove that every element of $H^i(E/G; R)$ for $i \leq n$ is in the image of the Weil homomorphism $I(G) \rightarrow H^*(E/G; R)$. Let T be a maximal torus of G . Then $(E, E/T, T)$ is an n -universal bundle for T . By Theorem A, the problem is now reduced to proving that every element of $H^i(E/T; R)$ for $i \leq n$ is in the image of the Weil homomorphism $I(T) \rightarrow H^*(E/T; R)$. For further reduction of the problem, let S^{2N+1} be an odd dimensional sphere whose dimension $2N + 1$ is greater than the dimension of E/T . Consider the principal fibre bundle $(S^{2N+1}, P_N(C), U(1))$ where the base space $P_N(C)$ is the complex projective space of complex dimension n and $U(1)$ is the circle group. Let m be the dimension of T . Let E' be the m -fold Cartesian product of S^{2N+1} . Then $T = U(1) \times \dots \times U(1)$ and $E'/T = P_N(C) \times \dots \times P_N(C)$. Thus we obtain a principal bundle $(E', E'/T, T)$. This is a $2N$ -universal bundle for T as the first $2N$ homotopy groups of E' vanish. We may assume $2N \geq n$. Then there exists a bundle map $E \rightarrow E'$. Since both E and E' are n -universal for T , we have a natural isomorphism $H^i(E'/T; R) \approx H^i(E/T; R)$ for $i \leq n$ (see, for instance, [1] p.166). By Proposition 1, it is sufficient to prove that every element of $H^i(E'/T; R)$ for $i \leq n$ is in the image of the Weil homomorphism $I(T) \rightarrow H^*(E'/T; R)$. Since $(E', E'/T, T)$ is the m -fold Cartesian product of $(S^{2N+1}, P_N(C), U(1))$, it suffices to prove that every element of $H^i(P_N(C))$ for $i \leq n$ is in the image of the Weil homomorphism $I(U(1)) \rightarrow H^*(P_N(C); R)$. The cohomology algebra $H^*(P_N(C); R)$ is gener-

rated by a single element c_1 of $H^2(P_N(C); R)$, the first Chern class of $P_N(C)$. The existence of an element of $I(U(1))$ which is mapped into c_1 by the Weil homomorphism is well known (see, for instance [3]). QED.

REVIEW OF THE PROOF. Step 1: the reduction from an n -universal bundle $(E, E/G, G)$ for G to an n -universal bundle $(E, E/T, T)$ for a maximal torus T . Step 2: the reduction from $(E, E/T, T)$ to the bundle $(E', E'/T, T)$ which is the m -fold Cartesian product of $(S^{2N+1}, P_N(C), U(1))$. Step 3: the reduction from $(E', E'/T, T)$ to $(S^{2N+1}, P_N(C), U(1))$.

BIBLIOGRAPHY

- [1] A. BOREL, Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupe de Lie, *Annals of Math.*, 57(1953)115-207.
- [2] S. S. CHERN, Lecture notes, Princeton (1951).
- [3] S. KOBAYASHI, Principal fibre bundles with 1-dimensional toroidal group, *Tôhoku Math. J.*, 8(1956)29-45.
- [4] J. LERAY, Sur l'homologie des groupes de Lie des espaces homogènes et des espaces fibrés principaux, *Colloque de topologie, Bruxelles* (1950)101-116.

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