FIBERINGS OF ENVELOPING SPACES II

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1. Introduction In a recent paper [5][2)], the author investigated the initial projection from the m-th enveloping space $E_m(X)$ of a topological space X into X and proved that, under some local conditions on X, the initial projection is a fibering.

The purpose of the present paper is to study the terminal projection from $E_m(X)$. It turns out that the terminal projection is a fibering without assuming the local conditions on X.

2. Residual Spaces and Enveloping Spaces. In the present section, we will recall for the sake of convenience the notion of the residual spaces and that of the enveloping spaces introduced in [3].

Let us first consider a given imbedding

$$i : X \to W$$

of a topological space X into a topological space W. By identifying such point x of X with its image $i(x)$ in W, we may consider X as a subspace of W and the map i as the inclusion map. Thus, we obtain a pair (W, X) of a space W and a subspace X of W.

By a *path* in W, we mean a continuous map $\sigma : I \to W$ of the closed unit interval $I = [0, 1]$ into W. The set of all paths in W together with the usual compact-open topology forms a topological space $P(W)$, called the *space of paths* in W. Then, the *enveloping space* $E(W, X)$ *of* X *in* W is defined to be the subspace of the space $P(W)$ which consists of all paths $\sigma : I \to W$ such that $\sigma(t) \in X$ if and only if $t = 0$. In other words, a path $\sigma \in P(W)$ is in $E(W, X)$ if and only if it issues from X and never comes back to X again. If X consists of a single point w_0 of W, then $E(W, X)$ becomes the tangent space $T(W, w_0)$ of the space W at the point w_0 as defined in [2].

Now, let X be an arbitrary topological space. Consider the *m-th (topological) power*

$$W = X^m$$

of the space X; in other words, W denotes the topological product $X \times \cdots \times X$

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2) Numbers in square brackets refer to the bibliography at the end of the paper.

of m copies of X. There is a natural imbedding

$$d: X \to W$$

defined by $d(x) = (x, \cdots, x) \in W$ for every $x \in X$. This imbedding d is called the *diagonal imbedding* of X into its m-th power X^m. By means of d, the space X can be identified with a subspace $d(X)$ of X^m, namely, the *diagonal* of $W = X^m$. Thus, we obtain a pair (W, X) of a space W and a subspace X of W.

For each integer $m > 1$, let us denote by

$$R_m(X) = W \backslash X = X^m \backslash X$$

the set-theoretic complement of X in its *m-th* power $W = X^m$ and denote by

$$E_m(X) = E(W, X) = E(X^m, X)$$

the enveloping space of the pair (W, X). The topological spaces $R_m(X)$ and $E_m(X)$ are called the *m-th residual space* and the *m-th enveloping space* of the space X respectively.

For completeness, we also define the *first residual space* $R_1(X)$ and the *first enveloping space* $E_1(X)$ of a space X by taking

$$R_1(X) = X,$$
$$E_1(X) = T(X),$$

where $T(X)$ denotes the *total tangent* space of X defined in [2] as the subspace of the space $P(X)$ of paths in X which consists of the totality of paths σ in X such that $\sigma(t) = \sigma(0)$ if and only if $t=0$. This space $T(X)$ was introduced by John Nash [6] in his proof of the topological invariance of the Stiefel-Whitney classes for a differentiable manifold.

According to [3, p. 343], the isotopy types of the residual spaces $R_m(X)$ and the enveloping spaces $E_m(X), m = 1, 2, \cdots$, are isotopy invariants of the given space X. Therefore, every isotopy invariant of $R_m(X)$ or $E_m(X)$ is an isotopy invariant of X. In particular, every homotopy invariant of $R_m(X)$ or $E_m(X)$ is an isotopy invariant of X. This introduces a huge family of algebraic isotopy invariants of topological spaces.

3. The Initial Projection. Let X be a given topological space and consider the m-th enveloping space $E_m(X)$, $m \geqq 1$, of X. In a recent paper [5], we studied the natural projection

$$p_m : E_m(X) \to X$$

defined as follows. Let $\sigma \in E_m(X)$. By the definition of $E_m(X)$, σ is a path in X^m with $\sigma(0) \in X$. Then, p_m is defined by taking

$$p_m(\sigma) = \sigma(0)$$

for every σ in $E_m(X)$.

Since $\sigma(0)$ is the initial point of the path σ, the natural projection p_m will be called the *initial projection* from $E_m(X)$ into X hereafter.

The following two lemmas are established in [5].

LEMMA 3.1. *If X is pathwise accessible, then the initial projection p_m maps $E_m(X)$ onto X for every $m \geqq 1$.*

LEMMA 3.2. *If X is locally homogeneous, then the initial projection p_m maps $E_m(X)$ onto an open subspace X_m of X and $E_m(X)$ is a sliced fiber space over X_m relative to p_m in the sense of* [1, p. 97].

Hence, $E_m(X)$ is a sliced fiber space over X relative to p_m if X is both pathwise accessible and locally homogeneous.

Let x_0 be an arbitrary point of X. The *fiber* over x_0, i.e. the inverse image $p_m^{-1}(x_0)$, is the subspace

$$F_m(X, x_0)$$

of $E_m(X)$ which consists of all $\sigma \in E_m(X)$ such that $\sigma(0) = x_0$. This subspace $F_m(X, x_0)$ of $E_m(X)$ is called the *fiber in $E_m(X)$ over the point x_0.*

In case $m > 1$, the fiber $F_m(X, x_0)$ is the subspace of $E_m(X)$ which consists of all paths $\sigma : I \to X^m$ such that

$$\sigma(0) = x_0,$$
$$\sigma(t) \in R_m(X), \ (0 < t \leqq 1).$$

For the case $m = 1$, we have $E_1(X) = T(X)$ and hence the fiber $F_1(X, x_0)$ becomes the tangent space $T(X, x_0)$ of the space X at the point x_0 as defined in [2].

If the space X is completely regular at the point x_0, then the homotopy type of $F_m(X, x_0)$ is a local invariant of the space X at the point x_0. Precisely, the following theorem is proved in [5, § 7].

THEOREM 3.3. *If X is completely regular at the point x_0 and if U is an arbitrary open neighborhood of x_0 in X, then the inclusion map*

$$i_m : F_m(U, x_0) \subset F_m(X, x_0)$$

is a homotopy equivalence.

If x_0 is a conic point of X and U a conic neighborhood of x_0 in X as defined in [5, § 8]. Then, the m-th residual space $R_m(\overline{U})$ of the closure $\overline{U}$ of U in X can be considered as a subspace of $F_m(X, x_0)$ by a natural imbedding

$$j_m : R_m(\overline{U}) \to F_m(X, x_0)$$

also defined in [5, § 8]. The following theorem is proved in [5, § 8].

THEOREM 3. 4. *If U is a conic neighborhood of a point x_0 in X, then the imbedding*

$$j_m : R_m(\bar{U}) \to F_m(X, x_0)$$

is a homotopy equivalence.

As an immediate consequence of (3. 4), we have the following corollary.

COROLLARY 3. 5. *If a topological space X is locally Euclidean and n-dimensional at a point $x_0 \in X$, then we have for $m > 1$:*

$$H_q(F_m(X, x_0)) \approx H_q(S^{(m-1)n-1}),$$

$$\pi_q(F_m(X, x_0)) \approx \pi_q(S^{(m-1)n-1}),$$

where S^r, $r = (m - 1)n - 1$, denotes the unit r-sphere in the $(r + 1)$-dimsenional Euclidean space R^{r+1}.

4. The Terminal Projection. The main objective of the present paper is the study of another natural projection of the m-th enveloping space $E_m(X)$ of a given topological space X, namely, the *terminal projection*

$$q_m : E_m(X) \to R_m(X)$$

which is defined for every $m \geqq 1$ by setting

$$q_m(\sigma) = \sigma(1) \in R_m(X), \ \sigma \in E_m(X).$$

Since $E_m(X)$ is a subspace of the space $P(X^m)$ of paths in X^m with the usual compact-open topology, q_m is obviously continuous.

The restriction of q_m on the subspace $F_m(X, x_0)$ of $E_m(X)$ defined in the preceding section for an arbitrarily given point x_0 of X will be denoted by

$$r_m : F_m(X, x_0) \to R_m(X)$$

and called the *terminal projection from $F_m(X, x_0)$ into $R_m(X)$.*

The topological space X is said to be *pathwise accessible at the point x_0* if there exists a path $\xi : I \to X$ such that $\xi(0) = x_0$ and $\xi(t) \neq x_0$ for every $t > 0$. If this is the case, the space $F_m(X, x_0)$ is non-empty because it contains the path $\sigma : I \to X^m$ defined by

$$\sigma(t) = (\xi(t), x_0, \cdots, x_0), \ (t \in I).$$

The point x_0 of a topological space X is said to be a *pathwise cut-point* of the space X if the residual space $X \backslash \{x_0\}$ obtained by deleting the point x_0 from X fails to be pathwise connected. The pathwise cut-point x_0 of X is said to be *regular* if the space $X \backslash \{x_0\}$ has exactly two path-components; otherwise, it is said to be *singular.*

PROPOSITION 4. 1. *If the topological space X is pathwise accessible at the point x_0 which is not a pathwise cut-point of X, then the terminal projection*

$$r_1 : T(X, x_0) \to X$$

maps the space $T(X, x_0)$ onto the residual space $X \setminus \{x_0\}$.

PROOF. Since X is pathwise accessible at the point x_0, it follows that the tangent space

$$F_1(X, x_0) = T(X, x_0)$$

at x_0 is non-empty and hence there is a path $\xi : I \to X$ such that $\xi(0) = x_0$ and $\xi(t) \neq x_0$ for every $t > 0$ in I. Let $x_1 = \xi(1)$. Then $x_1 \in X \setminus \{x_0\}$.

Consider an arbitrary point x of the residual space $X \setminus \{x_0\}$. Since x_0 is not a pathwise cut-point of X, the residual space $X \setminus \{x_0\}$ is pathwise connected. Hence, there exists a path

$$\alpha : I \to X \setminus \{x_0\}$$

such that $\alpha(0) = x_1$ and $\alpha(1) = x$. Define a path $\sigma : I \to X$ by setting

$$\sigma(t) = \begin{cases} \xi(2t), & \left(0 \leqq t \leqq \frac{1}{2}\right), \\ \alpha(2t-1), & \left(\frac{1}{2} \leqq t \leqq 1\right). \end{cases}$$

Then, obviously we have $\sigma \in T(X, x_0)$ and $r_1(\sigma) = \alpha(1) = x$.

Since it is clear that

$$r_1(\tau) = \tau(1) \neq x_0,$$

it follows that r_1 maps $T(X, x_0)$ onto $X \setminus \{x_0\}$. This completes the proof of (4. 1).

The following corollary is a direct consequence of (4. 1).

COROLLARY 4. 2. *If the topological space X is pathwise accessible at two distinct points x_0 and x_1 which are not pathwise cut-points of X, then the terminal projection*

$$q_1 : T(X) \to X$$

maps the tangent space $T(X)$ of X onto X.

PROPOSITION 4. 3. *If $m > 1$, then the terminal projection*

$$r_m : F_m(X, x_0) \to R_m(X)$$

maps $F_m(X, x_0)$ onto $R_m(X)$ provided that the following two conditions are *satisfied*:

(i) *The space X is pathwise accessible at the point x_0.*

(ii) *The m-th residual space $R_m(X)$ of the space X is pathwise connected.*

PROOF. Since X is pathwise accessible at the point x_0, it follows that the space $F_m(X, x_0)$ is non-empty. Let ξ be an arbitrary point of $F_m(X, x_0)$. Then, ξ is a path in X^m such that $\xi(0) = x_0$ and $\xi(t) \in R_m(X)$ for every $t > 0$ in I. Let $u = \xi(1) \in R_m(X)$.

Let v be an arbitrary point of $R_m(X)$. Since $R_m(X)$ is pathwise connected, there exists an arc α in $R_m(X)$ such that $\alpha(0) = u$ and $\alpha(1) = v$. Define a path $\sigma : I \to X^m$ by setting

$$\sigma(t) = \begin{cases} \xi(2t), & \left(0 \leqq t \leqq \frac{1}{2}\right), \\ \alpha(2t-1), & \left(\frac{1}{2} \leqq t \leqq 1\right). \end{cases}$$

Then, obviously we have $\sigma \in F_m(X, x_0)$ and

$$r_m(\sigma) = \sigma(1) = v.$$

Hence, r_m maps $F_m(X, x_0)$ onto $R_m(X)$. This completes the proof of (4. 3).

The following corollary is an immediate consequence of (4. 3).

COROLLARY 4. 4. *If $m > 1$, then the terminal projection*

$$q_m : E_m(X) \to R_m(X)$$

maps $E_m(X)$ onto $R_m(X)$ provided that the following two conditions are satisfied:

(i) *The space X is pathwise accessible at least one point of X.*

(ii) *The m-th residual space $R_m(X)$ of the space X is pathwise connected.*

5. Pathwise Connectedness of $R_m(X)$. In (4. 3) and (4. 4) of the preceding section, we have used the pathwise connectedness of the m-th residual space $R_m(X)$, $m > 1$, for the terminal projection to be subjective. In the present section, we will give a few sufficient conditions for $R_m(X)$ to be pathwise connected.

For convenience, we will introduce, for any two points u and v in $R_m(X)$, the notation $u \sim v$ to stand for the fact that u and v can be joined by a path in $R_m(X)$.

First of all, let us consider the case $m = 2$.

THEOREM 5. 1. *If X is a pathwise connected Fréchet space ($= T_1$-space) which has no regular pathwise cut-point, then the second residual space $R_2(X)$ of X is pathwise connected.*

PROOF. Let (a, b) and (c, d) be any two points of $R_2(X)$. Then $a \neq b$ and $c \neq d$. We are going to prove

$$(a, b) \sim (c, d).$$

For this purpose, let us first consider the special case where $a = c$. Since X has no regular pathwise cut-point, it follows that either the point a is not a pathwise cut-point or the point a is a singular pathwise cut-point of X.

If the point a is not a pathwise cut-point of X, then the space $X \setminus \{a\}$ is pathwise connected and hence there exists a path

$$\alpha : I \to X \setminus \{a\}$$

such that $\alpha(0) = b$ and $\alpha(1) = d$. Define a path

$$\xi : I \to R_2(X)$$

by taking $\xi(t) = (a, \alpha(t))$ for every t in I. Obviously, ξ is a path which joins (a, b) to (a, d).

On the other hand, if a is a singular pathwise cut-point of X, then $X \setminus \{a\}$ has at least three path-components.

If the points b and d are in one and the same path-component of $X \setminus \{a\}$, then there exists a path $\alpha : I \to X \setminus \{a\}$ with $\alpha(0) = b$ and $\alpha(1) = d$. As above, this gives rise to a path $\xi : I \to R_2(X)$ joining (a, b) to (a, d).

If the points b and d are in different path-components U and V of the space $X \setminus \{a\}$, then there exists at least one path-component W which contains neither b nor d. Pick a point e in W. Since X is pathwise connected, there exists a path

$$\beta : I \to X$$

with $\beta(0) = a$ and $\beta(1) = b$. Since X is a Fréchet space, the point a forms a closed subset $\{a\}$ of X. Hence, it follows from the continuity of β that the inverse image $\beta^{-1}(a)$ is a closed subset of the unit interval I. Let

$$\lambda = \sup \{\beta^{-1}(a)\}.$$

Then $\beta(\lambda) = a$ and $\beta(t) \in U$ for every $t > \lambda$ in I. Define a path

$$f : I \to X$$

by taking $f(t) = \beta[(1 - \lambda)t + \lambda]$ for every t in I. Then we have

$$f(0) = a,\ f(1) = b,\ f(I) \subset U \cup \{a\}.$$

Similarly, there exist paths

$$g, h : I \to X$$

satisfying the conditions:

$$g(0) = a, \quad g(1) = d, \quad g(I) \subset V \cup \{a\};$$
$$h(0) = a, \quad h(1) = e, \quad h(I) \subset W \cup \{a\}.$$

Define a path $\eta : I \to R_2(X)$ by taking

$$\eta(t) = \begin{cases} (h(4t), b), & \left(\text{if } 0 \leqq t \leqq \frac{1}{4}\right), \\ (e, f(2-4t)), & \left(\text{if } \frac{1}{4} \leqq t \leqq \frac{1}{2}\right), \\ (e, g(4t-2)), & \left(\text{if } \frac{1}{2} \leqq t \leqq \frac{3}{4}\right), \\ (h(4-4t), d), & \left(\text{if } \frac{3}{4} \leqq t \leqq 1\right). \end{cases}$$

Then η is a path which joins (a, b) to (a, d). Hence, $(a, b) \sim (a, d)$. This proves the special case that $a = c$.

Similarly, one can establish the special case that $b = d$.

Now, let us consider the general case where $a \neq c$ and $b \neq d$.

If $a \neq d$, then it follows from the special cases proved above that

$$(a, b) \sim (a, d) \sim (c, d).$$

This implies $(a, b) \sim (c, d)$ and proves the case $a \neq d$. Similarly, one can prove the case $b \neq c$.

It remains to prove that every point (a, b) of $R_2(X)$ can be joined to (b, a) by a path in $R_2(X)$. In fact, since X is a pathwise connected Fréchet space, it must consist of more than two points: There exists a point e of X with $a \neq e$ and $b \neq e$. Then, it follows from the cases which have already been proved that

$$(a, b) \sim (e, b) \sim (b, a).$$

Hence $(a, b) \sim (b, a)$. This completes the proof of (5. 1).

That the conditions in (5. 1) are essential can be observed from the following counter examples. The second residual space $R_2(X)$ of X consists of exactly two path-components if X is the real line R, the closed unit interval $I = [0, 1]$, or the space $\{a, b\}$ of two points topologized by either the discrete topology or the topology composed of three open sets: $\Box$, $\{a\}$, and $\{a, b\}$. With the exception of the discrete $\{a, b\}$, these spaces are pathwise connected.

Let Y denote the *star of order* 3, that is to say, Y is a simplicial complex which consists of four vertices v_0, v_1, v_2, v_3 together with three 1-simplices $v_0 v_1$, $v_0 v_2$, and $v_0 v_3$. The vertex v_0 is called the *center* of the star Y.

THEOREM 5. 2. *If X is a pathwise connected Hausdorff space into which* *the star Y of order* 3 *can be imbedded, then the second residual space* $R_2(X)$ *is pathwise connected.*

PROOF. Since Y is compact and can be imbedded into the Hausdorff space X, we may consider Y as a closed subspace of X.

As in the proof of (5. 1), it suffices to prove that any two points of the second residual space $R_2(X)$ of the form (a, b) and (a, d) can be joined by a path in $R_2(X)$, i. e.

(i) $$(a, b) \sim (a, d).$$

If a is not a regular pathwise cut-point of X, then one can prove (i) exactly as in the proof of (5. 1).

If a is a regular pathwise cut-point of X, then the space $X \backslash \{a\}$ has exactly two path-components, say U and V. If the points b and d are contained in one and the same path-component of $X \backslash \{a\}$, then (i) follows as in the proof of (5. 1).

It remains to prove the case that the points b and d are in different path-components of $X \backslash \{a\}$, say

$$b \in U, \qquad d \in V.$$

Now, consider the center v_0 of the star Y in X.

Assume that $v_0 = a$. Since $a \neq b$ and $a \neq d$, we may choose the star so small that $b \notin Y$ and $d \notin Y$. Since X is pathwise connected, there exists a path

$$\xi : I \rightarrow X$$

with $\xi(0) = a$ and $\xi(1) = b$. Since Y is closed in X, it follows from the continuity of ξ that the inverse image $\xi^{-1}(Y)$ is a closed set of the unit interval $I = [0, 1]$. Let

$$\lambda = \sup\{\xi^{-1}(Y)\}.$$

Then $0 \leqq \lambda < 1$, $\xi(\lambda) \in Y$ and $\xi(t) \notin Y$ for every $t > \lambda$ in I. Replacing the part $\xi|[0, \lambda]$ of the path ξ by the line-segment joining v_0 to $\xi(\lambda)$ in Y, we obtain a path

$$f : I \rightarrow X$$

such that $f(0) = a$, $f(1) = b$, and $f(I) \cap Y$ is either the point a or a segment of one of the three 1-simplices v_0v_1, v_0v_2, v_0v_3. Similarly, there exists a path

$$g : I \rightarrow Y$$

such that $g(0) = a$, $g(1) = d$, and $g(I) \cap Y$ is either the point a or a segment of one of the three 1-simplices v_0v_1, v_0v_2, v_0v_3. Since $b \in U$ and $d \in V$, it follows that

$$f(I) \cap g(I) = \{a\}.$$

Hence, there is an $i = 1, 2$ or 3 such that the 1-simplex v_0v_i meets $f(I) \cup g(I)$ only at the point $v_0 = a$. Let $e = v_i$ and $h : I \to X$ denote the path defined by the line-segment v_0v_i. Then the path $\eta : I \to R_2(X)$ defined as in the proof of (5.1) joins (a, b) to (a, d). This proves the case $v_0 = a$.

Next, assume that $v_0 \neq a$. Then v_0 is in one of the two path-components, say $v_0 \in V$. We can choose the star Y so small that

$$Y \subset V.$$

As in the proof of (5. 1), one can establish the existence of a path

$$f : I \to X$$

such that $f(0) = a$, $f(1) = v_0$ and $f(I) \subset V \cup \{a\}$. The inverse image $f^{-1}(Y)$ is a closed set of the unit interval I. Let

$$\mu = \inf \{f^{-1}(Y)\}.$$

Then $0 < \mu \leqq 1$, $f(\mu) \in Y$ and $f(t) \notin Y$ for every $t < \mu$ in I. Replacing the part $f|[\mu, 1]$ of the path f by the line-segment joining $f(\mu)$ to v_0 in Y, we obtain a path

$$g : I \to X$$

such that $g(0) = a$, $g(1) = v_0$, $g(I) \subset V \cup \{a\}$, and $g(I) \cap Y$ is either the point v_0 or a segment of a 1-simplex v_0v_i of Y. Let $j \neq i$, then

$$g(I) \cap (v_0v_j) = \{v_0\}.$$

The existence of the path g implies that

(ii) $$(a, b) \sim (v_0, b).$$

By the cases which have already been proved, we have

(iii) $$(v_0, b) \sim (v_0, v_j).$$

Again, the existence of the path g implies that

(iv) $$(v_0, v_j) \sim (a, v_j).$$

Finally, since v_j and d are in the same path-component V of $X \setminus \{a\}$, we have

(v) $$(a, v_j) \sim (a, d).$$

Since the relation $\sim$ is transitive, (ii)—(v) imply (i). This completes the proof of (5. 2).

Next, let us consider the case $m \geqq 3$.

THEOEM 5. 3. *If a topological space X is pathwise connected, then so is the m-th residual space $R_m(X)$ for every $m \geqq 3$.*

PROOF. Consider any two points

$$a = (a_1, \cdots, a_m), \qquad b = (b_1, \cdots, b_m)$$

of $R_m(X)$. We are going to prove $a \sim b$.

Since a is in $R_m(X)$, there exists an integer j with $1 < j \leqq m$ and $a_1 \neq a_j$. If $j > 2$, then the pathwise connectedness of X implies $a \sim a'$ where $a' = (a_1', \cdots, a_m')$ is defined by

$$a_i' = \begin{cases} a_i, & (\text{if } i \neq 2), \\ a_j, & (\text{if } i = 2). \end{cases}$$

Since $a_1' \neq a_2'$, we may assume without loss of generality that

$$a_1 \neq a_2.$$

Similarly, we may assume that

$$b_1 \neq b_2.$$

Since $a_1 \neq a_2$, the pathwise connectedness of X implies

$$a \sim c$$

where $c = (c_1, \cdots, c_m)$ is defined by

$$c_1 = \begin{cases} a_i, & (\text{if } i \neq 3), \\ a_1, & (\text{if } i = 3). \end{cases}$$

Then we have $c_2 \neq c_3$. This and the pathwise connectedness of X imply

$$c \sim d$$

where $d = (d_1, \cdots, d_m)$ is defined by

$$d_i = \begin{cases} b_1, & (\text{if } i = 1), \\ c_i, & (\text{if } i \neq 1). \end{cases}$$

Since d is in $R_m(X)$, there is an integer j with $1 < j \leqq m$ and $d_1 \neq d_j$. If $j = 2$, then $d \sim d'$ where $d' = (d_1', \cdots, d_m')$ is defined by

$$d_i' = \begin{cases} d_i, & (\text{if } i \neq 3), \\ d_j, & (\text{if } i = 3). \end{cases}$$

Hence, we may assume $j > 2$. This and the pathwise connectedness of X imply

$$d \sim e$$

where $e = (e_1, \cdots, e_m)$ is defined by

$$e_i = \begin{cases} d_i, & (\text{if } i \neq 2), \\ b_2, & (\text{if } i = 2). \end{cases}$$

One observes that $e_1 = b_1$ and $e_2 = b_2$. Since $b_1 \neq b_2$, the pathwise connectedness of X implies

$$e \sim b.$$

Thus we obtain $a \sim c \sim d \sim e \sim b$. This completes the proof of (5.3).

The pathwise connectedness of the given space X in the theorems (5.1)—(5.3) is necessary. In fact, we have the following proposition.

PROPOSITION 5.4. *If $R_m(X)$ is non-empty and pathwise connected for some $m \geqq 2$, then the given space X is pathwise connected.*

PROOF. Since $m \geqq 2$ and $R_m(X)$ is non-empty, it follows that X consists of more than one point. Since $R_m(X)$ is a subspace of the topological power X^m, the projection of X^m onto its first coordinate space X defines a continuous map

$$\pi : R_m(X) \to X.$$

Let x be an arbitrary point of X. Since X consists of more than one point, there exists a point y of X with $x \neq y$. Consider the point $u = (u_1, \cdots, u_m)$ of X^m with

$$u_i = \begin{cases} x. & (\text{if } i = 1), \\ y, & (\text{if } i > 1). \end{cases}$$

Since $x \neq y$, u is in $R_m(X)$. Since $\pi(u) = x$, it follows that π maps $R_m(X)$ onto X. As a continuous image of a pathwise connected space $R_m(X)$, the given space X must be pathwise connected. This completes the proof of (5.4).

6. The Fibering Theorem. First, let us recall the *path lifting property* (abbreviated PLP) as follows.

Roughly speaking, a map $p: E \to B$ is said to have the PLP if, for each $e \in E$ and each path $f: I \to B$ with $f(0) = p(e)$, there exists a path $g: I \to E$ such $g(0) = e$, $p \circ g = f$, and that g depends continuously on e and f. For a precise definition, let $P(B)$ and $P(E)$ denote the spaces of paths in B and E respectively. Let Z denote the subspace of the product space $E \times P(B)$ defined by

$$Z = \{(e, f) \in E \times P(B) \mid p(e) = f(0)\}.$$

Define a map $\pi : P(E) \to Z$ by taking

$$\pi(g) = (g(0), p \circ g)$$

for each $g: I \to E$ in $P(E)$. Then $p: E \to B$ is said to have the PLP if there exists a map

$$\lambda: Z \to P(E)$$

such that $\pi \circ \lambda$ is the identity map on Z.

It is well-known that, if a map $p: E \to B$ has the PLP, then E is a fiber space over B relative to p, [1, p. 83].

The main objective of the present section is to prove that the terminal projections

$$q_m: E_m(X) \to R_m(X)$$
$$r_m: F_m(X, x_0) \to R_m(X)$$

have the PLP for every $m \geqq 1$.

For simplicity of notation, let

$$B = R_m(X), \quad E = E_m(X), \quad F = F_m(X, x_0).$$

Further, let

$$Z = \{(e, f) \in E \times P(B) \mid q_m(e) = f(0)\},$$
$$W = \{(e, f) \in F \times P(B) \mid r_m(e) = f(0)\}.$$

Since $F \subset E$ and $r_m = q_m | F$, it follows that $W \subset Z$.

Since $F \subset E$, we have $P(F) \subset P(E)$. Let

$$\pi: (P(E), P(F)) \to (Z, W)$$

denote the map defined by

$$\pi(g) = (g(0), q_m \circ q)$$

for each $g: I \to E$ in $P(E)$.

LEMMA 6.1. *There exists a map*

$$\lambda: (Z, W) \to (P(E), P(F))$$

such that the composition $\pi \circ \lambda$ *is the identity map on the pair* (Z, W).

PROOF. Let M denote the topological product $\left[\frac{1}{2}, 1\right] \times I$ of the closed intervals $\left[\frac{1}{2}, 1\right]$ and $I = [0, 1]$. Consider the subspace

$$L = \left(\frac{1}{2} \times I\right) \cup \left(\left[\frac{1}{2}, 1\right] \times 0\right) \cup (1 \times I)$$

of M. Obviously, L is a retract of M; let

$$\rho : M \to L$$

be a fixed retraction of M onto L.

For the construction of λ, consider an arbitrary point $z = (e, f)$ of Z. Then

$$e : I \to X^m, \quad f : I \to R_m(X)$$

are points of $E = E_m(X)$ and $P(B) = P(R_m(X))$ satisfying $e(1) = f(0)$.

Define a map $\xi : L \to R_m(X)$ by taking

$$\xi(s, t) = \begin{cases} e\left(\frac{1}{2}\right), & \left(s = \frac{1}{2},\ t \in I\right), \\ e(s), & \left(\frac{1}{2} \leqq s \leqq 1,\ t = 0\right), \\ f(t), & (s = 1,\ t \in I). \end{cases}$$

By means of this map ξ and the retraction ρ, we define a map

$$g : I \to E_m(X)$$

by taking $g(t)$, for each $t \in I$, to be the path $g(t) : I \to X^m$ given by

$$[g(t)](s) = \begin{cases} e(s), & \left(\text{if } 0 \leqq s \leqq \frac{1}{2}\right), \\ \xi[\rho(s, t)], & \left(\text{if } \frac{1}{2} \leqq s \leqq 1\right). \end{cases}$$

Since $[g(t)](s) \in X$ if and only if $s = 0$, it verifies that $g(t) \in E_m(X)$. The continuity of g follows as usual, [1, p. 75].

It follows from the compact-open topology the assignment $z \to g$ defines a map

$$\lambda : Z \to P(E).$$

Since $[g(t)](0) = e(0)$ for every $t \in I$, it follows that λ maps W into $P(F)$. It remains to verify that $\pi \circ \lambda$ is the identity map on Z. For this purpose, we have

$$(\pi \circ \lambda)(z) = \pi[\lambda(z)] = \pi(g) = (g(0), q_m \circ g).$$

According to the definition of $g(t)$, we have:

$$[g(0)](s) = e(s) \qquad (s \in I);$$

$$[q_m \circ g](t) = [g(t)](1) = f(t), \qquad (t \in I).$$

Hence $(\pi \circ \lambda)(z) = (e, f) = z$. This completes the proof of (6. 1).

As an immediate consequence of (6. 1), we have the following theorem.

THEOREM 6.2. *The terminal projections*

$$q_m : E_m(X) \to R_m(X)$$

$$r_m : F_m(X, x_0) \to R_m(X)$$

have the PLP for every $m \geqq 1$, *and hence these are fiberings.*

7. The Fibers. Let $u = (u_1, \cdots, u_m)$ be an arbitrarily given point in $R_m(X)$. Consider the inverse images

$$J_m(X, u) = q_m^{-1}(u) \subset E_m(X),$$

$$K_m(X, x_0, u) = r_m^{-1}(u) \subset F_m(X, x_0).$$

These subspaces will be called the *fibers* over the point u in $E_m(X)$ and $F_m(X, x_0)$ respectively.

Let $v = (v_1, \cdots, v_m)$ be another point in $R_m(X)$ and

$$f : I \to R_m(X)$$

be a path in $R_m(X)$ with $f(0) = u$ and $f(1) = v$. Then, the path f induces a map

$$f_* : J_m(X, u) \to J_m(X, v)$$

defined as follows. Let $\sigma \in J_m(X, u)$. Then $\sigma : I \to X^m$ is a path such that $\sigma(0) \in X$, $\sigma(1) = u$, and $\sigma(t) \in R_m(X)$ for every $t > 0$ in I. The image $f_*(\sigma)$ is defined to be the path

$$f_*(\sigma) : I \to X^m$$

given by

$$[f_*(\sigma)](t) = \begin{cases} \sigma(2t), & \left(\text{if } 0 \leqq t \leqq \frac{1}{2}\right), \\ f(2t-1), & \left(\text{if } \frac{1}{2} \leqq t \leqq 1\right). \end{cases}$$

Obviously, f_* is an imbedding and sends $K_m(X, x_0, u)$ into $K_m(X, x_0, v)$.

THEOREM 7.1. *The imbedding*

$$f_* : (J_m(X, u),\ K_m(X, x_0, u)) \to (J_m(X, v),\ K_m(X, x_0, v))$$

is an isotopy equivalence, [4].

PROOF. Let $g : I \to R_m(X)$ denote the reverse of f, that is to say, g is the path defined by $g(t) = f(1-t)$ for every $t \in I$. Then, g induces an imbedding

$$g_* : (J_m(X. v),\ K_m(X, x_0, v)) \to (J_m(X, u),\ K_m(X, x_0, u)).$$

Consider the composed imbedding

$$g_*\circ f_* : (J_m(X, u),\ K_m(X, x_0, u)) \to (J_m(X, u),\ K_m(X, x_0, u)).$$

For each $\sigma \in J_m(X, u)$, the path $\tau = (g_*\circ f_*)(\sigma)$ is defined by

$$\tau(t) = \begin{cases} \sigma(4t), & \left(\text{if } 0 \leqq t \leqq \frac{1}{4}\right), \\ f(4t-1), & \left(\text{if } \frac{1}{4} \leqq t \leqq \frac{1}{2}\right), \\ g(2t-1) = f(2-2t), & \left(\text{if } \frac{1}{2} \leqq t \leqq 1\right). \end{cases}$$

Define an isotopy h_s, $0 \leqq s \leqq 1$, of the pair $(J_m(X, u),\ K_m(X, x_0, u))$ into itself by taking

$$[h_s(\sigma)](t) = \begin{cases} \sigma(4t), & \left(\text{if } 0 \leqq t \leqq \frac{1}{4}\right), \\ f(4st-s), & \left(\text{if } \frac{1}{4} \leqq t \leqq \frac{1}{2}\right), \\ f(2s-2st), & \left(\text{if } \frac{1}{2} \leqq t \leqq 1\right). \end{cases}$$

Then $h_1(\sigma) = \tau = (g_*\circ f_*)(\sigma)$. The path $\theta = h_0(\sigma)$ is defined by

$$\theta(t) = \begin{cases} \sigma(4t), & \left(\text{if } 0 \leqq t \leqq \frac{1}{4}\right), \\ u, & \left(\text{if } \frac{1}{4} \leqq t \leqq 1\right). \end{cases}$$

Next, define an isotopy k_s, $0 \leqq s \leqq 1$, of the pair $(J_m(X, u),\ K_m(X, x_0, u))$ into itself by taking

$$[k_s(\sigma)](t) = \begin{cases} \sigma\left(\frac{4t}{4-3s}\right), & \left(\text{if } 0 \leqq t \leqq \frac{4-3s}{4}\right), \\ u, & \left(\text{if } \frac{4-3s}{4} \leqq t \leqq 1\right). \end{cases}$$

Then $k_0(\sigma) = \sigma$ and $k_1(\sigma) = \theta = h_0(\sigma)$. Hence, $g_*\circ f_*$ is isotopic to the identity imbedding k_0 on the pair $(J_m(X, u),\ K_m(X, x_0, u))$.

Similarly, one can prove that the composed imbedding $f_*\circ g_*$ is isotopic to the identity imbedding on the pair $(J_m(X, v),\ K_m(X, x_0, v))$. This completes the proof of (7. 1).

The following corollary is a direct consequense of (7. 1).

COROLLARY 7.2. *If $R_m(X)$ is pathwise connected, then the isotopy types (and hence the homotopy types) of the spaces*

$$J_m(X, u), \quad K_m(X, x_0, u)$$

are independent of the choice of the point u from $R_m(X)$.

As an example of the consequences which can be deduced from the fiberings q_m and r_m, let us consider

$$r_m : F_m(X, x_0) \rightarrow R_m(X)$$

with assumptions that $F_m(X, x_0)$ is non-empty and $R_m(X)$ is pathwise connected. Pick a point u from $R_m(X)$ and let

$$F = F_m(X, x_0),\ B = R_m(X), K = K_m(X, x_0, u).$$

Pick a point $w \in K \subset F$. Since F is a fiber space over B with K as the fiber over the point u, we have the exact homotopy sequence

$$\cdots \rightarrow \pi_q(F, w) \rightarrow \pi_q(B, u) \rightarrow \pi_{q-1}(K, w) \rightarrow \pi_{q-1}(F, w) \rightarrow \cdots.$$

In particular, if X is locally Euclidean and n-dimensional at the point x_0, then we have

$$\pi_q(F, w) \approx \pi_q(S^{(m-1)n-1}),$$

where $S^{(m-1)n-1}$ denotes the unit sphere in the $(m-1)n$-dimensional Euclidean space, according to [5, (8.4)]. Hence, we have the following theorem.

THEOREM 7.3. *If X is locally Euclidean and n-dimensional at the point x_0, then*

$$\pi_q(B, u) \approx \pi_{q-1}(K, w)$$

for every $q < (m-1)n - 1$.

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