ON DERIVED GROUPS OF NORMAL SIMPLE ALGEBRA

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Introduction. Let A be a normal simple algebra with an identity element, denoted by 1, over a field k. According to Wedderburn's structure theorem, A is isomorphic with a total matrix algebra of certain degree n over a normal division algebra D over k. J. Dieudonné has extended the theory of determinants to non-commutative fields, including the ordinary one, and determined the structure of the general linear group $GL_n(D)$ of all regular elements in A ([1], [2]).

For example, for any $i \neq j$ and any λ of D, we denote by $B_{ij}(\lambda)$ the matrix obtained from the unit matrix by replacing the element $a_{ij} = 0$ of the unit matrix by λ. The matrices $B_{ij}(\lambda)$ (for all $i \neq j$ and all λ in D) generate a subgroup $SL_n(D)$, called the unimodular group. Then $SL_n(D)$ is the kernel of the determinant map and also the derived group of $GL_n(D)$. The centralizer of $SL_n(D)$ is the center of $GL_n(D)$, isomorphic with the multiplicative group of k, for $n \geqq 2$.[1] Moreover, when $n \geqq 2$, but the case, where $n = 2$ and D is $GF(2)$, if a subgroup G of $GL_n(D)$ which is not contained in the center of $GL_n(D)$, then G contains $SL_n(D)$.

From these facts, we have easily

THEOREM 1. *Let D be a normal division algebra over a field k. Suppose that either $n \geqq 3$ or that $n = 2$ but that D contains at least four elements. Then, the second derived group G'' of $GL_n(D)$ is the first derived group G' of $GL_n(D)$.*

When $n = 1$, if D is a commutative field, the above fact trivial. We shall consider the case where D is a division algebra of characteristic zero, and we have

THEOREM 2. *Let D be a normal division algebra over a field k, of characteristic zero, and G be the group of non-zero elements of D. Then, the first derived group G' of G is the minimal algebraic group including the second derived group G'' of G.*

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1) Also, this holds good for $n=1$. In fact, if D is a commutative field, then $SL_1(D)=1$, $GL_1(D)=D$, and the center of D is D itself. Therefore our assertion is clear. If D is non-commutative, then D is generated by the derived group of all non-zero elements of D. ([3]). And, it remains valid too.

In Theorem 3, we shall see the structures of the algebras associated to G and G', the groups denoted in Theorem 2.

Proofs of Theorems.

PROOF OF THEOREM 1. Clearly, G'' is an invariant subgroup of G', that is of $SL_n(D)$. An element of G'',

$$B_{ij}(1)B_{ji}(1)B_{ij}(1)^{-1}B_{ji}(1)^{-1}$$

is not a diagonal matrix, and so, G'' is not contained in the center of $GL_n(D)$. This means that G'' contains $SL_n(D)$, that is G', and then, G'' is nothing but G', as we asserted.

PROOF OF THEOREM 2. Firstly, we shall show that G may be regarded as an algebraic group. Namely, D has a square degree over k, say n^2, therefore, there exists a basis $(u_1 = 1, u_2, \ldots\ldots, u_{n^2})$ such that

$$u_i u_j = \sum_k c_{ijk} u_k, \tag{1}$$

where c_{ijk} in k. By this basis, every element a of D is uniquely expressible in the form

$$a = a_1 u_1 + \ldots\ldots + a_{n^2} u_{n^2}, \tag{2}$$

with a_i in k. We form

$$a u_j = \sum_i u_i a_{ij}, \tag{3}$$

with a_{ij} in k, and define

$$A_a = (a_{ij}), \tag{4}$$

an n^2-rowed square matrix. Then, we have the so-called first regular representation. Since D has an identity element, this representation is faithful, and we shall identify D and its image each other.

Being assumed that $u_1 = 1$, it holds that

$$a_{ij} = \sum_k c_{kji} a_k, \tag{5}$$

and especially,

$$a_{i1} = a_i, \tag{6}$$

therefore, there is a set $\mathfrak{S}$ of n^4 linear equations of n^4 variables a_{ij}:

$$\mathfrak{S} : a_{ij} - \sum_k c_{kji} a_{k1} = 0. \tag{7}$$

Thus, by the restriction ρ of the regular representation to G we regard G

as an algebraic subgroup, defined by $\mathfrak{S}$, of $GL(V)$ of dimension n^2 over k. If we exchange a_i for indeterminates ξ_i, then the general quantity A_ξ of D is a generic point of G. Consequently G is an irreducible algebraic group of dimension n^2 (II, p. 110, Théorème 14).

Now we shall prove that the Lie algebra of G is D with the bracket product in place of the ordinary one. Let $\mathfrak{a}$ be the associated ideal of G, and $\mathfrak{g}$ be the Lie algebra of G. For an endomorphism X of V,

$$X \in \mathfrak{g} \Leftrightarrow \forall(P \in \mathfrak{a})(dP)(I, X) = 0,$$

where I is the identity element of G, (II, p. 128, Proposition 1), and

$$\Leftrightarrow \forall(P \in \mathfrak{S})(dP)(I, X) = 0,$$

(because G is determined by linear equations),

$$\Leftrightarrow \forall(P \in \mathfrak{S})P(X) = 0, \qquad \text{(II, p. 35, Formula)},$$

$$\Leftrightarrow X \in D.$$

Therefore, the intersection of $\mathfrak{g}$ and $\mathfrak{E}$ (the vector space of all endomorphisms of V) is D, and we see that $\mathfrak{g}$ contains D. On the other hand, D contains $\mathfrak{g}$. In fact, the enveloping algebra E of G, considered as a subgroup of $GL(V)$, contains $\mathfrak{g}$ (II, p. 135, Proposition 6). Now that E is nothing but D itself, D will contain $\mathfrak{g}$, then $\mathfrak{g}$ is D, as was to be proved.

Next, we shall see that G' is an irreducible algebraic group, having the derived algebra $\mathfrak{g}'$ of $\mathfrak{g}$ as its Lie algebra. Namely, if we consider G as a subgroup of $GL(V)$, the representation $\rho: G \to GL(V)$ may be regarded as the identity map G into $GL(V)$, and is a semi-simple representation. Then, the corresponding representation $d\rho$ of $\mathfrak{g}$ into the Lie algebra $\mathfrak{gl}(V)$ of $GL(V)$ is also faithful and semi-simple (III, p. 28, Corollaire 4 de Théorème 1). This means that $\mathfrak{g}$ is reductive. (III, p. 76, Proposition 3). Consequently, $\mathfrak{g}$ is a direct sum of its center $\mathfrak{z}$ and the derived algebra $\mathfrak{g}'$, and $\mathfrak{g}'$ is semi-simple (III, p. 75, Proposition 1),

$$\mathfrak{g} = \mathfrak{z} + \mathfrak{g}', \qquad \text{(direct sum)}. \tag{8}$$

$\mathfrak{g}'$ is the Lie algebra of an (irreducible) minimal algebraic group H containing G' (II, p. 177, Théorème 15). If G' is an algebraic group, then H will turn into G', and our assertion holds good.

Let $N(x)$ be a reduced norm of element of x of D, then G' is defined as a group of all elements of D of reduced norm 1 [4], [5]. For a of D, $N(a)$ is a polynomial of a_i, that is of a_{i1} in (2) and (3). Since $N(x)-1$ is a functional polynomial over $\mathfrak{E}$ (the vector space of endomorphisms of V over k), we see that G' is an algebraic group, to which associated algebra is $\mathfrak{g}'$.

As was mentioned before, $\mathfrak{g}'$ is semi-simple, therefore, $\mathfrak{g}' = \mathfrak{g}''$, the derived algebra of $\mathfrak{g}'$. Since $\mathfrak{g}''$ is the algebra of the (irreducible) minimal algebraic group $\overline{H}$ containing G'', and G' and $\overline{H}$ are irreducible, $\mathfrak{g}' = \mathfrak{g}''$ yields our theorem (II, p. 156, Corollaire 1 de Théorème 8).

The Lie algebra $\mathfrak{sl}(V)$ of $SL(V)$ of all automorphisms of determinant 1 is a set of all endomorphisms of trace 0 of V(II, p. 144, Exemple III). Then we have

THEOREM 3. *The notation being as above,*

$$\mathfrak{g}' = \mathfrak{g} \cap \mathfrak{sl}(V).$$

PROOF. Since D is normal over k, the center $\mathfrak{z}$ of $\mathfrak{g}$ is isomorphic with k, and has a dimension 1 over k. On account of (8), we see that dimension of $\mathfrak{g}'$ over k is $n^2 - 1$. Moreover, G' is contained in $SL(V)$, because every element of G' has reduced norm 1, and naturally has determinant 1. Therefore, $\mathfrak{g}'$ is contained in the intersection of $\mathfrak{g}$ and $\mathfrak{sl}(V)$. On the other hand, an element $X = (a_{ij})$ of $\mathfrak{g}$ belongs to $\mathfrak{sl}(V)$, if and only if the trace of X is 0, that is, in the formula (5),

$$\sum_i a_{ii} = \sum_{i,k} c_{kii} a_{k1} = 0.$$

Accordingly, not only a_{ij}(for all $j \neq 1$) but also a_{11} are linear combination of a_{i1}, $(2 \leqq i \leqq n^2)$. Thus, the dimension of the intersection $\mathfrak{g}$ and $\mathfrak{sl}(V)$ is $n^2 - 1$. Consequently, $\mathfrak{g}'$ coincides with the intersection $\mathfrak{g}$ and $\mathfrak{sl}(V)$.

REMARK. If k is algebraically closed, $G' = G''$. Because the derived group of the irreducible algebraic group is algebraic and irreducible in that case (II, p. 122, Corollaire 2 de Proposition 2).

Dr. H. Kuniyoshi pointed me out that it happens that G' is really greater than G''. Namely, let D be a normal division algebra over a p-adic number field k, and $W = k(\omega)$ be a maximal unramified subfield of D, where ω is an appropriate root of unity. Then, we see that for any element a of G', it holds that

$$a \equiv \omega^{\pm(1-S)} \quad (\text{mod. } \mathfrak{p}),$$

by Speiser's theorem (since the norm of $a = 1$), and not necessarily $\equiv 1$. On the other hand, it holds that

$$\omega^{1-S}\omega^{1-T}(\omega^{1-S})^{-1}(\omega^{1-T})^{-1} \equiv 1 \quad (\text{mod. } \mathfrak{p}),$$

where, $\mathfrak{p}$ is the prime ideal of the maximal order of D, S, T are galois transformations of W over k, and i is a rational integer. Therefore, for any element b of G'', we have

$$b \equiv 1 \quad (\text{mod. } \mathfrak{p}).$$

Thus, G' is really greater than G''.

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