A CERTAIN CLASS OF COMMUTATIVE ALGEBRAIC GROUPS

TSUNEO KANNO

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Contents

0. Introduction and preliminaries. Recently the author constructed the Lie theory of algebraic groups in his papers [3] and [4] where a certain class of algebraic groups played an important part; let G be a connected algebraic group defined over a field of characteristic 0; let $\mathfrak{g}$ be the Lie algebra of G; then it is shown that for any D of $\mathfrak{g}$ there exists the minimal algebraic subgroup $G(D)$ of G whose Lie algebra contains D. This algebraic subgroup $G(D)$ corresponds to the closed subgroup generated by the one-parameter subgroup determined by the element D of Lie algebra in the classical theory of Lie groups. In the linear case C. Chevalley has obtained the structural properties of this class of algebraic groups $G(D)$: for any X of $\mathfrak{gl}(n, k)$, the subgroup $G(X)$ of $GL(n, k)$ is commutative; if t is an indeterminate, the point $\exp tX$ is a generic point over k on $G(X)$ where k is a field of characteristic 0. In this paper we shall study the structure of $G(D)$ in the general case.

We shall show that $G(D)$ is commutative in the section 1; where the structural theorems of algebraic groups which were given by C. Chevalley and M. Rosenlicht are useful; a characterization of the Lie algebra of the connected algebraic subgroup generated by some subgroups shall be given. The method of proving the commutativity of $G(D)$ is applicable in showing that, in the characteristic zero case, a connected algebraic group is commutative if its Lie algebra is commutative. The converse of this theorem has been known and its proof has been given by several methods.

In order to obtain the structural properties of $G(D)$ we shall employ the theory of M.Rosenlicht's generalized jacobian varieties [6], [7] and [8]. This seems to be similar to the fact that, in the classical theory of Lie groups, to generalize the exponential mapping of matrix Lie group to general one, we employed the principle of monodromy of simply connected topological spaces. In the section 2 we shall show that for any D of $\mathfrak{g}$ there exists a curve on G passing through the unit element such that a generalized jacobian variety J which is determined by this curve has its rational homomorphic image $G(D)$; further it shall be obtained that if G is linear, $G(D)$ is the image of the maximal linear subgroup of J. This results are due to the universal mapping property of the generalized jacobian variety. The rational homomorphism which is guaranteed by this property gives us a means of obtaining a generic point of $G(D)$ as the exponential mapping does in the linear case.

Further we shall treat explicitly the linear algebraic groups from the point of view of what we consider in the preceding two sections. In the linear case the maximal linear algebraic subgroup L is more important than J itself. The structure of L is stated completely and a generic point of $G(D)$ is obtained explicitly. Firstly, dividing the problem in two cases, i. e. in unipotent case and semi-simple one, we shall consider the relation between L and $G(D)$. Then combining the results of these two cases by the Jordan decomposition, we shall consider the linear case to show explicitly what we referred in the general case.

In this paper we may assume the algebraic closedness of the fields of definition for connected algebraic groups without loss of generality. We shall denote the Lie algebras by the small German letters, for example, the Lie algebras of G and H by $\mathfrak{g}$ and $\mathfrak{h}$ respectively. The universal domain is denoted by Ω.*)

1. Commutativity of algebraic groups. In this section we assume that the characteristic of Ω is 0. Let G be a connected algebraic group defined over a field k. At first we shall prove the following lemma which is useful in this section.

LEMMA 1. *Let H and M be connected algebraic subgroups of G such that H is contained in the normalizator of M; then the set HM is a connected algebraic subgroup of G whose Lie algebra is $\mathfrak{h}+\mathfrak{m}$.*

Since H normalizes M, the set HM is an abstract group; therefore the algebraic closure K of HM is an algebraic subgroup of G: in fact, the mapping $x \to x^{-1}$ being birational biregular, K^{-1} is the algebraic closure of $(HM)^{-1}$; HM being an abstract group, $K^{-1}=K$; for any $x \in HM$, $x \cdot HM$ is contained in HM and therefore $x \cdot K$ is contained in K; thus $HM \cdot K$ is contained in K and finally $K \cdot K$ is contained in K; thus K is a group.

The mapping $x \times y \to x \cdot y$ from $G \times G$ into G induces an everywhere defined rational mapping from $H \times M$ into G. So the set-theoretic image HM of this mapping is épais i. e. HM contains a non-empty open set in the algebraic closure K of HM. Now suppose that HM were distinct from K; then,

*) As for the fundamental properties of algebraic groups, cf:[2] and [8].

HM being épais, the coset of HM in K is contained in a proper closed subset of K; since the cosets are biregularly equivalent, this is a contradiction; thus we have that $K = HM$, that is, HM is an algebraic subgroup of G. On the other hand, since H and M are connectd, HM being the rational image of irreducible variety $H \times M$, HM is connected.

Since H normalizes M, for any x of H, $\mathrm{Ad}(x)$ maps $\mathfrak{m}$ into itself, where Ad means the adjoint representation of G; and therefore the corollary of the proposition 4 of [3] shows that $\mathfrak{h}$ normalized $\mathfrak{m}$; thus the subalgebra of $\mathfrak{g}$ generated by $\mathfrak{h}$ and $\mathfrak{m}$ is $\mathfrak{h} + \mathfrak{m}$. By the main theorem of [3] we have that $\mathfrak{h} + \mathfrak{m}$ is the Lie algebra of the connected algebraic group HM. Thus the proof of Lemma 1 is complete.

It is known that there exists the unique maximal connected linear algebraic subgroup L of G; the structure theorem of Chevalley tells us that L is normal and that the factor group G/L is abelian. On the other hand, let C be the center of G; then it is shown by Rosenlicht that the factor group G/C is linear. The first assertion of the lemma 1 shows that the set LC is a connected normal algebraic subgroup of G. The connected algebraic group G/LC being a rational homomorphic image of the abelian variety G/L, G/LC is abelian and, G/LC being a rational homomorphic image of the linear algebraic group G/C, G/LC is linear. Therefore G/LC reduces to the unit element, that is, $G = LC$. From the lemma 1 we have the following lemma:

LEMMA 2. *$G = LC$ and $\mathfrak{g} = \mathfrak{l} + \mathfrak{c}$, where L is the maximal connected linear algebraic subgroup of G and C is the center of G.*

Now suppose that $\mathfrak{g}$ is commutative; then $\mathfrak{l}$ is commutative. As the characteristic of k is 0, the linear algebraic group L is commutative. Since C is the center of G, the lemma 2 shows that G is commutative. Thus we have

THEOREM 1. *A connected algebraic group defined over a field of characteristic 0 is commutative if its Lie algebra is so.*

The converse is true without the condition of the characteristic: in fact, if G is commutative, the adjoint representation $\mathrm{Ad}(G)$ of G reduces to the unit element and the Lie algebra of $\mathrm{Ad}(G)$ does to the zero matrix; then the proposition 1 of [3] shows that $\mathfrak{g}$ is commutative. This method is orthodoxy; Rosenlicht [5] gave an interesting proof, using the invariant differential forms on G.

Let D be any element of $\mathfrak{g}$. By the lemma 2 we have an expression $D = D_1 + D_2$ where D_1 is in $\mathfrak{l}$ and D_2 is in $\mathfrak{c}$. Let $L(D_1)$ be the smallest algebraic subgroup of G whose Lie algebra contains D_1 and let $C(D_2)$ be that of which Lie algebra contains D_2. Then $L(D_1)$ is contained in L and $C(D_2)$ is in C. From the lemma 1 it follows that the set $L(D_1)\,C(D_2)$ is a connected algebraic subgroup of G whose Lie algebra is $\mathfrak{l}(D_1) + \mathfrak{c}(D_2)$ where $\mathfrak{l}(D_1)$ and $\mathfrak{c}(D_2)$ are the Lie algebras of $L(D_1)$ and $C(D_2)$ respectively.

$D = D_1 + D_2$ being contained in $\mathfrak{l}(D_1) + \mathfrak{c}(D_2)$, the smallest algebraic subgroup $G(D)$ of G whose Lie algebra contains D is contained in the group $L(D_1)C(D_2)$. But L being linear, $L(D_1)$ is commutative. On the other hand $C(D_2)$ is central in the group $L(D_1)\,C\,(D_2)$. Therefore $L(D_1)\,C\,(D_2)$ is commutative and $G(D)$ is commutative.

Thus we have

THEOREM 2. *Let G be a connected algebraic group defined over a field of characteristic* 0; *then for any element D of $\mathfrak{g}$ the smallest algebraic subgroup $G(D)$ of G whose Lie algebra contains D is commutative.*

This theorem is the generalization of the result which is stated in the theorem 10 of the chapter II of [1].

2. Generalized jacobian varieties and $G(D)$. Now that the commutativity of $G(D)$ has been shown in the section 1, we may assume that the ambient algebraic group G of $G(D)$ is commutative so far as the structure of $G(D)$ is studied. Thus in this section we treat a connected commutative algebraic group G.

At first, we have to make some preparation, giving two lemmas on the relation betweent angent spaces and curves. The methods of the proofs of these two lemmas are useful in the next three sections

LEMMA 1. *Let Q be k-rational point on the n-dimensional affine space S; let X be a nonzero tangent vector on S at Q which is rational over k; then there exists an irreducible linear variety C_X of dimension* 1 *on S, defined over k, passing through Q such that Q is simple on C_X and the tangent space on C_X at Q is spanned by X.*

The affine space S has a structure of algebraic group $(G_a)^n$; let $f_1,\dots\dots,f_n$ be a system of coordinate functions on S such that $f_i(x) = x_i$ for $x = (x_1,\dots\dots,x_n)$. Firstly take $Q = 0 = (0, 0,\dots\dots, 0)$ and put $d_i = Xf_i$, then d_i is contained in k. Let C_X be the set of those points $(ad_1,\dots\dots,ad_n) = a(d)$ of S for any $a \in \Omega$. Then C_X has a structure of algebraic subgroup of $(G_a)^n$ and therefore the point 0 is simple on C_X. It is easily seen that if t is a quantity which is transcendental over k, the point $t\cdot(d)$ is generic on C_X over k. So C_X is an irreducible curve defined over k.

Then we have that the tangent space $T(0, C_X)$ on C_X at 0 is spanned by X. In fact; let ι be the natural embedding of C_X into S; let f be a coordinate function of C_X defined by $f(t\cdot(d)) = t$; let T be an element of $T(0, C_X)$ such that $Tf = 1$. Then we have $\iota^* f_i = d_i f$ since for $t\cdot(d) \in C_X$,

$$\iota^* f_i(t\cdot(d)) = f_i(t\cdot(d)) = td_i = d_i\cdot f(t(d)).$$

Thus for an element $d_\iota T$ of $T(0, S)$,

$$d_\iota T\cdot f_i = T\cdot\iota^* f_i = T\cdot d_i f = d_i = X\cdot f_i.$$

So we have $d_\iota T = X$, which means that $T(0, C_X)$ is spanned by X, since 0 is simple on C_X.

Now, for any k-rational point Q on S, the translation T_Q maps the linear variety C_X onto $T_Q(C_X)$, biregularly and the differential dT_Q is an isomorphism of $T(0, S)$ onto $T(Q, S)$. Put $X' = dT_{-Q}\cdot X$, then from what we have seen it follows that there exists $C_{X'}$ passing through 0. It is easily seen that $T_Q(C_{X'})$ is the required linear variety.

Now generalizing the preceding lemma we have

LEMMA 2. *Let P' be a simple point on an irreducible variety U; then for any non-zero tangent vector X on U at P', there exists an irreducible curve C_X on U passing through P' such that P' is simple on C_X and the tangent space on C_X at P' is spanned by X.*

We may assume that U is affine; let k be a field of definition for U such that P' is rational over k; let (x) be a generic point on U over k; let $F_1,\ldots\ldots, F_n$ be a uniformizing set of linear forms for U at P' such that $F_i \in k[X_1,\ldots \ldots, X_N]$; put $y_i = F_i(x)$ and $y_i' = F_i(x')$; then the points $Q = (y)$ and $Q' = (y')$ are in the n-dimensional affine space S. From the definition we have $k(x, y) = k(x)$. It gives a rational mapping f from U into S such that $f(x) = (y)$. Let W be the graph of f in $U \times S$; then W has the projection U on U and the projection S on S; and f is regular at P'.

It is shown that $df\cdot X$ is not zero. In fact, suppose that $df\cdot X = 0$, then

$$X\cdot F_i(f_1,\ldots\ldots, f_N) = df\;\; X\cdot g_i = 0,$$

where $f_1,\ldots\ldots, f_N$ and $g_1,\ldots\ldots, g_n$ be systems of coordinate functions on U and S respectively such that $f_i(X) = x_i$ and $g_j(y) = y_j$. For any polynomial H of the prime ideal associated with U in $k[X_1,\ldots\ldots,X_N]$,

$$\begin{aligned} X(\Delta_{x'} H)(f_1,\ldots\ldots, f_N) &= \textstyle\sum_j (\partial H/\partial X_j)(x')\cdot Xf_j \\ &= X\cdot H(f_1,\ldots\ldots,f_N) = X\cdot 0 = 0. \end{aligned}$$

Since $F_1,\ldots\ldots, F_n$ and those $\Delta_{x'} H$ span the vector space $k\cdot X_1 + \cdots\cdots + k\cdot X_N$, X is the zero vector; this is the contradiction.

From the lemma 1 there exists an irreducible linear variety C_{dfX} on S passing through Q' such that $T(Q', C_{dfX}) = \Omega\cdot\; df\; X$. Then the proposition 10 of VIII_3 of [8] shows that there exists uniquely proper component Y of $W \cap$

$(U \times C_{afX})$ containing $P' \times Q'$ which is multiplicity 1 of $W \cap (U \times C_{afX})$ on $U \times S$. Since Y is proper, the dimension of Y is 1. Y has the projection C_{afX} on S. Let $(x, \bar{y})$ be a generic point on Y over a field k' of definition for Y containing k; f being defined by the linear forms, f is defined at $\bar{x}$ and $f(\bar{x}) = \bar{y}$, and Y has the projection C_X on U which is an irreducible curve passing through P'.

Now we shall show that P' is simple on C_X; we may suppose that the above uniformizing set of linear forms $F_1, \ldots\ldots, F_n$ are $X_1, \ldots\ldots, X_n$. Let $df\, X \cdot g_i = d_i$; then $df\, X$ being non-zero, the vector $(d_1, \ldots\ldots, d_n)$ is not zero vector; suppose that $d_1 \neq 0$. Let M be the linear variety defined by

$$d_1(X_i - x_i') = d_i(X_1 - x_1'),\ i = 2, \ldots\ldots, n.$$

From the definitions and what we have seen it follows that C_X is a component of $U \cap M$ which contains P'.

Since P' is simple on U, the dimension of $T(P', U)$ is n. Let H_i be the linear forms in $k[X_1, \ldots\ldots, X_N]$ such that the linear equations $H_i = a_i$ for some $a_i \in k$, $i = 1, \ldots\ldots, N-n$ define the tangent linear variety to U at P'. Then the set of linear forms $X_1, \ldots\ldots, X_n, H_1, \ldots\ldots, H_{N-n}$ are linearly independent over k. Now put

$$Q_i = d_1 X_i - d_i X_1,\ i = 2, \ldots\ldots, n.$$

Then $Q_2, \ldots\ldots, Q_n, H_1, \ldots\ldots, H_{N-n}$ are linearly independent over k; in fact, suppose that

$$b_2 Q_2 + \cdots\cdots + b_n Q_n + c_1 H_1 + \cdots\cdots + c_{N-n} H_{N-n} = 0$$

for some b_i and $c_i \in k$. Then since $Q_i \in k[X_1, \ldots\ldots, X_n]$, $c_j = 0$. Thus,

$$-(b_2 d_2 + \cdots\cdots + b_n d_n) X_1 + b_2 d_1 X_2 + \cdots\cdots + b_n d_1 X_n = 0.$$

Therefore $b_i = 0$, since $d_1 \neq 0$.

So M is transversal to the tangent linear variety to U at P'. And P' being simple on U, the proposition 21 of V_3 of [9] shows that P' is simple on C_X.

It is easily seen that $T(P', C_X)$ is spanned by X.

*) Let X be a non-singular irreducible projective curve; a divisor $\mathfrak{m} = \sum n_P \cdot P$ of X such that $n_P > 0$ is called a module. Let S be the support of $\mathfrak{m}$. For divisors D and D' of X that are independent of the places of S, it is said that D and D' are $\mathfrak{m}$-equivalent if there exists a rational function g such that

a) $v_P(1-g) \geqq n_P$ for any P of S,

*) As for the theory of the generalized jacobian variety, cf. [6], [7] and [8].

b) $D - D' = (g)$,

where v_P is the normalized valuation at P. Let $C_{\mathfrak{m}}$ be the group of $\mathfrak{m}$-equivalence classes of divisors independent of the places of S and let $C^0_{\mathfrak{m}}$ be the subgroup of $C_{\mathfrak{m}}$ of those classes of divisors of degree 0, independent of the places of S. Let $J_{\mathfrak{m}}$ be the generalized jacobian variety which is a commutative connected algebraic group determined by X and $\mathfrak{m}$. Then there exists the canonical rational mapping φ from X into $J_{\mathfrak{m}}$ such that $\varphi(X)$ generates $J_{\mathfrak{m}}$; for any divisor $D = \sum e_P{\cdot}P$ of X, put $\varphi(D) = \sum e_P{\cdot}\varphi(P)$, then it is known that this extended φ induces an isomorphism from $C^0_{\mathfrak{m}}$ onto $J_{\mathfrak{m}}$.

For any point P of X, let U_P be the multiplicative group of rational functions g such that $v_P(g) = 0$ and for positive integer n, let $U_P^{(n)}$ be the subgroup of U_P of those rational functions g such that $v_P(1-g) \geqq n$. Then $U_P/U_P^{(n)}$ is isomorphic to the product $G_m \times V_{(a)}$; where $V_{(n)}$ is a connected algebraic group isomorphic to the product $(G_a)^{n-1}$ if the characteristic of Ω is 0. Let $R_{\mathfrak{m}}$ be the direct product of $U_P/U_P^{(n_P)}$ for $P \in S$. Let Δ be the subgroup of $R_{\mathfrak{m}}$ of those elements $(a, \ldots\ldots, a)$ for non-zero $a \in \Omega$. Let $H_{\mathfrak{m}} = R_{\mathfrak{m}}/\Delta$. Then it is known that the mapping $g \to (g)$ from the function field of X into the principal divisors induces an isomorphism from $H_{\mathfrak{m}}$ onto the maximal linear connected algebraic subgroup $L_{\mathfrak{m}}$ of $J_{\mathfrak{m}}$.

Let G be a connected commutative algebraic group; let f be rational mapping from X into G; let S be the set of those points P of X such that f is not regular at P. Then there exists a module $\mathfrak{m} = \sum n_P{\cdot}P$ with the support S such that for rational function g, $f((g)) = 0$ if

$$v_P(1-g) \geqq n_P \qquad \text{for } P \in S,$$

where f is considered the as naturally induced mapping from the divisors into G. Such a module $\mathfrak{m}$ is called to be associated with the rational mapping f from X into G. If the characteristic of Ω is 0, a module associated with f is obtained as follows: let $w_1, \ldots\ldots, w_n$ be a base of the vector space of invariant differential forms of degree 1 on G; let $w_1^*, \ldots\ldots, w_n^*$ be the differential forms on X induced by f; then for $P \in S$ there exists a positive integer n_P such that

$$v_P(w_i^*) \geqq - n_P.\ i = 1, \ldots\ldots, n.$$

; then the module $\mathfrak{m} = n_P{\cdot}P$ is associated with f. For any module $\mathfrak{m}$ to be associated with f, it is necessary and sufficient that there exists the local symbol associated with f and $\mathfrak{m}$, i. e. a mapping $(f, g)_P$ from $X \times \Omega(X)^*$ into G satisfying the following conditions:

(i) $(f, gg')_P = (f, g)_P + (f, g')_P$ for $g, g' \in \Omega(X^*)$, $P \in X$,

(ii) $(f, g)_P = 0$ if $P \in S$ and $v_P(1-g) \geqq n_P$,

(iii) $(f, g)_P = v_P(g) f(P)$ if $P \in X - S$,

(iv) $\sum_{P \in X} (f, g)_P = 0$.

There exist many modules associated with f. But the local symbol is uniquely determined by f if it exists.

For a rational mapping f from X into G there exists a rational homomorphism $F_{\mathfrak{m}}$ from $J_{\mathfrak{m}}$ into G such that $f = F_{\mathfrak{m}} \cdot \varphi$. This property is the universal mapping property of generalized jacobian variety.

Henceforth we assume that the characteristic of Ω is 0. Let D be a non-zero element of the Lie algebra $\mathfrak{g}$ of a connected commutative algebraic group G; let D_e be the local component of D at the unit element e of G. Then by the lemma 2 there exists an irreducible curve C_D on G passing through e such that $T(e, C_D) = \Omega \cdot D_e$. Let X_D be the non-singular projective model of C_D. Then we have a rational mapping from X_D into G such that

$$f : X_D \xrightarrow{j} C_D \xrightarrow{i} G,$$

where i is the natural embedding and j is the birational mapping between X_D and C_D. Let $\mathfrak{m}$ be a module associated with f; let $J_{\mathfrak{m}}$ be the generalized jacobian variety determined by X_D and $\mathfrak{m}$. Then the universal mapping property gives a rational mapping $F_{\mathfrak{m}}$ such that the following diagram is commutative

$$\begin{tikzcd} f:\ X_D \arrow[r] \arrow[dr, "\varphi"'] & C_D \arrow[r] & G \\ & J_{\mathfrak{m}} \arrow[ur, "F_{\mathfrak{m}}"'] & \end{tikzcd}$$

Thus $F_{\mathfrak{m}}(J_{\mathfrak{m}})$ contains $f(X_D) = C_D$. Since an invariant derivation on G is determined by its local component at any point of G, D is contained in the Lie algebra of $F_{\mathfrak{m}}(J_{\mathfrak{m}})$ and therefore $F_{\mathfrak{m}}(J_{\mathfrak{m}})$ contains $G(D)$.

In particular, the above consideration in the case of $G = G(D)$ gives that $F_{\mathfrak{m}}(J_{\mathfrak{m}}) = G(D)$. Thus we have

THEOREM 3. *Let G be a connected algebraic group defined over a field of characteristic* 0; *then for any non-zero element D of $\mathfrak{g}$ there exists an irreducible curve C_D on G passing through the unit element e of G as follows: let X_D be a non-singular projective model of C_D; let f be a rational mapping from X_D into G such that $f: X_D \to C_D \to G$; let $\mathfrak{m}$ be a module associated with f; then $G(D)$ is a rational homomorphic image $J_{\mathfrak{m}}$,* *where $J_{\mathfrak{m}}$ is the generalized jacobian variety determined by X_D and $\mathfrak{m}$.*

Now suppose that G is commutative and linear; then $F_{\mathfrak{m}}(J_{\mathfrak{m}})$ is linear. Let $L_{\mathfrak{m}}$ be the maximal connected linear algebraic subgroup of $J_{\mathfrak{m}}$. Put $\overline{L} = F_{\mathfrak{m}}(L_{\mathfrak{m}})$, then $J_{\mathfrak{m}}/F_{\mathfrak{m}}^{-1}(\overline{L})$ and $F_{\mathfrak{m}}(J_{\mathfrak{m}})/\overline{L}$ are isomorphic. Therefore $J_{\mathfrak{m}}/F_{\mathfrak{m}}^{-1}(\overline{L})$ is linear. Since $F_{\mathfrak{m}}^{-1}(\overline{L})$ contains $L_{\mathfrak{m}}$, we have the natural rational homomorphism from $J_{\mathfrak{m}}/L_{\mathfrak{m}}$ onto $J_{\mathfrak{m}}/F_{\mathfrak{m}}^{-1}(\overline{L})$. But $J_{\mathfrak{m}}/L_{\mathfrak{m}}$ being abelian, $J_{\mathfrak{m}}/F_{\mathfrak{m}}^{-1}(\overline{L})$ is abelian. Thus $J_{\mathfrak{m}}/F_{\mathfrak{m}}^{-1}(\overline{L})$ reduces to the unit group, that is, $J_{\mathfrak{m}} = F_{\mathfrak{m}}^{-1}(\overline{L})$. Therefore $F_{\mathfrak{m}}(J_{\mathfrak{m}}) = \overline{L} = F_{\mathfrak{m}}(L_{\mathfrak{m}})$. Thus we have

COROLLARY. *The notations being as those of the theorem 3, if G is linear, $G(D)$ is the rational homomorphic image of the maximal connected linear algebraic subgroup $L_{\mathfrak{m}}$ of $J_{\mathfrak{m}}$.*

From the definition, for a generic point x over k' on $J_{\mathfrak{m}}$, $F_{\mathfrak{m}}(x)$ is generic over k' on $G(D)$ if D is defined over k'. So we may say that the mapping $F_{\mathfrak{m}}$ is a generalization of the exponential mapping of the linear case.

3. Unipotent algebraic groups. In the rest of this paper we assume that the characteristic of Ω is 0. A matrix x is called to be unipotent if $x-e$ is nilpotent, where e is the unit matrix. An algebraic group of unipotent matrices is called to be unipotent. For commutative unipotent algebraic groups the following proposition is important.*)

PROPOSITION 1. *A connected commutative unipotent algebraic group defined over a field of characteristic 0 is isomorphic to a product of the groups G_a.*

Generally a connected unipotent algebraic group G has a composition series:

$$G = G_0 \supseteqq G_1 \supseteqq \cdots\cdots \supseteqq G_{n-1} \supseteqq G_n = \{e\}$$

such that G_i/G_{i+1} is isomorphic to G_a. Thus if the dimension of G is 1, the proposition is shown. We prove the proposition by the induction on the dimension of G. Suppose that it is shown for the case of dimension less than n. Then, from the induction hypothesis and what we have mentioned above it follows that the n-dimensional group G is an extension of G_a by $(G_a)^{n-1}$. This extension determines a regular symmetric factor set from $G_a \times G_a$ into $(G_a)^{n-1}$, i.e. an everywhere regular rational mapping f from $G_a \times G_a$ into $(G_a)^{n-1}$ satisfying the following conditions:

(1) $df(x,y,z) = f(y,z) - f(x+y,z) + f(x,y+z) - f(x,y) = 0$, for $x,y,z \in G_a$,

(2) $f(x,y) = f(y,x)$ for $x,y \in G_a$.

*) Cf. [8] p.172 Cor. of Prop. 8.

It is sufficient to show that there exists a rational mapping g from G_a into $(G_a)^{n-1}$ such that

$$(3) \qquad f(x,y) = dg(x,y) = g(x+y) - g(x) - g(y).$$

Since the group $(G_a)^{n-1}$ is a direct product of G_a, we may restrict the problem to the case of $n=2$. Thus f is an everywhere defined rational mapping from $G_a \times G_a$ into G_a and therefore f is expressed as follows:

$$f(x,y) = \sum a_{ij} x^i y^j \in k[x,y].$$

From (2) we have $a_{ij} = a_{ji}$. Now we shall find an everywhere defined rational mapping g from G_a into G_a. So g is expressed as a polynomial in $k[X]$. Thus we may suppose that f is a homogeneous polynomial of degree q. So we have

$$f(x,y) = \sum_{i=0}^{q} a_{iq-i} \cdot x^i y^{q-i}.$$

Firstly we have from (1)

$$df(x,0,0) = f(0,0) - f(x,0) = 0,$$
$$df(0,0,z) = f(0,z) - f(0,0) = 0.$$

and therefore

$$a_{q0} = a_{0q} = 0.$$

Calculating the coefficient of $x^i y^i z^l$ in the equation (1), we have

$$\binom{i+j}{j} \cdot a_{i+j\, l} = \binom{j+l}{j} \cdot a_{i\, j+l}$$

for $i+j+l = q$ and $1 \leqq i, l \leqq q-1$.

Then putting $i=1$,

$$(4) \qquad i \cdot a_{i\, q-i} = \binom{q-1}{i-1} \cdot a_{1\, q-1} \text{ for } 1 \leqq i \leqq q-1.$$

Now putting $C(x,y) = (x+y)^q - x^q - y^q$, find an element a of k such that in the polynomial $f(x,y) - a \cdot C(x,y)$ the coefficient of xy^{q-1} is 0, that is

$$a_{1\, q-1} = a \cdot \binom{q}{1}.$$

By a simple calculation, it follows from (4) that

$$a_{i\, q-i} = a \cdot \binom{q}{i}$$

and we have $f(x,y) = a \cdot C(x,y)$. Thus $g(x) = a \cdot x^q$ satisfies (3).

The proposition shows that, in treating our problem, we may assume that $G = (G_a)^n$ without loss of generality. Let $\mathfrak{g}$ be the Lie algebra of G; let D be a non-zero element of $\mathfrak{g}$; let $\xi_1, \ldots\ldots, \xi_n$ be a system of coordinate functions on G; let D_0 be the local component of D at the unit element $0 = (0, \ldots\ldots, 0)$, then

D_0 is determined by $(d_1,\ldots\ldots,d_n)$ where $d_i = (D\xi_i)(0)$.

The proof of the lemma 1 of the section 2 shows that the linear variety C_D consisting of those points $a\cdot(d) = (ad_1,\ldots\ldots,ad_n)$ for $a\in\Omega$ is a connected algebraic group whose Lie algebra contains D. So the following lemma shows that $C_D = G(D)$.

LEMMA. *Let G be a connected algebraic group defined over a field of characteristic 0; then if H is a one-dimensional connected algebraic subgroup of G whose Lie algebra contains D, $H = G(D)$.*

In fact, H contains $G(D)$ from the definition of $G(D)$, therefore $\dim H \geqq \dim G(D)$. But $\dim H = 1$ and $\dim G(D) > 0$, so $\dim H = \dim G(D) = 1$. Since H and $G(D)$ are connected, we have $H = G(D)$.

Now let ξ be a coordinate function on G_a; let D be an element of the Lie algebra of G_a which is defined over k; then we have

(5) $\quad R_x\cdot D\xi = D\cdot R_x\xi \quad$ for any x of G_a.

Since ξ is everywhere defined on G_a, $D\xi$ is everywhere defined on G_a and therefore $D\xi$ is a polynomial $\sum_i a_i\xi^i$ in $k[\xi]$; then from (5) we have

$$a_i(x+y)^i = \sum^i a_i y^i \text{ for any } x \text{ and } y\in G_a,$$

therefore $D\xi = a\in k$. Let D_i be the derivations of the function field of $G = (G_a)^n$ such that

(6) $\quad D_i\xi_j = \delta_{ij}$ (Kronecker delta).

then by what is mentioned above, $D_1,\ldots\ldots,D_n$ form a base of $\mathfrak{g}$. Let $<D, w>$ be the dual operation from the product of the space of derivations on $G=(G_a)^n$ and the space of differential forms of degree 1 on G into Ω. Then (6) means $<D_i, d\xi_j> = \delta_{ij}$. That is, $d\xi_1,\ldots\ldots,d\xi_n$ form a dual base of the space of differential forms. But $D_1,\ldots\ldots,D_n$ are invariant and therefore $d\xi_1,\ldots\ldots,d\xi_k$ form a base of the space of invariant differential forms.

We have a rational mapping from the projective line Λ into G defined by

$$f(y_0,y_1) = y_0/y_1\cdot(d)\,;$$

f is defined over k and everywhere regular on $\Lambda - P_0$ where $P_0 = (1,0)$. Then $\mathfrak{m} = 2\cdot P_0$ is a module associated with f; in fact, let η be a uniformizing variable at P_0 on Λ such that $\eta(y_0,y_1) = y_1/y_0$; let T be a derivation of the function field of Λ such that $T\eta = 1$; let $f^* d\xi_i$ be the differential forms on Λ induced by f; put $g(\eta)\cdot d\eta = f^* d\xi_i$; then

$$<T, g(\eta)\cdot d\eta> = <T, f^*d\xi_i> = <df\ T, d\xi_i> = df\ T\cdot\xi_i = T\cdot d_i/\eta = -d_i/\eta^2.$$

Thus $f^*d\xi_i = -d_i/\eta^2\cdot d\eta$ and we have

$$\begin{aligned} v_{P_0}(f^*d\xi_i) &= v_{P_0}(-d_i/\eta^2) = -2 && \text{if } d_i \neq 0,\\ &= \infty && \text{if } d_i = 0.\end{aligned}$$

The local symbol determined by f is given by $(f,g)_P = \mathrm{Res}_P(\bar{f}\cdot dg/g)\cdot(d_1,\ldots\ldots, d_n)$ for $P\in\Lambda$, $g\in\Omega(\Lambda)^*$, where $\bar{f}$ is a rational function on Λ such that $\bar{f}(y_0,y_1) = y_0/y_1$, (cf. [8] p.43 Prop.5).

Now let $J_{\mathfrak{m}}$ be the generalized jacobian variety determined by the projective line Λ and $\mathfrak{m} = 2\cdot P_0$; let $L_{\mathfrak{m}}$ be the maximal connected linear algebraic subgroup of $J_{\mathfrak{m}}$. Then $U_{P_0}/U^{(2)}{}_{P_0}$ is isomorphic to $G_m\times G_a$ and $H_{\mathfrak{m}}$ is isomorphic to G_a. Therefore $L_{\mathfrak{m}}$ is isomorphic to G_a. So $\dim L_{\mathfrak{m}}\geqq\dim F_{\mathfrak{m}}(L_{\mathfrak{m}})>0$ and $F_{\mathfrak{m}}(L_{\mathfrak{m}})=C_D$.

Let $\mathfrak{M}$ be the maximal ideal of the local ring at P_0 in the function field of Λ. For any $a_0+a_1\eta$ and $b_0+b_1\eta$, $a_i,b_j\in\Omega$, we have

$$(a_0+a_1\eta)(b_0+b_1\eta)\equiv a_0b_0+(a_1+b_1)\eta \qquad \mathrm{mod}\ \mathfrak{M}^2.$$

Thus, for a representative $1+a\eta$ of a class of $H_{\mathfrak{m}} = R_{\mathfrak{m}}/\Delta$, the quantity a is the coordinate of the corresponding point of $L_{\mathfrak{m}}$ in the faithful representation guaranteed by the isomorphism between $L_{\mathfrak{m}}$ and G_a. Let t be a quantity transcendental over k; let $g = 1-t\cdot\eta$, then the principal divisor $(g) = Q-p$, where $Q=(t,1)$ and $P=(0,1)$. Let z be a generic point over k on $L_{\mathfrak{m}}$ which corresponds to g by the representation; then

$$F_{\mathfrak{m}}(z) = f((g)) = f(Q)-f(P) = t\cdot(d_1,\ldots\ldots,d_n).$$

We know that G_a has a matrix representation γ as follows:

$$\gamma : G_a\ni x\rightarrow \begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\in GL(2,\Omega),$$

therefore the direct product Γ of n-copies of γ is a matrix representation of $(G_a)^n$. On the other hand for an indeterminate t, $\exp tN$ is in $GL(n, k[t])$ if the matrix N is nilpotent in $\mathfrak{gl}(n,k)$. Thus we may suppose that t is a quantity. We have

$$\begin{aligned}\Gamma\cdot(t(d)) &= \text{direct product of } \begin{pmatrix}1 & td_i\\ 0 & 1\end{pmatrix} = \exp t\cdot(\text{direct product of } \begin{pmatrix}0 & d_i\\ 0 & 0\end{pmatrix})\\ &= \exp t\cdot d\Gamma D_0.\end{aligned}$$

This implies $\Gamma\cdot F_{\mathfrak{m}}(z) = \exp t\cdot d\Gamma D_0$. That is, $F_{\mathfrak{m}}$ may be considered as an exponential mapping.

Considering the isomorphism between the unipotent group G and $(G_a)^n$, we have

PROPOSITION 2. *Let G be a connected unipotent algebraic group defined over a field k of characteristic 0; then for non-zero D of $\mathfrak{g}$, defined over k, there exists a rational mapping f from the projective line Λ into G as follows: f is defined over k and regular outside $P_0=(1,0)$; $f(0,1)=e$; $f(\Lambda)$ generates $G(D)$; $\mathfrak{m}=2\cdot P_0$ is a module associated with f; let $J_{\mathfrak{m}}$, $L_{\mathfrak{m}}$, and $F_{\mathfrak{m}}$ be those defined by Λ and $\mathfrak{m}$ as in the section 2, $L_{\mathfrak{m}}$ is isomorphic with G_a; $F_{\mathfrak{m}}$ induces an isomorphism between $L_{\mathfrak{m}}$ and $G(D)$; if z is generic on $L_{\mathfrak{m}}$ over k, $F_{\mathfrak{m}}(z) = \exp tD_e$, where D_e is the local component of D at the unit element e and* *qnautity t is a transcendental over k which corresponds to z by the isomorphism between L_m and G_a.*

Non-trivial rational homomorphism from G_a into itself is an isomorphism $x \to ax$ for some $a \in \Omega^*$. So the proof is completed.

4. Algebraic groups of semi-simple matrices. A matrix x is called to be semi-simple if the minimal polynomial of x is a product of distinct irreducible separable polynomials. A commutative algebraic group of semi-simple matrices is transformed into an algebraic subgroup of the group of diagonal matrices $D(n) = D(n,\Omega)$. For such a group the following proposition is fundamental*).

PROPOSITION 1. *Let G be an r-dimensional connected algebraic subgroup of $D(n)$. Then there exists a unimodular matrix $(m_{ij}) \in GL(n,Z)$ such that, let*

$$F_i = X_1^{m_{i1}} \cdot X_2^{m_{i2}} \cdot \cdots\cdots\cdots\cdots \cdot X_n^{m_{in}}$$

then the automorphism σ of $D(n)$ defined by

$$(1) \qquad \sigma(x) = (F_1(x), \ldots\ldots\ldots\ldots, F_n(x))$$

maps G onto the subgroup of $D(n)$ of those elements $(a_1, \ldots\ldots, a_r, 1, \ldots\ldots, 1)$ for $a_i \in \Omega^*$, where Z is the ring of integers.

Let (ξ_{ij}) be the coordinate functions on $\mathfrak{gl}(n,\Omega)$. Then for $x = (x_1, \ldots\ldots, x_n) \in D(n)$,

$$R_x^* \ \xi_{ij} = x_j \xi_{ij}.$$

Thus the space R_q which is the homogeneous component of degree q of $\Omega[\xi]$ is G-invariant; if we take a base of R_q consisting the monomials, R_x^* gives a diagonal representation of G whose matrix-coefficients are

$$x_1^{e_1} \cdots\cdots x_n^{e_n}, \qquad e_i \text{ are non-negetive integers.}$$

Let $P_1, \ldots\ldots, P_m$ be the semi-invariant polynomials of G with the same weight χ which define G; take an integer q such that $\sum_{i=1}^q R_i$ contains all P_j; let V be the subspaces of $\sum_{i=1}^q R_i$ of those which are semi-invariant of G with the weight χ. Since $D(n)$ is commutative, V is $D(n)$- invariant. And, for $x \in D(n)$, x is in G if and only if R_x^* induces a mapping $a \cdot 1$ on V for some $a \in \Omega^*$. Let $\mathfrak{E}(V)$ be the ring of endomorphisms of V; put ρ be the representation of $D(n)$ with the representation space $\mathfrak{E}(V)$ such that

$$\rho(x) : X \to R_x^* X R_{x^{-1}}^*,$$

then ρ is a diagonal representation whose coefficients are

$$x_1^{m_1} \cdots\cdots x_n^{m_n}, \ m_i \text{ are integers.}$$

*) Cf. [2] p. 43 Prop. 7.2 and Prop. 7.4.

Thus G is defined by the finite number of the equations of the type

$$x_1^{m_1}\cdots\cdots x_n^{m_n}=1.$$

Let x be a generic point on G over a field k of definition for G, then for any integer $d>0$, x^d is generic on G over k. Thus we may suppose that the greatest common divisor of $m_1,\ldots\ldots,m_n$ is 1. Then there exists a unimodular matrix $(m_{ij})\in GL(n,Z)$ such that $m_{nj}=m_j$. Now let τ be such that

$$\tau(x)_i=x_1^{m_{i1}}\cdots\cdots\cdots\cdots x_n^{m_{in}}.$$

Then τ is an automorphism of $D(n)$ and if x is in G, $\tau(x)_n=1$. Succeeding this method we have the proposition.

This proposition shows that so far as the structure of $G(D)$ is studied in the case of semi-simple matrices, we may suppose $G=D(n)$. Further if we assume $G(D)=G$, then we obtain that $s_1,\ldots\ldots,s_n$ are linearly independent over the ring of integers, i.e.

(C) There exists no non-trivial relation such that

$$e_1s_1+\cdots\cdots+e_ns_n=0,\ e_i\text{: integers},$$

where s_i are such quantities that $(D\xi_i)(e)=s_i$, $(\xi_1,\ldots\ldots,\xi_n)$ being the coordinate functions of G. In fact, if such non-trivial relation existed, the local component D_e of D at the unit element e is contained in the tangent space at e on the algebraic subgroup G' of G defined by $X_1^{e_1}\cdots\cdots X_n^{e_n}=1$. Therefore D is contained in the Lie algebra of G'. But the proof of the proposition 1 shows that $\dim G'<\dim G=\dim G(D)$, and it is the contradiction.

Thus, let $G=D(n)$ and let D be an element of $\mathfrak{g}$, defined over k, satisfying the condition (C). $X_1,\ldots\ldots,X_n$ form a uniformizing set of linear forms for G at the unit element $e=(1,\ldots\ldots,1)$. Let γ be a birational mapping between G and the n-dimensional affine space S such that $\gamma(x)=(x_1-1,\ldots\ldots,x_n-1)$. Then the differential $d\gamma$ is an isomorphism from $T(e,G)$ onto $T(0,S)$. Let D_e be the local component of D at e, then $d\gamma D_e$ is in $T(0,S)$ and $d\gamma D_e=(s_1,\ldots\ldots,s_n)$; in fact, if $\eta_1,\ldots\ldots,\eta_n$ are the coordinate functions on S, then

$$d\gamma D_e\eta_j=D_e(\xi_j+1)=s_j.$$

Let $C_{d\gamma D}$ be the curve on S consisting of those points $a\cdot(s)$ for any $a\in\Omega$; let $\gamma^{-1}(C_{d\gamma D})=C_D$; then C_D is the irreducible curve on those points

$$(as_1+1,\ as_2+1,\ldots\ldots\ldots\ldots,as_n+1),\ \text{for } a\in\Omega$$

such that $T(e,C_D)=\Omega\cdot D_e$.

Let ξ be the coordinate function on G_m. Then, the derivation D on G_m such that $D\xi=\xi$ is invariant: in fact any x and y of G_m,

$(R_x^*D\xi)(y)=(R_x^*\xi)(y)=xy,$

$(D\cdot R_x^*\ \xi,\ R_x^*)(y)=(D\cdot x\xi)(y)=x\cdot D\xi(y)=x\cdot\xi(y)=xy.$

Let D_i be the derivations on G such that

$$(2) \qquad D_i\xi_j = \delta_{ij}\xi_j,$$

then by what is mentioned above $D_1,\ldots\ldots,D_n$ form a base of $\mathfrak{g}$. Further the dual base to $(D_1,\ldots\ldots,D_n)$ of the space of invariant differential forms of degree 1 on G is $(\xi_1^{-1}\cdot d\xi_1,\ldots\ldots,\xi_n^{-1}\cdot d\xi_n)$; in fact, from (2)

$$< D_i,\xi_j^{-1}\cdot d\xi_j > \ = \xi_j^{-1}\cdot < D_i, d\xi_j > \ = \xi_j^{-1}\cdot\delta_{ij}\,\xi_i = \delta_{ij}.$$

We have a rational mapping f from the projective line Λ into G defined by

$$f(y_0,y_1) = (y_0y_1^{-1}s_1+1,\ldots\ldots\ldots\ldots,y_0y_1^{-1}s_n+1);$$

f is defined over k and everywhere regular on $\Lambda - S$ where $S = \{P_0 = (1,0), P_i = (1,-s_i),\ i = 1,\ldots\ldots,n\}$. From the condition (C) those points P_i are distinct n points on Λ. Then $\mathfrak{m} = P_0+P_1+\cdots\cdots+P_n$ is a module associated with f. In fact, let η be the uniformizing variable at P_0 on Λ such that $\eta(y_0,y_1) = y_1/y_0$; let T be the derivation of the function field of Λ such that $T\eta = 1$; let $f^*\xi_i^{-1}\cdot\ d\xi_i = g(\eta)\cdot d\eta$; then

$$\begin{aligned} < T,g(\eta)\,d\eta > &= < T,f^*\xi_i^{-1}\cdot d\xi_i > \ = (f^*\xi_i)^{-1}\cdot < df\ T,d\xi_i > \\ &= (f^*\xi_i)^{-1}Tf^*\xi_i = (s_i\eta^{-1}+1)^{-1}T(s_i\eta^{-1}+1) \\ &= -s_i(s_i\eta\ +\eta^2)^{-1}. \end{aligned}$$

Thus,

$$f^*\xi_i^{-1}\cdot d\xi_i = -s_i(s_i\eta + \eta^2)^{-1}d\eta,$$

and we have $v_{P_0}(f^*\xi_i^{-1}\cdot d\xi_i) = -1$, since, from the condition (C), $s_i \neq 0$. Next, for $i = 1,,\ldots\ldots n$, let τ be the uniformizing variable at P_i on Λ such that $\tau(y_0,y_1) = y_1y_0^{-1}+s_i$. Then $\tau = \eta+s_i$ and $d\tau = d\eta$. And we have

$$\begin{aligned} f^*\xi_j^{-1}\cdot d\xi_j &= -s_j(s_j\eta+\eta^2)^{-1}\cdot d\eta \\ &= -s_j(\tau-s_i)^{-1}\cdot(\tau-s_i+s_j)^{-1}\cdot d\tau \end{aligned}$$

Thus we have

$$\begin{aligned} v_{P_i}(f^*\xi_j^{-1}\cdot d\xi_j) &= 0 \qquad \text{if } i \neq j, \\ &= -1 \qquad \text{if } i = j, \end{aligned}$$

since, from the condition (C), $s_i \neq 0$ and $s_i \neq s_j$ if $i \neq j$.

From the definition of the local symbol we have

LEMMA. *Let f_i be rational mappings from $X - S$ into commutative algebraic groups G_i for $i = 1,2$; let $(f_i,g)_P$ be the local symbols associated with f_i. Then*

$$(f_1\times f_2,g)_P = (f_1,g)_P \times (f_2,g)_P$$

is the local symbol associated with the rational mapping $f_1\times f_2$ from $X-S$ into $G_1\times\ G_2$ such that $f_1\times f_2\cdot(P) = f_1(P)\times f_2(P)$.

Thus, in our case, the local symbol associated with f is

$$(f,g)_P = (-1)^l((-1)^{m_1}\bar{f}_1^l g^{-m_1}(P),\ldots\ldots,(-1)^{m_n}\bar{f}_n^l g^{-m_n}(P)) \text{ for } P\in\Lambda,\ g\in\Omega(\Lambda)^*,$$

where $\bar{f}_i$ is rational functions on Λ such that

$\bar{f}_i(y_0,y_1) = y_0 y_1^{-1}s_i+1$ and $l = v_P(g)$, $m_i = v_P(f_i)$, (cf.[8]p.44 Prop.6).

Let $J_{\mathfrak{m}}, L_{\mathfrak{m}}$ and $F_{\mathfrak{m}}$ be those determined by the projective line Λ and $\mathfrak{m} = P_0+P_1+\ldots+P_n$. Then $R_{\mathfrak{m}}$ is isomorphic to $(G_m)^{n+1}$. So, Le is isomorphic to $(G_m)^n$. Let $t_1,\ldots\ldots,t_n$ be independent transcendental quantities over k; let $g = \prod_{i=1}^n(\eta-t_i)$, then the principal divisor $(g)=Q_1+\cdots\cdots+Q_n-n\cdot P$, where $Q_j=(1, t_j)$ and $P=(0,1)$. Let z be a point of $L_{\mathfrak{m}}$ which corresponds to the divisor (g). Then

$$F_{\mathfrak{m}}((z))=f((g)) = f(Q_1)\ldots\ldots f(Q_n)f(P)^{-n}$$
$$= \left(\prod_{j=1}^n(s_1t_j+1),\ldots\ldots,\prod_{j=1}^n(s_nt_j+1)\right).$$

Let $u_i = f(Q_i) = (s_1t_i+1,\ldots\ldots,s_nt_i+1)$; then $u_1,\ldots\ldots,u_n$ are independent generic points on C_D over k. Thus $f((g))$ is generic over k on the product C_D^n in G. Now let $u_{n+1},\ldots\ldots,u_{2n}$ be independent generic points on C_D over $k(u_1,\ldots\ldots,u_n)$; for $1\leqq s\leqq 2n$, let H_s be the locus of the product $u_1\ldots\ldots u_s$ on G over k, then since C_D contains e, we have

$$C_D = H_1 \subseteqq H_2 \subseteqq\ldots\ldots\ldots\ldots\subseteqq H_{2n}.$$

Since $\dim G = n$, there exists an integer $r\leqq n$ such that $H_r = H_{r+1}$. Then $H_r = H_{r+1} = \ldots\ldots = H_{2n}$; in fact; since the two points $x_2\cdots\cdots x_{r+1}$ and $x_2\cdots\cdots x_{r+2}$ are generic on H_r over $k(x_1)$, the two points $x_1\cdots\cdots x_{r+1}$ and $x_1\cdots\cdots x_{r+2}$ have the same locus on G over k. Thus $H_r = H_{2r}$ and H_r is a connected algebraic subgroup of G. Since $n\geqq r$, the point $f((g))$ is generic over k on $H=H_r$.

Suppose that $\dim H < \dim G$, then there exist non-trivial integers $(e_1,\ldots\ldots,e_n)$ such that the polynomial

$$M(X) = X_1^{e_1}\cdots\cdots X_n^{e_n}-1$$

vanishes on H. Let A be the ideal of $k[t_1,\ldots\ldots,t_n]$ which is spanned by the monomials of degree $\geqq 2$. Then

$$\begin{aligned}
0 = M(f((g))) &= \prod_{i=1}^n\left(\prod_{j=1}^n(s_it_j+1)\right)e_i-1 & \\
&\equiv \prod_{i=1}^n\left(\prod_{j=1}^n(1+e_i\ s_it_j)\right)-1 & \text{mod } A\\
&\equiv \prod_{i=1}^n(1+e_is_it_1+\quad +e_is_it_n)-1 & \text{mod } A\\
&\equiv \sum_{i,j=1}^n e_is_it_j & \text{mod } A\\
&= \sum_{j=1}^n\left(\sum_{i=1}^n e_is_i\right)\cdot t_j).
\end{aligned}$$

Thus, $e_1s_1+\ldots\ldots+e_ns_n=0$. This is a contradiction. Therefore $H=G$, i.e. $F_{\mathfrak{m}}(z)$ is a generic point on G over k.

Now, we shall go back to the propostion 1; let σ be the automorphism of

$D(n)$ of the type (1) in the proposition. Let D be an element of the Lie algebra of $D(n)$; let $(D\xi_i)(e) = (s_1,\cdots\cdots,s_n)$; then

$$(d\sigma D\xi_i)(e) = (D\sigma^*\xi_i)(e) = DF_i\ (\xi)(e) = \sum_{j=1}^{n} m_{ij}s_j\,.$$

Let G be the group in the proposition; then D is contained in $\mathfrak{g}$ if and only if

$$\sum_{j=1}^{n} m_{ij}s_j = 0 \qquad \text{for } i = r+1,\cdots\cdots\cdots\cdots,n.$$

Thus we have

PROPOSITION 2. *Let G be a connected algebraic group of semi-simple matrices defined over a field k of characteristic* 0 *let D be an element of* $\mathfrak{g}$ *defined over k; let $s_1,\cdots, s_n$ be the eigen-values of the local component D_e of D at the unit element e; then there exists a unimodular matrix (m_{ij}) of integer coefficients such that, let*

$$s_i' = m_{i1}s_1 + \cdots\cdots\cdots\cdots + m_{in}s_n,$$

$s_1',\cdots\cdots,s_r'$ are linearly independent over integers and $s_{r+1}' = \cdots\cdots = s_n' = 0$. Further there exists a rational mapping f from the projective line Λ into G as follows; f is defined over k; let $P_0 = (1,0)$ and $P_i = (1, -s_i')$, f is everywhere regular on $\Lambda - \{P_0,P_1,\cdots\cdots,P_r\}$; $f(0,1) = e$; $f(\Lambda)$ generates $G(D)$; $\mathfrak{m} = P_0 + \cdots \cdots + P_r$ is a module associated with f; let *$J_{\mathfrak{m}}$, $L_{\mathfrak{m}}$ let and $F_{\mathfrak{m}}$ be those defined by Λ and $\mathfrak{m}$ as in the section* 2, *then $L_{\mathfrak{m}}$ is isomorphic to $(G_m)^r$; $F_{\mathfrak{m}}(L_{\mathfrak{m}}) = G(D)$ is isomorphic to $(G_m)^r$ and if z is generic on $L_{\mathfrak{m}}$ over k, $F_{\mathfrak{m}}(z)$ and* exp *tD_e are generic specialization of each other over k where t is an indeterminate.*

5. Linear algebraic groups. Now we shall discuss the problem in the case of linear algebraic groups. Let G be a connected commutative linear algebraic group defined over k. Then G is the direct product of the semi-simple part G_s of G and the unipotent part G_u of G. G_s is the algebraic subgroup of G of the semi-simple matrices in G and G_u is that of the unipotent matrices. For $x\in G$, let $x = x_s x_u$ be the multiplicative Jordan decomposition. The mappings $x\to x_s$ from G onto G_s and $x\to x_u$ from G onto G_u are rational. Since G_s and G_u are commutative, the propositions of the sections 3 and 4 show that we may assume that $G_s = (G_m)^r$ and $G_u = (G_a)^q$. Then $\mathfrak{g}$ is the direct sum of $\mathfrak{g}_s$ and $\mathfrak{g}_u$ where $\mathfrak{g}_s$ and $\mathfrak{g}_u$ are the Lie algebra of G_s and G_u.

Let D be an element of $\mathfrak{g}$ defined over k; let $D_e = S+N$ be the Jordan decomposition of the local component D_e of D at e where S is semi-simple and N is nilpotent. For simplicity, put $D_e = (s_1,\cdots\cdots,s_r,d_1,\cdots\cdots,d_q)$ and suppose that $s_1,\cdots\cdots,s_r$ are linearly independent over integers. Let C_D be the curve on G of those points

$$(as_1+1,\cdots\cdots,as_r+1,\ ad_1,\cdots,ad_q) \qquad\qquad \text{for } a\in\Omega,$$

then C_D is irreducible and $T(e, C_D) = \Omega\cdot D_e$.

We have a rational mapping f from the projective line Λ into G defined by

$$f(y_0, y_1) = (y_0 y_1^{-1} s_1 + 1, \ldots\ldots, y_0 y_1^{-1} s_r + 1,\ y_0 y_1^{-1} d_1, \ldots\ldots, y_0 y_1^{-1} d_q).$$

Suppose that $S \neq 0$ and $N \neq 0$. Let $P_0 = (1,0)$ and $P_i = (1, -s_i)$, then the arguments in the sections 3 and 4 give that f is everywhere regular on $\Lambda - \{P_0, \ldots\ldots, P_r\}$ and that $\mathfrak{m} = 2 \cdot P_0 + P_1 + \cdots\cdots + P_r$ is a module associated with f. From the lemma of the section 4 it follows that the local symbol associated with f is

$$(f,g)_P = (-1)^l((-1)^{m_1}\bar{f}_1^l g^{-m_1} \cdot (P), \ldots\ldots, (-1)^{m_r}\bar{f}_r g^{-m_r} \cdot (P)) \times$$
$$\mathrm{Res}_P(\bar{f} \cdot dg/g) \cdot (d_1, \ldots\ldots, d_q)$$
$$\text{for } P \in \Lambda,\ g \in \Omega(\Lambda)^*,$$

where $\bar{f}_i$ and $\bar{f}$ are rational functions on Λ such that

$$\bar{f}_i(y_0, y_1) = y_0 y_1^{-1} s_i + 1 \text{ and } \bar{f}(y_0, y_1) = y_0/y_1 \text{ and } l = v_P(g),\ m_i = v_P(\bar{f}_i).$$

Let $J_{\mathfrak{m}}$, $L_{\mathfrak{m}}$ and $F_{\mathfrak{m}}$ be those defined by Λ and $\mathfrak{m}$ as in the section 2, then $H_{\mathfrak{m}}$ is isomorphic to $(G_m)^{r+1}\ G_a$ and $L_{\mathfrak{m}}$ is isomorphic to $(G_m)^r \times G_a$.

Thus, combining the results of the sections 3 and 4, we have

THEOREM 4. *Let G be connected linear algebraic group defined over a field k of characteristic 0. Let D be an element of $\mathfrak{g}$ defined over k. Let D_e be the local component of D at the unit element e. Let $D_e = S + N$ be the Jordan decomposition where S is semi-smple and N is nilpotent. Suppose that $S \neq 0$ and $N \neq 0$. Let $s_1 \ldots\ldots s_n$ be the eigen-values of D_e. Then there exists a rational mapping f, defined over k, from the projective line Λ into G as follows: $f(\Lambda)$ generates $G(D)$; f is everywhere regular on $\Lambda - \{P_0, P_1, \cdots\cdots, P_r\}$ where $P_0 = (1, 0)$ and $P_i = (1, -s_i')$ for $i = 1, \ldots\ldots, r$, the vector $(s_1', \ldots\ldots, s_r', 0, \ldots\ldots, 0)$ being a transformed one of $(s_1, \ldots\ldots, s_r, s_{r+1}, \cdots\cdots, s_n)$ by a unimodular matrix of integer coefficients such that $s_1', \ldots\ldots, s_r'$ are linearly independent over integers: $f(0, 1) = e$; $\mathfrak{m} = 2 \cdot P_0 + P_1 + \cdots + P_r$ is a module associated with f; let $J_{\mathfrak{m}}$, $L_{\mathfrak{m}}$ and $F_{\mathfrak{m}}$ be those defined by Λ and $\mathfrak{m}$ as in the theorem 3, then $L_{\mathfrak{m}}$ is isomorphic to $(G_m)^r \times G_a$ and $F_{\mathfrak{m}}(L_{\mathfrak{m}}) = G(D)$ is isomorphic to $(G_m)^r \times G_a$; if z is generic point over k on $L_{\mathfrak{m}}$, $F_{\mathfrak{m}}(z)$ and $\exp tD_e$ are generic specialization of each other over k, where t is an indeterminate.*

As for the assumption that $S \neq 0$ and $N \neq 0$, if $S = 0$, the theorem is the proposition 2 of the section 3 and if $N = 0$, the proposition 2 of the section 4.

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KAWAUCHI COLLEGE, TÔHOKU UNIVERSITY.