DIFFERENTIABLE IMBEDDING AND COBORDISM OF ORIENTABLE MANIFOLDS

YASURÔ TOMONAGA

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Introduction. It is known that if a compact orientable differentiable n-manifold is differentiably imbedded in a euclidean space whose dimension is less than $\frac{3}{2}n$, then some of the dual-Pontryagin classes must vanish. Meanwhile the cobordism coefficients of a manifold are determined by the Pontryagin classes and the Pontryagin classes are explicitly expressed by the dual-Pontryagin classes. Therefore, if a compact orientable differentiable n-manifold is imbedded in a euclidean space whose dimension is less than $\frac{3}{2}n$, then its cobordism decomposition takes a special form. In this paper we shall deal with this problem.

1. A compact orientable differentiable $4n$-manifold admits the cobordism decomposition of the form:

$$M_{4n} \sim \sum_{i_1+\cdots+i_t=n} A^n_{i_1\cdots\cdots i_t} P_{2i_1}(c)\cdots\cdots P_{2i_t}(c) \text{ mod torsion,} \tag{1. 1}$$

where $P_{2i}(c)$ denotes the complex projective space of complex dimension $2i$ and A's denote some rational numbers. The torsions have been completely made clear by Wall ([8]).

It is known that

$$\tau = \text{index} = \sum_{i_1+\cdots+i_t} A^n_{i_1\cdots\cdots i_t}, \tag{1. 2}$$

$$\begin{cases} A^2_2 = \frac{1}{5}(-2p_2 + p_1^2)[M_8], \\ A^2_{11} = \frac{1}{9}(5p_2 - 2p_1^2)[M_8], \\ \tau = \frac{1}{45}(7p_2 - p_1^2)[M_8], \end{cases} \tag{1. 3}$$

$$(1.\ 4)\quad \begin{cases} A_3^3 = \dfrac{1}{7}(3\,p_3 - 3\,p_2p_1 + p_1^3)[M_{12}], \\[2mm] A_{21}^3 = \dfrac{1}{15}(-21\,p_3 + 19\,p_2p_1 - 6\,p_1^3)[M_{12}], \\[2mm] A_{111}^3 = \dfrac{1}{27}(28\,p_3 - 23\,p_2p_1 + 7p_1^3)\,[M_{12}], \\[2mm] \tau = \dfrac{1}{3^3\cdot 5\cdot 7}(62\,p_3 - 13\,p_2p_1 + 2\,p_1^3)\,[M_{12}], \end{cases}$$

$$(1.\ 5)\quad \begin{cases} A_4^4 = \dfrac{1}{9}(-4p_4 + 4p_3p_1 + 2p_2^2 - 4p_2p_1^2 + p_1^4)\,[M_{16}], \\[2mm] A_{31}^4 = \dfrac{1}{21}(36\,p_4 - 33\,p_3p_1 - 18p_2^2 + 33\,p_2p_1^2 - 8p_1^4)\,[M_{16}], \\[2mm] A_{22}^4 = \dfrac{1}{25}(18\,p_4 - 18\,p_3p_1 - 7p_2^2 + 16\,p_2p_1^2 - 4p_1^4)\,[M_1{}^6], \\[2mm] A_{211}^4 = \dfrac{1}{45}(-180p_4 + 159p_3p_1 + 80p_2^2 - 150p_2p_1^2 + 36p_1^4)\,[M_{16}], \\[2mm] A_{1111}^4 = \dfrac{1}{81}(165p_4 - 137\,p_3p_1 - 70p_2^2 + 127\,p_2p_1^2 - 30p_1^4)\,[M_{16}], \\[2mm] \tau = \dfrac{1}{3^4\cdot 5^2\cdot 7}(381\,p_4 - 71p_3p_1 - 19p_2^2 + 22\,p_2p_1^2 - 3p_1^4)\,[M_{16}], \end{cases}$$

([6])

$$(1.\ 6)\quad \tau\,[M_{20}] = \frac{1}{3^5\cdot 5^2\cdot 7\cdot 11}(5110\,p_5 - 919\,p_4p_1 - 336\,p_3p_2 + 237\,p_3p_1^2 + 127p_2^2p_1 - 83\,p_2p_1^3 + 10p_1^5)\,[M_{20}],$$ ([3] p. 13)

where p_i denotes the Pontryagin class of dimension $4i$.

It is known that *all cobordism coefficients of* M_8 *and* M_{12} *and* $3\,A_4^4, A_{31}^4, A_{22}^4, A_{211}^4, 3\,A_{1111}^4$ *are integers* ([7]). It is also well-known that *if a compact orientable differentiable manifold* M_n *is differentiably imbedded in the euclidean space* E_{n+q} *it must be that*

$$(1.\ 7)\qquad \bar{p}_k = 0 \qquad 2k \geqq q+1,$$

where $\bar{p}_k$ *denotes the dual-Pontryagin class of dimension* $4\,k$. Between the Pontryagin classes and the dual-Pontryagin classes there exists a relation such that ([1])

$$(1.\ 8)\qquad p\cdot\bar{p} = 1$$

where

$$(1.\ 9)\qquad p=\sum_{k\geqq 0}(-1)^k p_k t^k$$

and

$$(1.10)\qquad \bar{p}=\sum_{k\geqq 0}\bar{p}_k t^k.$$

It follows from (1. 8) that

$$(1.11)\qquad \begin{cases} p_1=\bar{p}_1, \\ p_2=-\bar{p}_2+\bar{p}_1^2, \\ p_3=\bar{p}_3-2\bar{p}_2\cdot\bar{p}_1+\bar{p}_1^3, \\ p_4=-\bar{p}_4+2\bar{p}_3\cdot\bar{p}_1+\bar{p}_2^2-3\bar{p}_2\cdot\bar{p}_1^2+\bar{p}_1^4. \end{cases}$$

2. Hereafter M_n always denotes a compact orientable differentiable n-manifold and the imbedding means the differentiable one. We denote by E_k the euclidean k-space. It is known that *if an M_n is imbedded in the E_{n+2}, then it is "bord"*. Therefore we shall deal with the case where an M_n is imbedded in an E_{n+q} $(q\geqq 3)$ and $n=4k$.

The case $M_8\subset E_{8+q}$.

We have from (1. 3) and (1. 11)

$$(2.\ 1)\qquad \begin{cases} A_2^2=\dfrac{1}{5}(2\bar{p}_2-\bar{p}_1^2)[M_8], \\ A_{11}^2=\dfrac{1}{9}(-5\bar{p}_2+3\bar{p}_1^2)[M_8], \\ \tau=\dfrac{1}{45}(-7\bar{p}_2+6\bar{p}_1^2)[M_8]. \end{cases}$$

If the M_8 is imbedded in the E_{11} we have from (1. 7)

$$\bar{p}_2=0.$$

Hence we have from (2. 1)

$$(2.\ 2)\qquad A_2^2=-\frac{1}{5}\bar{p}_1^2[M_8],\ A_{11}^2=\frac{1}{3}\bar{p}_1^2\,[M_8],\ \tau=\frac{2}{15}\bar{p}_1^2\,[M_8],$$

from which we have

$$(2.\ 3)\qquad M_8\sim-\frac{\tau}{2}\{3\,P_4(c)-5\,P_2(c)^2\}.$$

Hence we have the

THEOREM 1. *If an M_8 is imbedded in the E_{11}, then its index is even and it admits the cobordism decomposition of the form* (2. 3).

The case $M_{12} \subset E_{12+q}$.

We have from (1. 4) and (1. 11)

$$(2.\ 4)\qquad \begin{cases} A_3^3 = \dfrac{1}{7}(3\bar{p}_3 - 3\bar{p}_2 \cdot \bar{p}_1 + \bar{p}_1^3)[M_{12}], \\ A_{21}^3 = \dfrac{1}{15}(-21\bar{p}_3 + 23\bar{p}_2\bar{p}_1 - 8\bar{p}_1^3)[M_{12}], \\ A_{111}^3 = \dfrac{1}{27}(28\bar{p}_3 - 33\bar{p}_2\bar{p}_1 + 12\bar{p}_1^3)[M_{12}]. \end{cases}$$

We consider the case where $M_{12} \subset E_{15}$. In this case we have from (2. 4)

$$(2.\ 5)\qquad \begin{cases} A_3^3 = \dfrac{1}{7}\bar{p}_1^3[M_{12}],\ A_{21}^3 = -\dfrac{8}{15}\bar{p}_1^3[M_{12}],\ A_{111}^3 = \dfrac{4}{9}\bar{p}_1^3[M_{12}] \\ \tau = \dfrac{17}{7 \cdot 45}\bar{p}_1^3[M_{12}] \end{cases}$$

from which we have

$$(2.\ 6)\qquad M_{12} \sim \frac{\tau}{17}\{45\,P_6(c) - 168\,P_4(c)P_2(c) + 140\,P_2(c)^3\}.$$

Hence we have the

THEOREM 2. *If an M_{12} is imbedded in the E_{15}, then its index is divisible by* 17 *and it admits the cobordism decomposition of the form* (2. 6).

In the case where $M_{12} \subset E_{17}$ we have from (1. 7)

$$\bar{p}_3 = 0.$$

Hence we have from (2. 4)

$$(2.\ 5)\qquad \begin{cases} A_3^3 = \dfrac{1}{7}(-3\bar{p}_2 \cdot \bar{p}_1 + \bar{p}_1^3)[M_{12}], \\ A_{21}^3 = \dfrac{1}{15}(23\bar{p}_2 \cdot \bar{p}_1 - 8\bar{p}_1^3)[M_{12}], \\ A_{111}^3 = \dfrac{1}{27}(-33\bar{p}_2 \cdot \bar{p}_1 + 12\bar{p}_1^3)[M_{12}]. \end{cases}$$

It follows from (2. 5) that

$$(2.\ 6)\qquad 28\,A_3^3 + 15\,A_{21}^3 + 9\,A_{111}^3 = 0.$$

Meanwhile we have

(2. 7) $$\tau = A^3_3 + A^3_{21} + A^3_{111}.$$

We have from (2. 6) and (2. 7)

(2. 8) $$13\,A^3_{21} + 19\,A^3_{111} = 28\,\tau.$$

Solving this equation we have

(2. 9) $$A^3_3 = -27\,\tau - 6\,m,\ A^3_{21} = 84\,\tau + 19\,m,\ A^3_{111} = -56\,\tau - 13\,m,$$

where m denotes some integer. Thus we have the

THEOREM 3. *If an M_{12} is imbedded in the E_{17}, then it admits the following cobordism decomposition*:

(2. 10) $$M_{12} \sim \tau\,\{-27\,P_6(c) + 84\,P_4(c)\,P_2(c) - 56\,P_2(c)^3\} + m\{6\,P_6(c) - 19\,P_4(c)P_2(c) + 13\,P_2(c)^3\}$$

where m denotes some integer.

COROLLARY 1. *Neither $P_4(c)P_2(c)$ nor $P_2(c)^3$ can be imbedded in the E_{17}.*

Next we consider the 12-dimensional submanifold of $P_7(c)$.
Let

(2. 11) $$p = (1 + g^2)^8 \qquad g \in H^2(P_7(c), Z)$$

be the Pontryagin class of $P_7(c)$ and let $v = \lambda g$ (λ; integer) be the cohomology class corresponding to such a submanifold. Then its cobordism coefficients are given by

(2. 12) $$\begin{cases} A^3_3 = \dfrac{1}{7}(8\,\lambda - \lambda^7), \\ A^3_{21} = \dfrac{8}{15}(\lambda^7 - \lambda^5 - \lambda^3 + \lambda), \\ A^3_{111} = \dfrac{1}{9}(-4\lambda^7 + 8\,\lambda^5 - 4\,\lambda^3). \end{cases}$$ ([6])

Comparing (2 .12) and (2. 6) we have the

COROLLARY 2. *If an M_{12} is a submanifold of the $P_7(c)$, it cannot be imbedded in the E_{17}.*

3. *The case $M_{16} \subset E_{16+q}$.*

We have from (1. 5) and (1. 11)

$$\begin{cases} A^4_4 = \dfrac{1}{9}(4\bar{p}_4 - 4\bar{p}_3\bar{p}_1 - 2\bar{p}_2^2 + 4\bar{p}_2\cdot\bar{p}_1^2 - \bar{p}_1^4)\,[M_{16}], \\ A^4_{31} = \dfrac{1}{21}(-36\bar{p}_4 + 39\bar{p}_3\bar{p}_1 + 18\bar{p}_2^2 - 39\bar{p}_2\bar{p}_1^2 + 10\bar{p}_1^4)\,[M_{16}], \end{cases}$$

$$(3.\ 1)\begin{cases} A^4_{22} = \dfrac{1}{25}(-18\overline{p}_4 + 18\overline{p_3 p_1} + 11\overline{p}^2_2 - 20\overline{p}_2\cdot\overline{p}^2_1 + 5\overline{p}^4_1)\,[M_{16}], \\ A^4_{211} = \dfrac{1}{45}(180\,\overline{p}_4 - 201\,\overline{p_3 p_1} - 100\,\overline{p}^2_2 + 212\,\overline{p_2 p}^2_1 - 55\,\overline{p}^4_1)\,[M_{16}], \\ A^4_{1111} = \dfrac{1}{81}(-165\,\overline{p}_4 + 193\,\overline{p_3 p_1} + 95\overline{p}^2_2 - 208\,\overline{p_2 p}^2_1 + 55\overline{p}^4_1)\,[M_{16}]. \end{cases}$$

First of all we deal with the case where $M_{16}\subset E_{19}$, In this case we have from (1. 7)

$$\overline{p}_2 = \overline{p}_3 = \overline{p}_4 = 0.$$

Hence we have from (3. 1)

$$(3.\ 2)\begin{cases} A^4_4 = -\dfrac{1}{9}\overline{p}^4_1[M_{16}],\ A^4_{31} = \dfrac{10}{21}\overline{p}^4_1[M_{16}],\ A^4_{22} = \dfrac{1}{5}\overline{p}^4_1[M_{16}], \\ A^4_{211} = -\dfrac{11}{9}\overline{p}^4_1[M_{16}],\ A^4_{1111} = \dfrac{55}{81}\overline{p}^4_1[M_{16}],\ \tau = \dfrac{62}{5\cdot 7\cdot 81}\overline{p}^4_1[M_{16}]. \end{cases}$$

Therefore we have the

THEOREM 6. *If an* M_{16} *is imbedded in the* E_{19}, *then its index is divisible by* 62 *and it admits the cobordism decomposition of the form*

$$(3.\ 3)\qquad M_{16}\sim\frac{\tau}{62}\{-315\,P_8(c) + 1350\,P_6(c)P_2(c) + 567\,P_4(c)^2$$

$$-3465\,P_4(c)P_2(c)^2 + 1925\,P_2(c)^4\}\qquad \text{mod } torsion.$$

Next we consider the case where $M_{16}\subset E_{21}$. In this case we have from (1. 7)

$$\overline{p}_3 = \overline{p}_4 = 0.$$

Eliminating $\overline{p}^2_2$, $\overline{p}_2\cdot\overline{p}^2_1$ and $\overline{p}^4_1$ from (3. 1) we have

$$(3.\ 4)\qquad \text{rank}\begin{pmatrix} 9\,A^4_4 & -2 & 4 & -1 \\ 21\,A^4_{31} & 18 & -39 & 10 \\ 25\,A^4_{22} & 11 & -20 & 5 \\ 45\,A^4_{211} & -100 & 212 & -55 \\ 81\,A^4_{1111} & 95 & -208 & 55 \end{pmatrix}\leqq 3$$

i. e.

$$(3.\ 5)\begin{cases} \text{(i)}\ \ 165\,A^4_4 + 56\,A^4_{31} + 50\,A^4_{22} + 15\,A^4_{211} = 0, \\ \text{(ii)}\ \ 110\,A^4_4 + 28\,A^4_{31} + 25\,A^4_{22} - 9\,A^4_{1111} = 0, \\ \text{(iii)}\ \ -55\,A^4_4 + 15\,A^4_{211} + 18\,A^4_{1111} = 0, \\ \text{(iv)}\ \ 30\,A^4_{211} + 27\,A^4_{1111} + 28\,A^4_{31} + 25\,A^4_{22} = 0. \end{cases}$$

Two of these equations are independent. From (iii) we see that A^4_4 is an integer which is divisible by 3 and hence A^4_{1111} is also an integer. We put as follows:

$$A^4_4 = 3\alpha \qquad (\alpha\text{: integer}). \tag{3. 6}$$

We have from (iii)

$$\begin{cases} A^4_{1111} = 55\,\alpha + 5\,\beta \\ A^4_{211} = -\,55\,\alpha - 6\,\beta \qquad (\beta\text{: integer}). \end{cases} \tag{3. 7}$$

Hence (iv) becomes

$$28\,A^4_{31} + 25A^4_{22} = 165\,\alpha + 45\,\beta. \tag{3. 8}$$

Solving (3. 8) we have

$$\begin{cases} A^4_{22} = 9\,(165\,\alpha + 45\,\beta) + 28\gamma, \\ A^4_{31} = -\,8\,(165\,\alpha + 45\,\beta) - 25\,\gamma \qquad (\gamma\text{: integer}). \end{cases} \tag{3. 9}$$

Thus we have the

THEOREM 7. *If an M_{16} is imbedded in the E_{21}, then it admits the cobordism decomposition of the form*:

$$\begin{aligned} M_{16} \sim \alpha\,\{&3P_8(c) - 8 \times 165\,P_6(c)P_2(c) + 9 \times 165\,P_4(c)^2 - 55\,P_4(c)P_2(c)^2 \\ &+ 55\,P_2(c)^4\} + \beta\{-\,360\,P_6(c)P_2(c) + 405\,P_4(c)^2 - 6P_4(c)P_2(c)^2 \\ &+ 5\,P_2(c)^4\} + \gamma\{-\,25\,P_6(c)P_2(c) + 28P_4(c)^2\} \qquad \text{mod } \textit{torsion} \end{aligned} \tag{3. 10}$$

where α, β and γ denote some integers.

In the case where $M_{16} \subset E_{23}$ we have from (1. 7)

$$\bar{p}_4 = 0.$$

Hence we have from (3. 1)

$$\begin{vmatrix} 9\,A^4_4 & -\,4 & -\,2 & 4 & -\,1 \\ 21\,A^4_{31} & 39 & 18 & -\,39 & 10 \\ 25\,A^4_{22} & 18 & 11 & -\,20 & 5 \\ 45\,A^4_{211} & -\,201 & -\,100 & 212 & -\,55 \\ 81\,A^4_{1111} & 193 & 95 & -\,208 & 55 \end{vmatrix} = 0 \tag{3. 11}$$

i. e.

$$55\,A^4_4 + 28\,A^4_{31} + 25\,A^4_{22} + 15\,A^4_{211} + 9\,A^4_{1111} = 0. \tag{3. 12}$$

Hence we have the

THEOREM 8. *If an M_{16} is imbedded in E_{23}, then its cobordism coefficients must satisfy* (3. 12), *in particular A^4_4 and A^4_{1111} are integers.*

COROLLARY 1. *$P_6(c)P_2(c)$, $P_4(c)^2$, $P_4(c)P_2(c)^2$ and $P_2(c)^4$ cannot be imbedded in the E_{23}.*

Next we consider the Cayley plane $W = F_4/\mathrm{Spin}(9)$ ([5], p. 534). Its cobordism coefficients are as follows: ([6])

$$A^4_4 = -\frac{28}{3},\ A^4_{31} = 36,\ A^4_{22} = 18,\ A^4_{21} = -92,\ A^4_{1111} = \frac{145}{3}. \tag{3. 13}$$

Thus the A^4_4 of W is not an integer. Hence we have the

COROLLARY 2. *The Cayley plane cannot be imbedded in the E_{23}.*

Next we consider the submanifolds of $P_9(c)$. Its Pontryagin class is given by

$$p = (1 + g^2)^{10},\ g \in H^2(P_9(c), Z). \tag{3. 14}$$

Let $v = \lambda g$ (λ: integer) correspond to the submanifold. Then we have from the formula given in [7]

$$\left\{\begin{aligned} &A^4_4 = \frac{\lambda}{9}(10 - \lambda^8),\ A^4_{211} = -\frac{2}{3}\lambda^3 + \frac{\lambda^5}{9} + \frac{16}{9}\lambda^7 - \frac{11}{9}\lambda^9, \\ &A^4_{31} = \frac{10\lambda}{21}(1 - \lambda^2)(1 - \lambda^6),\ A^4_{1111} = \frac{5\lambda^3}{81}(11\lambda^2 - 2)(\lambda^2 - 1)^2, \\ &A^4_{22} = \frac{1}{5}(\lambda - 2\lambda^5 + \lambda^9). \end{aligned}\right. \tag{3. 15}$$

We have from (3. 15) and (3. 12)

$$\frac{\lambda}{9}(715 - 220\lambda^2) = 0. \tag{3. 16}$$

This equation has no integral solution other than $\lambda = 0$.
Hence we have the

COROLLARY 3. *If an M_{16} is a submanifold of $P_9(c)$, then it cannot be imbedded in the E_{23}.*

The case $M_{20} \subset E_{20+q}$.

Though we have no concrete knowledge about the cobordism coefficients of M_{20}, the expression for the index of M_{20} is known ((1. 6)).
Hence, if $M_{20} \subset E_{23}$ we have from (1. 7)

$$\bar{p}_2 = \bar{p}_3 = \bar{p}_4 = \bar{p}_5 = 0.$$

Therefore we have from (1. 6) and (1. 8)

$$\tau = \frac{1382}{3^4 \cdot 5^2 \cdot 7 \cdot 11} \overline{p}_1^5 \, [M_{20}]. \qquad (3.17)$$

Thus we have the

THEOREM 9. *If an M_{20} is imbedded in the E_{23}, then its index is divisible by* 1382.

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UTSUNOMIYA UNIVERSITY, JAPAN.