EXTENSIONS OF RINGS OF OPERATORS ON HILBERT SPACES

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Introduction. In the development of the extension theory of rings of operators on a Hilbert space, the study of the direct product of factors[1] initiated by Murray and von Neumann [6], [7] has been developed successively for general rings of operators by Dixmier [1] and Misonou [5]. In recent years, we have brought the notion of the crossed product of division algebras or simple algebras into rings of operators, more particularly, finite factors and established the foundation of the theory of the crossed product. Subsequently it has been examined by Saitô [10], [12] and the other authors. The initial impetus which led us to these investigation was provided by the so-called factor construction created by Murray and von Neuman [6], [8]. Making use of the crossed product, we have seen that the factors of different algebraical types from the original one are constructed by varying the groups of automorphisms[2] [11], [14].

The purpose of this paper is to present a unified account of these development and to study the more general types of extensions of finite factors. Let $\mathbf{A}$ be a finite factor on a Hilbert space, then a factor $\mathbf{M}$ containing $\mathbf{A}$ as a subfactor is called an extension of $\mathbf{A}$. The first class of the extension $\mathbf{M}$ of $\mathbf{A}$ which is called the discrete extension was indicated by the classification of the dimension type for rings of operators, in particular, the class of type I. In the first step it will be shown that the discrete extension involves not only the $n \times n$ matrix algebra over $\mathbf{A}$, but also the crossed product of $\mathbf{A}$ by a certain group of automorphisms.

The second class of the extension $\mathbf{M}$ which is called the splitting extension of $\mathbf{A}$ by a group G is, roughly speaking, as follows; $\mathbf{M}$ is decomposed for the suitable topology in the form

$$\mathbf{M} = \sum_{\alpha \in G} \mathbf{A}\, U_\alpha,$$

where $\{U_\alpha\}_{\alpha \in G}$ is a unitary representation of G in $\mathbf{M}$ such that $U^*\mathbf{A}U \subset \mathbf{A}$. This class of the extension was inspired from the group extension theory and

1) A W^*-algebra means a weakly closed self-adjoint operator algebra with the identity on a Hilbert space and a factor means a W^*-algebra whose center consists of scalar multiples of the identity.

2) By an isomorphism between W^*-algebras, we always understand a $*$-isomorphism, i.e. an automorphism of a factor means a $*$-automorphism.

the theory of the crossed product is completely covered by the theory of the splitting extension. Actually, the theorems on the crossed product are extended to those on the splitting extension. In the final step our observation will be concentrated on the special splitting extension in which G is a cyclic group of order n. In the case where $n=2$ or 3, it is fully investigated by the employment of the concept of discrete extension.

1. Fundamental concepts and notations. Throughout this paper, we shall deal with only finite factors. Let $\mathbf{M}$ be a finite factor on a Hilbert space $\mathbf{H}$ and $\mathbf{A}$ a subfactor of $\mathbf{M}$, i.e., the identities of $\mathbf{M}$ and $\mathbf{A}$ coincide and we denote it by I, then we say that $\mathbf{M}$ is an extension of $\mathbf{A}$. That is, by an extension $\mathbf{M}$ of a factor $\mathbf{A}$, we always understand that $\mathbf{M}$ is a finite factor and $\mathbf{A}$ is a subfactor of $\mathbf{M}$, without confusions. It is very important to keep this convention in mind in what follows. The elements of $\mathbf{M}$ are, in general, denoted by $A, B, \cdots\cdots$, and in particular the elements of $\mathbf{A}$ are denoted by $a, b, \cdots\cdots$. An extension $\mathbf{M}$ of a factor $\mathbf{A}$ is said to be finite (infinite) dimensional if the commutant $\mathbf{A}'$ of $\mathbf{A}$ is finite (properly infinite) resp. and is simply called a finite (infinite) extension of $\mathbf{A}$ resp.. For a finite extension $\mathbf{M}$ of a factor $\mathbf{A}$ we shall define the so-called degree of the extension by $C_{\mathbf{A}}/C_{\mathbf{M}}$ where $C_{\mathbf{A}}$, $C_{\mathbf{M}}$ is the spatial invariant of $\mathbf{A}$, $\mathbf{M}$, resp., and denote it by $[\mathbf{M};\mathbf{A}]$[3]. A subfactor $\mathbf{B}$ of an extension $\mathbf{M}$ of a factor $\mathbf{A}$ such that $\mathbf{A}\subset\mathbf{B}\subset\mathbf{M}$ is called a subextension of the extension $\mathbf{M}$ of $\mathbf{A}$. Let $\mathbf{M}$, $\mathbf{N}$ be two extensions of a factor $\mathbf{A}$, we say that the extensions $\mathbf{M}$ and $\mathbf{N}$ are equivalent to each other if there exists an isomorphism of $\mathbf{M}$ and $\mathbf{N}$ which coincides on $\mathbf{A}$ with the identity automorphism.

Let $\mathbf{M}$ be an extension of a factor $\mathbf{A}$, then for each element A in $\mathbf{M}$, there exists uniquely an element A^{ε} in $\mathbf{A}$ such that

$$\tau(Ab)=\tau(A^{\varepsilon}b) \qquad \text{for all } b\in\mathbf{A},$$

where τ is a (normalized) faithful normal trace of $\mathbf{M}$. The mapping ε has been described largely in [2], [15] and is occasionally called the conditional expectation of $\mathbf{M}$ relative to $\mathbf{A}$. The mapping ε has the following fundamental properties :

(i) The mapping ε is a faithful, normal, positive linear mapping of $\mathbf{M}$ onto $\mathbf{A}$.

(ii) $a^{\varepsilon}=a$ for all $a\in\mathbf{A}$.

(iii) $(aAb)^{\varepsilon}=aA^{\varepsilon}b$ for all $a, b\in\mathbf{A}$, $A\in\mathbf{M}$.

(iv) $(AB)^{\varepsilon}=(BA)^{\varepsilon}$ for all $A\in\mathbf{M}$, $\mathbf{B}\in\mathbf{A}'\cap\mathbf{M}$.

Since the faithful normal trace τ of $\mathbf{M}$ is unique, such a mapping ε is uniquely determined by $\mathbf{M}$ and $\mathbf{A}$. Using the mapping ε, we shall introduce the

3) For the spatial invariant C_A of a W^*-algebra $\mathbf{A}$, see [9]. It is called "fonction de liaison" in [1].

notion of "inner product type" on $\mathbf{M}$ associated to $\mathbf{A}$ as follows:

$$[A, B] = (AB^*)^{\natural}.$$

Then we say that A and B in $\mathbf{M}$ are mutually orthogonal over $\mathbf{A}$ if $[A, B] = 0$ (trivially $[B, A] = 0$), and the set $\{A \in \mathbf{M}; [a, A] = 0 \text{ for all } a \in \mathbf{A}\}$ is denoted by $\mathbf{A}^{\perp}$. In connection with this, we mention the fact that $\mathbf{M}$ is uniquely decomposed in the form $\mathbf{M} = \mathbf{A} \oplus \mathbf{A}^{\perp}$ as seen in [2].

An extension $\mathbf{M}$ of a factor $\mathbf{A}$ is regarded as a module over $\mathbf{A}$ with the $\mathbf{A}$-valued inner product [,] and so we use the term "base" as follows. By a base of $\mathbf{M}$ over $\mathbf{A}$, we understand a family $\{U_\alpha\}_{\alpha \in \Lambda}$ of unitary elements in $\mathbf{M}$ containing the identity I such that U_α are mutually orthogonal over $\mathbf{A}$ and for each element A in $\mathbf{M}$ there exists a family $\{a_\alpha\}_{\alpha \in \Lambda}$ in $\mathbf{A}$ such that $A = \sum_{\alpha \in \Lambda} a_\alpha U_\alpha$, where $\sum$ is taken in sense of the metrical convergence[4]. Here it is easy to see that a family $\{a_\alpha\}_{\alpha \in \Lambda}$ is unique for each $A \in \mathbf{M}$. In connection with this, there arises a significant problem whether there exists a base or not for any extension of a factor, but we do not concern with this in the present paper, because we study only the certain class of the extension. We shall provide a lemma by which the natural connection between the base and the degree is illustrated and which will be of use to us later.

LEMMA 1. *If a finite extension* $\mathbf{M}$ *of a factor* $\mathbf{A}$ *has a base, the following statements for a family* $\mathfrak{U} = \{U_\alpha\}_{\alpha \in \Lambda}$ *of unitary elements in* $\mathbf{M}$ *are equivalent to each other.*

(1) $\mathfrak{U}$ *is a base.*

(2) *The elements in* $\mathfrak{U}$ *are mutually orthogonal over* $\mathbf{A}$ *and the number of* Λ *equals to* $[\mathbf{M}; \mathbf{A}]$.

PROOF. Note first that the degree $[\mathbf{M}; \mathbf{A}]$ is invariant by the ampliation of $\mathbf{M}$, we may assume to be $C_{\mathbf{M}} \geqq 1$. Now, select a projection E' of $\mathbf{M}'$ such that $D'(E') = 1/C_M$ for the dimension function D' of $\mathbf{M}'$, we have from [1; Chap. III, p.282] that $C_{\mathbf{M}_{E'}} = C_{\mathbf{M}} \cdot D'(E') = 1$ and $C_{\mathbf{A}_{E'}} = C_{\mathbf{A}} \cdot D'(E') = [\mathbf{M}; \mathbf{A}]$. Thus we may confine ourselves to the case where $C_{\mathbf{M}} = 1$ and $C_{\mathbf{A}} = [\mathbf{M}; \mathbf{A}]$. Let x be a separating trace vector of $\mathbf{M}$ (that is, of $\mathbf{A}$), and let $\mathbf{D}, \mathbf{D}'$ be the dimension functions of $\mathbf{A}$, $\mathbf{A}'$ resp. in what follows then, since $D([\mathbf{A}'x]) = 1$, $C_{\mathbf{A}} = 1/D'([\mathbf{A}x])$. If the elements in $\mathfrak{U}$ are mutually orthogonal over $\mathbf{A}$, $[\mathbf{A}U_\alpha x]_{\alpha \in \Lambda}$ are evidently mutually orthogonal, and further mutually equivalent. In fact, $C_{\mathbf{M}} = 1$ yields that there exists a unitary element U'_α in $\mathbf{M}'$, that is, in $\mathbf{A}'$ such that $U_\alpha x = U'_\alpha x$ for each $\alpha \in \Lambda$. From the above comments the equi-

4) Let τ be the faithful normal trace of $\mathbf{M}$, we define $[[A]] = \sqrt{\tau(A^*A)}$ for each $A \in \mathbf{M}$. Then $\mathbf{M}$ becomes a new topological space with a purely algebraical metric [[ ]].

valence of our statements is easily deduced. (1) → (2); The fact that $\mathcal{U}$ is a base implies that $[\mathbf{A}U_\alpha x]_{\alpha\in\Lambda}$ are mutually equivalent, orthogonal and $\mathbf{H} = \sum_{\alpha\in\Lambda} [\mathbf{A}U_\alpha x]$, since x is generating for $\mathbf{M}$. Namely, since $\mathbf{A}'$ is finite, the number of Λ is finite and equals to $1/D'([\mathbf{A}x]) = C_A = [\mathbf{M}; \mathbf{A}]$. (2) → (1); $D'(\sum_{\alpha\in\Lambda}[\mathbf{A}U_\alpha x])$

$=\sum_{\alpha\in\Lambda} D'[\mathbf{A}U_\alpha x]) = [\mathbf{M};\mathbf{A}]\cdot D'([\mathbf{A}x]) = C_A\cdot D'([\mathbf{A}x]) = 1$ yields $\mathbf{H} = \sum_{\alpha\in\Lambda}[\mathbf{A}U_\alpha x]$, and so $\mathcal{U}$ is a base.

We make use the notion indicated from the minimal projection of a factor of type I. A non-zero projection E in $\mathbf{M}$ is called an $\mathbf{A}$-projection if there exists an element a in $\mathbf{A}$ such that $EAE = aE$ for each $A \in \mathbf{M}$. In this case, obviously $EAE = EaE = aE$ and there exists, of course, an element a in $\mathbf{A}$ such that $EAE = Ea$. We shall take a special interest in the extensions having such a projection which is expounded in the following section.

Among the special extension there exists the direct product of factors. This extension will occupy the basic rôle in our exposition. For this, we refer to [1], [5], but certain notations are used on this account. Let $\mathbf{A}$ be a factor on a Hilbert space $\mathbf{H}$ and let S be an arbitrary set, then each vector of the direct product $\mathfrak{H} = \mathbf{H} \otimes l_2(S)$ of $\mathbf{H}$ and $l_2(S)$ is expressed in the form $\sum_{\alpha\in S} x_\alpha \otimes \varepsilon_\alpha$, where x_α are vectors of $\mathbf{H}$ such that $\sum_{\alpha\in S} \|x_\alpha\|^2$ is finite and $\{\varepsilon_\alpha\}_{\alpha\in S}$ is a complete orthonormal system of $l_2(S)$. An operator $a \otimes I (a \in \mathbf{M})$ means an operator on $\mathfrak{H}$ defined by

$$(a \otimes I)\left(\sum_{\alpha\in S} x_\alpha \otimes \varepsilon_\alpha\right) = \sum_{\alpha\in S} a x_\alpha \otimes \varepsilon_\alpha.$$

Then $a \to a \otimes I$ is an isomorphism of $\mathbf{A}$ into the full operator ring on $\mathfrak{H}$ which is called the ampliation of $\mathbf{A}$, and a set of operators $a \otimes I$ is a W^*-algebra on $\mathfrak{H}$, denoted by $\mathbf{A} \otimes \mathbf{I}$. Furthermore, $\mathbf{A} \otimes \mathbf{B}$ is frequently denoted by $\mathbf{A} \otimes \mathrm{I}_n$ in the case where $\mathbf{B}$ is a factor of type I_n. Let τ be a faithful normal trace of the direct product of two factors $\mathbf{A}$ and $\mathbf{B}$, then there exist faithful normal traces τ_1, τ_2 of $\mathbf{A}$, $\mathbf{B}$ resp. such that $\tau(a \otimes b) = \tau_1(a)\tau_2(b)$ for $a \in \mathbf{A}$, $b \in \mathbf{B}$. According to this fact, the following lemma relative to the mapping ε in $\mathbf{A} \otimes \mathbf{B}$ is directly verified.

LEMMA 2. *Let $\mathbf{A}_0$, $\mathbf{B}_0$ be subfactors of $\mathbf{A}, \mathbf{B}$ respectively and let $\varepsilon, \varepsilon_1, \varepsilon_2$ the conditional expectations of $\mathbf{A} \otimes \mathbf{B}$, $\mathbf{A}, \mathbf{B}$ relative to $\mathbf{A}_0 \otimes \mathbf{B}_0$, $\mathbf{A}_0$, $\mathbf{B}_0$ respectively, then for all $a \in \mathbf{A}, b \in \mathbf{B}$*

$$(a \otimes b)^{\varepsilon} = a^{\varepsilon_1} \otimes b^{\varepsilon_2}.$$

2. Discrete extensions. The present section is devoted to introduce an extension of discrete type which is taken an interest in the simple structure.

Actually, it will be shown that the $n \times n$ matrix algebra over a factor is the special case of such an extension.

DEFINITION 1. An extension $\mathbf{M}$ of a factor $\mathbf{A}$ is said to be *discrete* over $\mathbf{A}$ if there exists a family of mutually orthogonal $\mathbf{A}$-projections $\{E_i\}$ $(i = 1, 2, \cdots\cdots, n)$ in $\mathbf{M}$ such that $I = \sum_{i=1}^{n} E_i$ and $(E_1 aE_1)^{\varepsilon} = (E_2 aE_2)^{\varepsilon} = \cdots\cdots = (E_n aE_n)^{\varepsilon}$ for all $a \in \mathbf{A}$.

Here, n is called the order of the discrete extension $\mathbf{M}$ and a family $\{E_i\}$ is called a discrete base of $\mathbf{M}$ over $\mathbf{A}$.

Before being shown an example of the discrete extension, there will be deduced the basic properties of the discrete base $\{E_i\}$ $(i = 1, 2, \ldots\ldots n)$ which is of frequent use. Let $\mathbf{D}_0$ be a W^*-algebra generated by elements in $\mathbf{M}$ commuting with all $E_i (i = 1, 2, \cdots\cdots, n)$, then we obtain

LEMMA 3. *Let $\{E_i\}$ $(i = 1, 2, \ldots\ldots, n)$ be the discrete base of the discrete extension $\mathbf{M}$ of $\mathbf{A}$, then an element $a_i \in \mathbf{A}$ such that $E_i A E_i = a_i E_i$ is uniquely determined for each $A \in \mathbf{M}$ and for each i, and lies in $\mathbf{A} \cap \mathbf{D}_0$.*

PROOF. First we shall prove the uniqueness of a_i. In fact, if $a_i E_i = 0$, $E_i a_i^* a_i E_i = 0$, and so $(E_k a_i^* a_i E_k)^{\varepsilon} = 0$ for all k under our assumption. Since the mapping ε is faithful, $E_k a_i^* a_i E_k = 0$. Hence, for the faithful normal trace τ of $\mathbf{M}$,

$$\tau(a_i^* a_i) = \sum_{k=1}^{n} \tau(a_i^* a_i E_k) = \sum_{k=1}^{n} \tau(E_k a_i^* a_j E_k) = 0,$$

so that $a_i^* a_i = 0$, $a_i = 0$.

Next, let A be an element in $\mathbf{A}$. Using the assumption $(E_1 AE_1)^{\varepsilon} = \cdots\cdots = (E_n AE_n)^{\varepsilon}$ and in particular $E_1^{\varepsilon} = E_2^{\varepsilon} = \cdots\cdots = E_n^{\varepsilon} = \dfrac{1}{n} I$, we see that the equality $(E_i AE_i)^{\varepsilon} = a_i E_i^{\varepsilon}$ implies $a_1 = a_2 = \cdots\cdots = a_n$ (denoted by a). Thereby $E_i AE_i = aE_i$ for all i. Summing up for $i = 1, 2, \cdots\cdots, n$, we obtain $a = \sum_{i=1}^{n} E_i AE_i \in \mathbf{D}_0 \cap \mathbf{A}$. Now, let A be any element in $\mathbf{M}$. Applying the above fact for an element a_i in $\mathbf{A}$ such that $E_i AE_i = a_i E_i$, we know that there exists an element a_i' in $\mathbf{A} \cap \mathbf{D}_0$. such that $a_i E_i = E_i a_i E_i = a_i' E_i$. By the first part of our proof, $a_i = a_i' \in \mathbf{A} \cap \mathbf{D}_0$. This completes the proof.

Clearly, the projections E_i of the discrete base except the trivial case $n = 1$ do not belong to $\mathbf{A}$ and it will be easily seen that E_i are minimal with respect to $\mathbf{A}$ in the sense that there is no non-zero projection e in $\mathbf{A}$ such that $e < E_i$.

LEMMA 4. *A projection $E \neq I$ in $\mathbf{M}$ such that $E^{\varepsilon} = \lambda I$ $(\lambda < 1)$ is minimal with respect to $\mathbf{A}$.*

PROOF. Assume that there exists a projection e in $\mathbf{A}$ with $e < E$, then $E-e>0$ implies $(E-e)^{\varepsilon}=E^{\varepsilon} - e = \lambda I - e > 0$. This means $<ex, x> \leqq <\lambda x, x>$ for all vector $x \in \mathbf{H}$, and so $\|ex\|^2 \leqq \lambda \ \|x\|^2$ for all vectors $x \in \mathbf{H}$. Since $\lambda < 1$, this contradicts to $\| e \| = 1$.

Now, we shall observe that the $n \times n$ matrix algebra over $\mathbf{A}$ is the most special discrete extension of $\mathbf{A}$.

THEOREM 1. *Let* $\mathbf{A}$ *be a factor, then an extension* $\mathbf{A} \otimes \mathrm{I}_n$ *is discrete over* $\mathbf{A} \otimes \mathbf{I}$ *with the order* n.

PROOF. Choose a family of mutually orthogonal minimal projections $\{e_i\}$ $(i = 1, 2, \cdots\cdots, n)$ in factor $\mathbf{B}$ of type I_n and put $E_i = I \otimes e_i$, then it may be shown that $\{E_i\}$ $(i = 1, 2, \cdots\cdots, n)$ is a discrete base over $\mathbf{A} \otimes \mathbf{I}$. For any element $a \otimes b$ in $\mathbf{A} \otimes \mathbf{B}$, we obtain $(I \otimes e_i)\,(a \otimes b)(I \otimes e_i) = a \otimes e_i b e_i = a \otimes \lambda e_i = \lambda a \otimes e_i = (\lambda a \otimes I)\,(I \otimes e_i)$ where λ is a scalar and $\lambda a \otimes I$ lies in $\mathbf{A} \otimes \mathbf{I}$. It follows that there exists an element a_i in $\mathbf{A}$ such that $E_i A E_i = a_i E_i$ for each $A \in \mathbf{A} \otimes \mathbf{B}$ and for each i. Moreover, using Lemma 2,

$$[(I \otimes e_i)\,(a \otimes I)(I \otimes e_i)]^{\varepsilon} = (a \otimes e_i)^{\varepsilon} = a^{\varepsilon_1} \otimes e_i^{\varepsilon_2} = a \otimes \frac{1}{n} I$$

where $\varepsilon_1, \varepsilon_2$ means the expectation of $\mathbf{A}, \mathbf{B}$ relative to $\mathbf{A}$, the center (λI) of $\mathbf{B}$ respectively. Hence it holds that $(E_1 a E_1)^{\varepsilon} = (E_2\, a E_2)^{\varepsilon} = \cdots\cdots = (E_n a E_n)^{\varepsilon}$ for all $a \in \mathbf{A} \otimes \mathbf{I}$. Therefore, we have seen that $\{E_i\}$ $(i= 1,2,\cdots\cdots,n)$ is a discrete base of $\mathbf{A} \otimes \mathrm{I}_n$ over $\mathbf{A} \otimes \mathbf{I}$.

Now, although we have seen a typical example of the discrete extension, it is exceedingly desirable to show the existence of the discrete extension which is defferent from the extension as in Theorem 1.

LEMMA 5. *Let* $\mathbf{M}$ *be an extension of a factor* $\mathbf{A}$ *with* $[\mathbf{M}\,;\,\mathbf{A}] = 2$, *then there exists a projection* E *in* $\mathbf{M}$ *such that* $E^{\varepsilon} = \frac{1}{2} I$ *and* $\mathbf{A}^{\perp} = \mathbf{A}(2E - I)$.

PROOF. From the proof of Lemma 1, we may assume that $C_M = 1$ and $C_A = 2$. Let x be a normalized separating trace vector of $\mathbf{M}$, then there exists a projection E in $\mathbf{M}$ such that $D(E) = \frac{1}{2}$ and Ex is a separating trace vector of $\mathbf{A}$ as seen in the proof of [3; Theorem 1]. Thus, for all $a \in \mathbf{A}$, there is a scalar λ such that $<Eax, x> = <aEx, Ex> = <\lambda\, ax, x>$. Here, since $D(E) = \frac{1}{2}$, $\lambda = \frac{1}{2}$. On the other hand, $< Eax, x> = <E^{\varepsilon}ax, x>$ for all $a \in \mathbf{A}$. Therefore $< \frac{1}{2} ax, x> = <E^{\varepsilon}ax, x>$ for all $a \in \mathbf{A}$ implis $E^{\varepsilon} = \frac{1}{2} I$. Since $(2E - I)^{\varepsilon} = 0$, we see that the unitary elements I and $2E - I$ in $\mathbf{M}$ are mutually orthogonal over $\mathbf{A}$. It follows from Lemma 1 that $\{I, 2E - I\}$ is a base of $\mathbf{M}$ over $\mathbf{A}$, that is to say, $\mathbf{A}^{\perp} = \mathbf{A}(2E - I)$.

THEOREM 2. *Let* $\mathbf{M}$ *be an extension of a factor* $\mathbf{A}$ *with* $[\mathbf{M}; \mathbf{A}] = 2$, *then* $\mathbf{M}$ *is discrete over* $\mathbf{A}$ *with the order* 2.

PROOF. $\mathbf{M}$ is decomposed in the form $\mathbf{M} = \mathbf{A} \oplus \mathbf{A}^{\perp}$ as described in the section 1. Choosing a projection E in $\mathbf{M}$ such that $E^{\varepsilon} = \frac{1}{2} I$ by Lemma 5, we obtain $\mathbf{M} = \mathbf{A} \oplus \mathbf{A}(2E - I)$. That is to say, each $A \in \mathbf{M}$ is uniquely written in the form $A = a_1 + a_2 E\,(a_1, a_2 \in \mathbf{A})$. Hence $AE = (a_1 + a_2)E$ for each $A \in \mathbf{M}$. This shows that there exists an element $a \in \mathbf{A}$ such that $EAE = aE$ for each $A \in \mathbf{M}$ by considering EA as the preceding A. That is, E is an $\mathbf{A}$-projection in $\mathbf{M}$. Moreover, since $(I - E)^{\varepsilon} = \frac{1}{2} I$, the same argument shows that $I - E$ is an $\mathbf{A}$-projection in $\mathbf{M}$. Now it remains only to show that $[EaE]^{\varepsilon} = [(I - E)a(I - E)]^{\varepsilon}$ for each $a \in \mathbf{A}$. The following equality is directly computed;

$$a = [E + (I - E)]a[E + (I - E)] = (I - E)a(I - E) - EaE + aE + Ea.$$

Using the mapping $\mathcal{E}$, we obtain

$$a = a^{\varepsilon} = [(I - E)a(I - E)]^{\varepsilon} - [EaE]^{\varepsilon} + a.$$

Consequently $[EaE]^{\varepsilon} = [(I - E)a(I - E)]^{\varepsilon}$ for each $a \in \mathbf{A}$. We now established that $\{E, I - E\}$ is a discrete base of $\mathbf{M}$ over $\mathbf{A}$ as desired.

We ought to mention that the discrete extension in Theorem 2 is different from that in Theorem 1. This follows evidently from the fact that $\mathbf{A}' \cap \mathbf{M} = (\lambda I)$ as regarded in the section 4. Proceeding as in Lemma 1, 2, we shall examine the structure of the discrete extension.

LEMMA 6. *Let* $\mathbf{A}_0$ *be the intersection of* $\mathbf{A}$ *and* $\mathbf{D}_0$, *then the discrete extension* $\mathbf{M}$ *of* $\mathbf{A}$ *is isomorphic to* $\mathbf{A}_0 \otimes \mathrm{I}_n$.

This is quickly concluded from Lemma 3 and we see that the $n \times n$ matrix algebra over $\mathbf{A}$ i.e. the discrete extension in Theorem 1 is nothing but the case where $\mathbf{A} = \mathbf{A}_0$.

PROOF. By Lemma 3, we have already known that $E_1 \mathbf{M} E_1 = \mathbf{A}_0 E_1$. If we correspond for each $E_1 A E_1 \in \mathbf{M}$ an element $a \in \mathbf{A}_0$ such that $E_1 A E_1 = aE_1$, we get an isomorphism of $E_1 \mathbf{M} E_1$ onto $\mathbf{A}_0$. However, $E_1 \mathbf{M} E_1$ is isomorphic to $\mathbf{M}_{E_1}$ and $\mathbf{M}$ is isomorphic to $\mathbf{M}_{E_1} \otimes \mathrm{I}_n$ where $\mathbf{M}_{E_1}$ is the restriction of $\mathbf{M}$ on the range of E_1. Consequently, $\mathbf{M}$ is isomorphic to $\mathbf{A}_0 \otimes \mathrm{I}_n$.

COROLLARY. *The discrete extension* $\mathbf{M}$ *of* $\mathbf{A}$ *is of type I, type II if* $\mathbf{A}$ *is of type I, type II respectively.*

In fact, a subfactor of type I_n does not contain a subfactor of type II as seen in the proof of Lemma 8(2)

DEFINITION 2. An extension $\mathbf{M}$ of a factor $\mathbf{A}$ is *solvable over* $\mathbf{A}$ if there exists a chain of subfactor $\mathbf{A}_i$ of $\mathbf{M}$ with the following properties;

(i) $\mathbf{A} = \mathbf{A}_0 \subset \mathbf{A}_1 \subset \mathbf{A}_2 \subset \cdots\cdots \subset \mathbf{M}$.

(ii) $\mathbf{M}$ is generated by $\mathbf{A}_0, \mathbf{A}_1 \cdots\cdots$ i. e. $\mathbf{M} = \mathbf{R}(\mathbf{A}_i;\ i = 0, 1, \cdots\cdots)$.

(iii) For each i, $\mathbf{A}_{i+1}$ is discrete over $\mathbf{A}_i$.

In the final step of this section, we shall generalize Theorem 1 as indicated from the above definition. It is essentially suggested by the following Lemma.

LEMMA 7. *Let* $\mathbf{A}, \mathbf{B}$ *be two factors of type* I_p, I_q *respectively such that* $\mathbf{A} \subset \mathbf{B}$, *then* $\mathbf{B}$ *is discrete over* $\mathbf{A}$.

PROOF. Let D_A, D_B be the dimension functions of $\mathbf{A}, \mathbf{B}$ respectively, then $D_A(E) = D_B(E)$ for all $E \in \mathbf{A}$ by the uniqueness of the dimension function. Hence, the range $\left\{0, \frac{1}{p}, \cdots\cdots, 1\right\}$ of D_A is contained in the range $\left\{0, \frac{1}{q}, \frac{2}{q} \cdots\cdots 1\right\}$ of $D_B(E)$, so that p is a divisor of q, i. e. $q = pr$, $r = 1, 2, \cdots\cdots$. Now choose mutually orthogonal minimal projections $e_1, e_2, \cdots\cdots, e_p$ in $\mathbf{A}$ such that $I = \sum_{i=1}^{p} e_i$, and put $\mathbf{C} = \mathbf{B}_{e_1}$, $\mathbf{C}$ is of type I_r and there exists a spatial isomorphism Φ of $\mathbf{B}$ onto $\mathbf{C} \otimes \mathrm{I}_p$ which maps $\mathbf{A}$ onto $\mathbf{I} \otimes \mathrm{I}_p$. By Theorem 1, $\mathbf{C} \otimes \mathrm{I}_p$ is discrete over $\mathbf{I} \otimes \mathrm{I}_p$ since $\mathbf{C}$ is of type I_r. Let us keep in mind that the transposition $\{\Phi(E_i)\}$ of a discrete base $\{E_i\}$ $(i = 1, 2, \cdots\cdots, q)$ of $\mathbf{C} \otimes \mathrm{I}_p$ over $\mathbf{I} \otimes \mathrm{I}_p$ forms a discrete base of $\mathbf{B}$ over $\mathbf{A}$, we conclude that $\mathbf{B}$ is discrete over $\mathbf{A}$.

THEOREM 3. *Let* $\mathbf{A}$ *be a factor and let* $\mathbf{B}$ *a hyperfinite factor, then* $\mathbf{A} \otimes \mathbf{B}$ *is solvable over* $\mathbf{A} \otimes \mathbf{I}$.

PROOF. It needs only to prove in the case where $\mathbf{B}$ is of type II. That is, there exists a chain of subfactors $\mathbf{B}_i$ such that $\mathbf{B}_1 \subset \mathbf{B}_2 \subset \cdots\cdots \subset \mathbf{B}, \mathbf{B} = \mathbf{R}(\mathbf{B}_i;$ $i = 1, 2, \cdots\cdots)$ and each $\mathbf{B}_i$ is of type I_{2^i}. Thus, we obtain a chain of subfactors $\mathbf{A} \otimes \mathbf{B}_i$ in $\mathbf{A} \otimes \mathbf{B}$ such that $\mathbf{A} \otimes \mathbf{I} = \mathbf{A} \otimes \mathbf{B}_0 \subset \mathbf{A} \otimes \mathbf{B}_1 \subset \cdots\cdots \subset \mathbf{A} \otimes \mathbf{B}$ and $\mathbf{A} \otimes \mathbf{B} = \mathbf{R}(\mathbf{A} \otimes \mathbf{B}_i; i = 1, 2, \cdots\cdots)$. Lemma 7 implies that $\mathbf{B}_i$ is discrete over $\mathbf{B}_{i-1}$ for each i. Hence, we can select a discrete base $\{E_1, E_2\}$ of $\mathbf{B}_i$ over $\mathbf{B}_{i-1}$, and then it is directly verified that $\{I \otimes E_1, I \otimes E_2\}$ gives a discrete base of $\mathbf{A} \otimes \mathbf{B}_i$ over $\mathbf{A} \otimes \mathbf{B}_{i-1}$.

Finally, we take an interest in the fact that our observation illustrate the structure of the hyperfinite factor of type II.

THEOREM 4. *For each positive integer* n, *the hyperfinite factor* $\mathbf{M}$ *of type II contains a subfactor* $\mathbf{A}$ *of type* I_n *such that* $\mathbf{M}$ *is solvable over* $\mathbf{A}$. *Conversely, if a factor* $\mathbf{M}$ *of type II is solvable over a factor* $\mathbf{A}$ *of type* I_n *for some* n, $\mathbf{M}$ *is hyperfinite.*

PROOF. Let $\mathbf{B}$ be an arbitrary subfactor of type I_n, then we obtain by a repeated application of [7; Lemma 4, 2, 2] a chain of subfactors $\mathbf{B}_i$ such that $\mathbf{B} = \mathbf{B}_0 \subset \mathbf{B}_1 \subset \cdots\cdots \subset \mathbf{M}$ and (i) each $\mathbf{B}_i$ is of type I_{p_i} (ii) p_{i+1} is divisible by p_i. Obviously, $\mathbf{N} = \mathbf{R}(\mathbf{B}_i;\ i = 0, 1, 2 \cdots\cdots) \subset \mathbf{M}$ and $\mathbf{N}$ is hyperfinite. Thus, since there exists an isomorphism of $\mathbf{N}$ onto $\mathbf{M}$ (cf. [7; Theorem XIV]), we obtain a chain of subfactors $\mathbf{A}_i$ of $\mathbf{M}$ such that $\mathbf{A} = \mathbf{A}_0 \subset \mathbf{A}_1 \subset \cdots\cdots \subset \mathbf{M}, \mathbf{M} = \mathbf{R}(\mathbf{A}_i;\ i = 0, 1, 2, \cdots\cdots)$ and further the properties (ii) (iii) as above hold. By Lemma 7, $\mathbf{A}_{i+1}$ is discrete over $\mathbf{A}_i$ for each i and so $\mathbf{M}$ is solvable over $\mathbf{A}$ of type I_n. Next, suppose that $\mathbf{M}$ is solvable over a factor $\mathbf{A}$ of type I_n for some n, then there is a chain of subfactors $\mathbf{B}_i$ of $\mathbf{M}$ such that $\mathbf{A} = \mathbf{B}_0 \subset \mathbf{B}_1 \subset \cdots\cdots \subset \mathbf{M}$, $\mathbf{M} = \mathbf{R}(\mathbf{B}_i;\ i = 0, 1, 2, \cdots\cdots)$ and $\mathbf{B}_{i+1}$ is discrete over $\mathbf{B}_i$ for each i. By Lemma 6, Cor., we see that $\mathbf{B}_i$ is of type I_{p_i}. Hence $\mathbf{M}$ is hyperfinite.

3. Splitting extensions. In this section, we shall study the certain class of the extension, which is the more general extension than the crossed product of rings of operators developed in [13], [14], The splitting extension we are going to define was inspired from the extension theory of the group (cf. [4]). Some of the elementary properties of such an extension are obtained as more general theorems than those on the crossed product.

DEFINITION 3. An extension $\mathbf{M}$ of a factor $\mathbf{A}$ is called a *splitting extension* of $\mathbf{A}$ by a group G if there exists a faithful unitary representation $\{U_\alpha\}_{\alpha\in G}$ in $\mathbf{M}$ of G with the following properties;

(1) $\{U_\alpha\}_{\alpha\in G}$ is a base of $\mathbf{M}$ over $\mathbf{A}$.

(2) For each $\alpha \in G$ and $a \in \mathbf{A}$ there is an element a^α in $\mathbf{A}$ such that $aU_\alpha = U_\alpha a^\alpha$.

We note the basic properties of the splitting extension derived immediately from the definition.

(1) Any element A in $\mathbf{M}$ is uniquely expressed in the form

$$A = \textstyle\sum_{\alpha\in G} a_\alpha U\ ,$$

where $\{a^\alpha\}_{\alpha\in G}$ is a family of elements in $\mathbf{A}$ and $\sum$ is taken in the sense of the metrical convergence. The condition (1) may be replaced by the following statement: $U_\alpha(\alpha \in G)$ are mutually orthogonal over $\mathbf{A}$ and $\mathbf{M}$ is generated by $\mathbf{A}$ and $\{U_\alpha\}_{\alpha\in G}$. Indeed, this fact is seen by the similar proof to [13; Theorem 1].

(2) G is homomorphic to the group $\widetilde{G}$ of automorphisms of $\mathbf{A}$ which are induced by $\{U_\alpha\}_{\alpha\in G}$ by the property (2). In particular, if $\widetilde{G}$ is isomorphic to G, such a splitting extension is called the crossed product of $\mathbf{A}$ by G, which is not essentially distinct from one in [13] (cf. The section 4).

The initial stage in our discussion is to determine the type of the splitting extension. It is completely answered.

LEMMA 8. *Let* $\mathbf{M}$ *be a splitting extension of* $\mathbf{A}$ *by a group* G.

(1) *Let* $\mathbf{A}$ *be a factor of type I.*

(I_1) *If* G *is finite,* $\mathbf{M}$ *is of type I.*

(I_2) *If* G *is infinite,* $\mathbf{M}$ *is of type II.*

(I_3) *If* G *is countable and locally finite,* $\mathbf{M}$ *is hyperfinite.*

(2) *If* $\mathbf{A}$ *is of type II,* $\mathbf{M}$ *is of type II.*

PROOF. (I_1) Choose a set of an ordinary matrix units $\{W_{ij}\}$ $(i,j=1,2,\ldots\ldots,n)$ of $\mathbf{A}$, each element $a\in\mathbf{A}$ is uniquely expressed in the form $\sum_{i,j=1}^{n}\lambda_{ij}W_{ij}$. Namely, each element $A\in\mathbf{M}$ is expressed in the form;

$$\sum_{i,j,k}\lambda_{ijk}W_{ij}U_k \qquad (k\in G).$$

Hereupon, the system $\{W_{ij}U_k\}$ is linearly independent in $\mathbf{M}$. In fact, $\sum_{i,j,k}\lambda_{ijk}W_{ij}U_k=0$ yields $\left(\sum_{i,j}\lambda_{ijk}W_{ij}\right)U_k=0$ for each $k\in G$, and so $\sum_{i,j}\lambda_{ijk}W_{ij}=0$, $\lambda_{ijk}=0$, for each $i.j,k$. Thus, $\mathbf{M}$ is of finite rank. Since $\mathbf{M}$ is a factor, this means that $\mathbf{M}$ is of type I.

(I_2) Let S be any finite set in an infinite group G, then $\{U_\alpha\}_{\alpha\in S}$ is linearly independent in $\mathbf{M}$. Hence $\mathbf{M}$ is impossible to be of finite rank and so $\mathbf{M}$ is of type II.

(I_3) We may consider only the case where G is countably infinite and locally finite. Then G is generated by a family of finite subgroups G_n such that $G_1\subset G_2\subset\cdots\cdots\subset G$. Therefore, it follows from (I_1) that $\mathbf{M}$ is generated by a family of subfactors $\mathbf{M}_n$ of type I_{p_n} such that $\mathbf{M}_1\subset\mathbf{M}_2\subset\cdots\cdots\subset\mathbf{M}$. This proves that $\mathbf{M}$ is hyperfinite.

(2) Since $\mathbf{A}$ is of type II, we choose a strictly monotone decreasing infinite directed set of projections $\{e_i\}_{i\in I}$ in $\mathbf{A}$. If $\mathbf{M}$ is of type I, $\mathbf{M}$ is considered to be the ring of all bounded operators on n-dimensional Hilbert space and then $\{e_i\}_{i\in I}$ is impossible to be infinite, strictly decreasing. Thus $\mathbf{M}$ is of type II.

REMARK. Let $\mathbf{B}$ be a factor generated by the regular representations $\{U_\alpha\}_{\alpha\in G}$ of a group G with the unit e whose non-trivial conjugate classes are infinite (cf. [7]), then the direct product $\mathbf{A}\otimes\mathbf{B}$ of factors $\mathbf{A}$ and $\mathbf{B}$ is a splitting extension of $\mathbf{A}\otimes\mathbf{I}$ by G. Indeed, let τ be a faithful normal trace of $\mathbf{A}\otimes\mathbf{B}$ and let τ_1, τ_2 faithful normal traces of $\mathbf{A}$ and $\mathbf{B}$, then $\tau((a\otimes I)(I\otimes u_\alpha))=\tau(a\otimes u_\alpha)=\tau_1(a)\tau_2(u_\alpha)=0$ $(\alpha\neq e)$ since $\tau_2(u_\alpha)=0$. Therefore, setting $U_\alpha=I\otimes u_\alpha$, we know that $\{U_\alpha\}_{\alpha\in G}$ is a base of $\mathbf{A}\otimes\mathbf{B}$ over $\mathbf{A}\otimes\mathbf{I}$ and $U_\alpha(a\otimes\mathrm{I})=(a\otimes I)U_\alpha$ for all $\alpha\in G$, that is, it is nothing but the case where all automorphisms of $\mathbf{A}\otimes\mathbf{I}$ induced by U_α are the identity one.

In examining the structure of the splitting extension $\mathbf{M}$ of $\mathbf{A}$, it will be need to determine an intermediate subfactor of **M** and **A**. Actually, it is possible for a finite group G.

LEMMA 9. *Let* **M** *be a splitting extension of a factor* **A** *by a finite group* G, *then a subextension* **N** *of* **A** *is a splitting extension of* **A** *by a subgroup* G_0 *of* G.

PROOF. Each element A in **N** is uniquely written in the form; $A = \sum_{\alpha\in G} a_\alpha U_\alpha$. Denote by $\mathbf{N}_\alpha$ the set of α-components a_α for $A \in \mathbf{N}$, it is immediately verified that each $\mathbf{N}_\alpha$ is a two-sided ideal of **A**. However, **A** is algebraically simple and so $\mathbf{N}_\alpha = \{0\}$ or **A**. Now, set $G_0 = \{\alpha \in G; \mathbf{N}_\alpha \neq \{0\}\}$, **N** is expressed as the direct sum of $\{\mathbf{A}U_\alpha\}_{\alpha\in G_0}$; $\mathbf{N} = \sum_{\alpha\in G_0} \mathbf{A}U_\alpha$ which completes the proof since G_0 is obviously a group.

4. Constructions of splitting extensions. Let **A** be a factor with the spatial invariant $C=1$ on a Hilbert space **H** and let G be a group whose non-trivial conjugate classes are infinite. In addition, let us assume that there is a homomorphism φ of G onto a group $\widetilde{G}$ of automorphisms of **A**. We shall actually construct the splitting extension of $\mathbf{A}\otimes\mathbf{I}$ by G as a factor on $\mathbf{H}\otimes l_2(G)$. The method utilized is the slight generalization of the construction of the crossed product in [13]. For this reason we shall frequently refer to [13] throughout the present section. Using the unitary representation on **H** of a group $\widetilde{G}$ of automorphisms of **A** in [13; Lemma 1], we obtain a unitary representation $\{u_\sigma\}_{\sigma\in G}$ (not necessarily faithful) of G on **H** such that $u_\sigma{}^* a u_\sigma = a^{\tilde{\sigma}}$ for all $a \in \mathbf{A}$ where $\tilde{\sigma} = \varphi(\sigma)$ for each $\sigma \in G$. Now, define unitary operators U_σ on $\mathbf{H} \otimes l_2(G)$ as follows;

$$U_\sigma\left(\sum_{\alpha\in G} x_\alpha \otimes \varepsilon_\alpha\right) = \sum_{\alpha\in G} u_\sigma x_\alpha \otimes \varepsilon_{\sigma\alpha}$$

for each vector $\sum_{\alpha\in G} x_\alpha \otimes \varepsilon_\alpha$ of $\mathbf{H}\otimes l_2(G)$. Then $\sigma \to U_\sigma$ gives a faithful unitary representation of G on $\mathbf{H}\otimes l_2(G)$ such that $U_\sigma^*(a\otimes \mathrm{I})\,U_\sigma = a^{\tilde{\sigma}} \otimes \mathrm{I}\,(a \in \mathbf{A})$ and $\{(\mathbf{A}\otimes\mathbf{I})U^\tau(x\otimes\varepsilon_\alpha)\}_{\tau\in G}$ are mutually orthogonal for each vector $x\otimes\varepsilon_\alpha$ (a fixed $\alpha\in G$) as in [13; Lemma 2]. We consider a system $\mathfrak{S}$ of all linear forms

$$\sum_{\alpha\in G} A_\alpha U_\alpha$$

where A_α are elements of $\mathbf{A}\otimes\mathbf{I}$ and all but a finite number of them are zero. Then the system $\mathfrak{S}$ is a *-algebra. We shall denote by **M** a W^*-algebra generated by the system $\mathfrak{S}$. Now, passing on the similar proof to [13; Thearem 1], we arrive at the fact that **M** *is a finite* W^**-algebra with* $C=1$ *on* $\mathbf{H}\otimes l_2(G)$ *and each element* A *in* **M** *is expressed by a unique family* $\{A_\alpha\}_{\alpha\in G}$ *in the form*

$$A = \sum_{\alpha\in G} A_\alpha U_\alpha$$

where $\sum$ *is taken in the sense of the metrical convergence.* A W^*-algebra $\mathbf{M}$ defined above seems to depend on the choice of the representations of G on $\mathbf{H}$, but the proof of [13; Lemma 5] holds in our case and shows that $\mathbf{M}$ is uniquely determined by $\mathbf{A}$, G and φ within unitary equivalence, and we denote it by $(\mathbf{A}, G, \varphi)$. Since we confine ourselves to a factor in the present paper, we must examine whether $\mathbf{M}$ is a factor or not. To do so [12; Theorem 2] will be slightly modified in our case. In fact, for an arbitrary element $A = \sum_{\alpha\in G} A_\alpha U_\alpha$ $(A_\alpha \in \mathbf{A}\otimes\mathbf{I})$ in the center of $\mathbf{M}$, $U_\sigma^* A U_\sigma = A$ for all $\sigma \in G$. That is, keeping in mind that setting $A_\alpha^{\tilde{\sigma}} = a_\alpha^{\tilde{\sigma}} \otimes I$ for $A_\alpha = a_\alpha \otimes I$, $U_\sigma^* A_\alpha U_\sigma = A_\alpha^{\tilde{\sigma}}$, we obtain

$$\sum_{\alpha\in G} A_\alpha U_\alpha = \sum_{\alpha\in G} A_\alpha^{\tilde{\sigma}} U_{\sigma^{-1}\alpha\sigma}$$

If $A_\alpha \neq 0$ for some $\alpha \neq e$ (e; the unit of G), the uniqueness of a family $\{A_\alpha\}_{\alpha\in G}$ leads that $A_\alpha = A_{\sigma\alpha\sigma^{-1}}^{\tilde{\sigma}}$ for all $\sigma \in G$. Hence $[[A_\alpha]] = [[a_{\sigma\alpha\sigma^{-1}}]]$ for all $\sigma \in G$. Since the conjugate classes $\{\sigma^{-1}\alpha\sigma\}_{\sigma\in G}$ are infinite, this contradicts to $\sum_{\alpha\in G}[[A_\alpha]] < \infty$. Therefore, $A_\alpha = 0$ for all $\alpha \neq e$, in other words, A lies in $\mathbf{A}\otimes\mathbf{I}$. Moreover, since A commutes with all elements in $\mathbf{A}\otimes\mathbf{I}$, A must be scalar multiples of the identity and so $\mathbf{M}$ is a factor. Consequently, since $\{U_\sigma\}_{\sigma\in G}$ is a base of $\mathbf{M}$ over $\mathbf{A}\otimes\mathbf{I}$ as described above, $\mathbf{M}$ is a splitting extension of $\mathbf{A}\otimes\mathbf{I}$ by G. When the restrictions of some kind are imposed on φ, the splitting extension we have obtained is simplified as follows:

The Case I. *$\varphi(\alpha)$ are the identity for all $\alpha \in G$.* Let $\{v_\sigma\}_{\sigma\in G}$ be a regular representation of G on $l_2(G)$, then the equality $A_\sigma U_\sigma(\sum_{\alpha\in G} x_\alpha \otimes \varepsilon_\alpha) = \sum_{\alpha\in G} a_\sigma x_\sigma \otimes \varepsilon_{\sigma\alpha} = \sum_{\alpha\in G} a_\sigma x_\alpha \otimes v_\sigma x_\alpha (A_\sigma = a_\sigma \otimes I)$ implies $A_\sigma U_\sigma = a_\sigma \otimes v_\sigma$. Accordingly $(\mathbf{A}, G, \varphi)$ is isomorphic to the direct product $\mathbf{A}\otimes\mathbf{B}$ of $\mathbf{A}$ and factor $\mathbf{B}$ generated by $\{v_\sigma\}_{\sigma\in G}$.

The Case II. *φ is an isomorphism.* In this case, it will be easily seen that $(\mathbf{A}, G, \varphi)$ is the crossed product of $\mathbf{A}\otimes\mathbf{I}$ by G and nothing but the crossed product $(\mathbf{A}, \widetilde{G})$ of $\mathbf{A}$ by $\widetilde{G}$ defined in [13].

In closing this section, we shall point out that for *any group G homomorphic to a group of outer automorphisms of* $\mathbf{A}$, *our construction is possible* (cf. [13]).

5. Cyclic extensions. There will be considered the cyclic extension which is the simplest and the most interesting splitting extension, and we wish to study the structure of this extension. We first make the following

DEFINITION 4. A splitting extension $\mathbf{M}$ of a factor $\mathbf{A}$ by a group G is called a *cyclic extension* of $\mathbf{A}$ by G if G is cyclic.

In the case where G is of order two or three, we are able to answer completely for the question: What is the necessary and sufficient condition under which $\mathbf{M}$ is a cyclic extension of $\mathbf{A}$ by G?

The case $[\mathbf{M}; \mathbf{A}] = 2$. If $\mathbf{M}$ is the cyclic extension $\mathbf{A}$ by a group G of order two, evidently $[\mathbf{M}; \mathbf{A}] = 2$ by Lemma 1. Conversely, if $[\mathbf{M}, \mathbf{A}] = 2$, we have known by Theorem 2 that $\mathbf{M}$ is discrete over $\mathbf{A}$ with the order two. Further it has been shown in [3; Theorem 3] that $\mathbf{M}$ is cyclic over $\mathbf{A}$. Now, we shall give the proof of this result by making use of Theorem 2.

THEOREM 5. *Let* $\mathbf{M}$ *be an extension of a factor* $\mathbf{A}$, *then* $\mathbf{M}$ *is a cyclic extension of* $\mathbf{A}$ *by a group* G *of order two if and only if* $[\mathbf{M}; \mathbf{A}] = 2$.

PROOF. It is sufficient to prove that if $[\mathbf{M}; \mathbf{A}] = 2$, $\mathbf{M}$ is a cyclic extension of $\mathbf{A}$ by a group of order two. Suppose $[\mathbf{M}; \mathbf{A}] = 2$, then we know by Theorem 2 that $\mathbf{M}$ is discrete over $\mathbf{A}$ with the order two. That is to say, there is a discrete base $\{E_i\}$ $(i = 1, 2)$ of $\mathbf{M}$ over $\mathbf{A}$. Setting $U = 2E_1 - I$, $U^2 = I$ and it is directly seen from Lemma 1 that $\{I, U\}$ is a base of $\mathbf{M}$ over $\mathbf{A}$. Now, for each $a \in \mathbf{A}$, we denote by $a^{\#}$ an element in $\mathbf{A}$ such that $E_iaE_i = a^{\#}E_i$ $(i = 1, 2)$ as regarded in proving Lemma 3, and then $E_1aE_1 + E_2aE_2 = a^{\#}(E_1 + E_2) = a^{\#}$. Moreover, the equality $a = (E_1 + E_2)a(E_1 + E_2) = E_1aE_1 + E_2aE_2 + E_1aE_2 + E_2aE_1$ yields $E_1aE_2 + E_2aE_1 = a - a^{\#}$. Therefore, we obtain, for each $a \in \mathbf{A}$,

$$\begin{aligned} UaU &= (E_1 - E_2)a(E_1 - E_2) = E_1aE_1 + E_2aE_2 - E_1aE_2 - E_2aE_1 \\ &= a^{\#} - (a - a^{\#}) = 2\,a^{\#} - a. \end{aligned}$$

This means that UaU belongs to $\mathbf{A}$ for each $a \in \mathbf{A}$. Thus we complete the proof.

REMARK. In the preceding theorem, put $a^\sigma = UaU$ for all $a \in \mathbf{A}$, σ is a non-identity automorphism of $\mathbf{A}$ and $\mathbf{M}$ is the crossed product of $\mathbf{A}$ by a cyclic group $\{\sigma\}$ of order two. In fact, if $a^\sigma = a$ for all $a \in \mathbf{A}$, U commutes with all $\mathbf{A} \in \mathbf{M}$, this contradicts to the fact that $\mathbf{M}$ is a factor. Further, we know that σ is outer. Because, if σ is inner, $U = vw$ $(v \in \mathbf{A}, w \in \mathbf{A}' \cap \mathbf{M})$ and all $A \in \mathbf{M}$ are written in the form $A = a_1 + a_2vw$ $(a_1, a_2 \in \mathbf{A})$. Thus, w commutes with all $A \in \mathbf{M}$ and so w must be a scalar multiple of the identity, which contradicts to $U \notin \mathbf{A}$. Accordingly, as in [13; Theorem 3], $\mathbf{A}' \cap \mathbf{M} = \{$scalar multiples of the identity$\}$.

The case $[\mathbf{M}; \mathbf{A}] = 3$. It has been shown in [3] that there are factors $\mathbf{M}$, $\mathbf{A}$ such that for any unitary element $U \in \mathbf{M}$, $U^*\mathbf{A}U \subset \mathbf{A}$ implies $U \in \mathbf{A}$. This means that the similar theorem as before is not valid in our case. However, with the aid of the notion of the discrete extension, we derive the following

THEOREM 6. *Let* $\mathbf{M}$ *be an extension of a factor* $\mathbf{A}$, *then* $\mathbf{M}$ *is a cyclic extension of* $\mathbf{A}$ *by a group* G *of order three if and only if* $\mathbf{M}$ *is discrete over* $\mathbf{A}$ *with the order three and* $[\mathbf{M}; \mathbf{A}] = 3$.

PROOF. Suppose that $\mathbf{M}$ is a cyclic extension of $\mathbf{A}$ by a group G of order three. Let $\{I, U, U^2\}$ be a base of $\mathbf{M}$ over $\mathbf{A}$ and put $E_1 = \frac{1}{3}(I + U + U^2)$, $E_2 = \frac{1}{3}(I + \omega U + \omega^2 U^2)$, $E_3 = \frac{1}{3}(I + \omega^2 U + \omega U^2)$ where ω is the primitive cubic root of unity, then it is immediately verified that $E_i (i = 1, 2, 3)$ are projections in $\mathbf{M}$. Moreover, the equality

$$E_i E_j = \frac{1}{9}(I + \omega + \omega^2)(I + U + U^2) = 0 \qquad (i \neq j)$$

yields that E_i are mutually orthogonal and in addition

$$\sum_{i=1}^{3} E_i = \frac{1}{3}\{3 \cdot I + (1 + \omega + \omega^2)(U + U^2)\} = I.$$

Therefore we have shown that $\{E_i\}$ $(i = 1, 2, 3)$ is a family of mutually orthogonal projections such that $\sum_{i=1}^{3} E_i = I$.

Now we shall assert that $\{E_i\}$ $(i = 1, 2, 3)$ is a discete base of $\mathbf{M}$ over $\mathbf{A}$. For each $a \in \mathbf{A}$, the direct computation yields

$$E_i a E_i = \frac{1}{3}(a + a^{\sigma} + a^{\sigma^2}) E_i \ (i = 1, 2, 3)$$

where $a^{\sigma} = UaU^*$. Thus, since $\frac{1}{3}(a + a^{\sigma} + a^{\sigma^2}) \in \mathbf{A}$, E_i are $\mathbf{A}$-projections in $\mathbf{M}$. Further $U^{\varepsilon} = 0$ implies $E_i^{\varepsilon} = \frac{1}{3} I$ for all i and so for each $a \in \mathbf{A}$

$$(E_i a E_i)^{\varepsilon} = \frac{1}{3}(a + a + a^{\sigma_2}) E_i^{\varepsilon} = \frac{1}{9}(a + a^{\sigma} + a^{\sigma^2}) \qquad (i = 1.2, 3),$$

that is, $(E_1 a E_1)^{\varepsilon} = (E_2 a E_2)^{\varepsilon} = (E_3 a E_3)^{\varepsilon}$. Therefore, we conclude that $\mathbf{M}$ is discrete over $\mathbf{A}$ with the order three, and it follows obviously from Lemma 1 that $[\mathbf{M}; \mathbf{A}] = 3$.

Conversely, suppose that $\mathbf{M}$ is discrete over $\mathbf{A}$ with the order three and $[\mathbf{M}; \mathbf{A}] = 3$. Let $\{E_i\}$ $(i = 1, 2, 3)$ be a discrete base of $\mathbf{M}$ over $\mathbf{A}$ and put

$$U = E_1 + \omega E_2 + \omega^2 E_3,$$

where ω is the primitive cubic root of unity, then U is a unitary element in $\mathbf{M}$ such that $U^3 = I$ and $(U^i)^{\varepsilon} = \frac{1}{3}(1 + \omega + \omega^2) I = 0$ $(i = 1, 2)$, since $E_k^{\varepsilon} = \frac{1}{3} I$ for all k. Hence $[U^i, U^j] = 0$ $(i \neq j;\ i, j = 1, 2, 3)$, that is, $U^i (i = 1, 2, 3)$ are mutually orthogonal over $\mathbf{A}$. Under the assumption $[\mathbf{M}; \mathbf{A}] = 3$ it follows from Lemma 1 that $\{I, U, U^2\}$ is a base of $\mathbf{M}$ over $\mathbf{A}$. To complete the proof, it remains only to show that $U^* \mathbf{A} U \subset \mathbf{A}$. As seen in Lemma 3, for each $a \in \mathbf{A}$ there exists an element $a^{\#}$ in $\mathbf{A}$ such that $E_k a E_k = a^{\#} E_k$ $(k = 1, 2, 3)$. Thus it holds that

$$\textstyle\sum_{k=1}^{3} E_k a E_k = a^{\#}(\sum_{k=1}^{3} E_k) = a^{\#}. \qquad (1)$$

Now, making use of the equality $4B^*A = (A+B)^*(A+B) - (A-B)^*(A-B) + i(A+iB)^*(A+iB) - i(A-iB)^*(A-iB)$, we obtain, for each $a \in \mathbf{A}$,

$$\begin{aligned} 4E_h a^* a E_k = 4\{(aE_h)^*(aE_k)\} &= (aE_k + aE_h)^*(aE_k + aE_h) - (aE_k \\ &\quad - aE_h)^*(aE_k - aE_h) + i(aE_k + iaE_h)^*(aE_k + iaE_h) \\ &\quad - i(aE_k - iaE_h)^*(aE_k - iaE_h) \\ &= (E_h + E_k)a^*a(E_h + E_k) - (E_k - E_h)a^*a(E_k - E_h) \\ &\quad + i(E_k + iE_h)a^*a(E_k + iE_h) - i(E_k - iE_h)a^*a(E_k - iE_h). \end{aligned}$$

Therefore, applying (1) and $\sum_{k=1}^{3} E_k = I$,

$$\begin{aligned} 4(E_1a^*aE_2 + E_2a^*aE_3 + E_3a^*aE_1) &= 4\{a^*a + (a^*a)^{\#} - 2(a^*a)^{\#} \\ + \sum_{h\neq k} E_h a^* a E_k + i^2 \sum_{h\neq k} E_h a^* a E_k &- i\sum_{h\neq k} E_h a^* a E_k = 4\{a^*a - (a^*a)^{\#} \\ - i\sum_{h\neq k} E_h a^* a E_k\} \end{aligned}$$

where h, k run over 1, 2, 3. Here, note that

$$\sum_{h\neq k} E_h a^* a E_k = a^*a - \textstyle\sum_k E_k a^* a E_k = a^*a - (a^*a)^{\#}.$$

We have

$$\begin{aligned} 4(E_1a^*aE_2 + Ea^*aE_3 + E_3a^*aE_1) &= 4\{a^*a - (a^*a)^{\#} - i(a^*a - (a^*a)^{\#})\} \\ &= 4(1-i)\{a^*a - (a^*a)^{\#}\} \qquad (2) \end{aligned}$$

Now,
$$\begin{aligned} U^*a^*aU &= (E_1 + \omega^2E_2 + \omega E_3)a^*a(E_1 + \omega E_2 + \omega^2 E_3) \\ &= \textstyle\sum_k E_k a^* a E_k + \omega(E_1a^*aE_2 + E_2a^*aE_3 + E_3a^*aE_1) \\ &\quad + \{\omega(E_1a^*aE_2 + E_2a^*aE_3 + E_3a^*aE_1)\}^*. \end{aligned}$$

Combining (1) and (2),

$$U^*a^*aU = (a^*a)^{\#} + \omega(1-i)(a^*a - (a^*a)^{\#}) + \omega^2(1+i)(a^*a - (a^*a)^{\#}).$$

Consequently, we have shown that $U^*a^*aU \in \mathbf{A}$ for all $a \in \mathbf{A}$, that is to say, $U^*aU \in \mathbf{A}$ for all positive elements $a \in \mathbf{A}$. Since any element in $\mathbf{A}$ is expressed as a linear combination of positive elements in $\mathbf{A}$, we have that $U^*aU \in \mathbf{A}$ for all $a \in \mathbf{A}$ as desired. This completes the proof.

The above theorems give us a method of examining the cyclic extension of

A by a group of order n and so it seems that Theorem 6 is perhaps true for any positive integer n.

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