SOME REMARKS ON SPACE WITH A CERTAIN CONTACT STRUCTURE

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Introduction. Recently S. Sasaki [3][1] defined the notion of (ϕ, ξ, η, g) structure of a differentiable manifold. Further, S. Sasaki and Y. Hatakeyama [4] [5] showed that the structure is closely related to contact structure. By means of this notion, it is shown that a space with a contact structure can be dealt with as we deal with an almost complex space. So, by similar manner, some problems discussed in the latter space may be considered in the former. On the other hand, S. Tachibana [6] [7] proved many interesting theorems in an almost complex space. In this paper, the present author tries to study, in the space with a certain contact structure, the problem corresponding to S. Tachibana's results. We shall devote § 1 to preliminaries and in this section introduce a normal contact structure. In § 2, we ennumerate identities which will be useful in the later sections. We shall prove in § 3 that a space with a normal contact structure satisfying $\nabla_k R_{ji} = 0$ be necessarily an Einstein one and that a symmetric space with a normal contact structure reduces to the space of constant curvature respectively. The differential form $\widehat{R}$ is dealt with in § 4, and in this section, we shall show a necessary and sufficient condition that the space be an Einstein space by means of the form $\widehat{R}$. Finally in § 5, we introduce a certain type of (ϕ, η, g)-connection with respect to which the fundamental tensors ϕ_{ji}, η_j and g_{ji} are all covariant constant.

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1. Preliminaries. Let M be an n-dimensional real differentiable manifold. If there exist a tensor field $\phi_j{}^i$, contravariant and covariant vector fields ξ^i, η_j over M such that

$$\xi^i \eta_i = 1, \tag{1. 1}$$

$$\text{rank } |\phi_j{}^i| = n - 1, \tag{1. 2}$$

$$\phi_j{}^i \xi^j = 0, \tag{1. 3}$$

$$\phi_j{}^i \eta_i = 0, \tag{1. 4}$$

$$\phi_j{}^i \phi_k{}^j = -\delta_k{}^i + \xi^i \eta_k, \tag{1. 5}$$

1) Numbers in brackets refer to the references at the end of the paper.

then the manifold M is called to have a (ϕ, ξ, η)-structure. In the space with (ϕ, ξ, η)-structure, it is known that the space is odd dimensional. There are four important tensors, in the space, which correspond to the Nijenhuis tensor of an almost complex space and they are defined by the following equations.

(1. 6)[2]
$$N_{kj}{}^{i} = \phi_k{}^h(\partial_h\phi_j{}^i - \partial_j\phi_h{}^i) - \phi_j{}^h(\partial_h\phi_k{}^i - \partial_k\phi_h{}^i) + (\partial_j\xi^i)\eta_k - (\partial_k\xi^i)\eta_j,$$

(1. 7)
$$N_{kj} = \phi_k{}^h(\partial_j\eta_h - \partial_h\eta_j) - \phi_j{}^h(\partial_k\eta_h - \partial_h\eta_k),$$

(1. 8)
$$N_j{}^i = \xi^h(\partial_h\phi_j{}^i - \partial_j\phi_h{}^i) - \phi_j{}^h\partial_h\xi^i,$$

(1. 9)
$$N_j = \xi^h(\partial_j\eta_k - \partial_h\eta_j).$$

Furtheremore, we have known that a manifold M with a (ϕ, ξ, η)-structure always admits a positive definite Riemannian metric tensor g_{ji} such that

(1. 10)
$$g_{ji}\xi^j = \eta_i,$$

(1. 11)
$$g_{ji}\phi_h{}^i\phi_k{}^j = g_{kh} - \eta_h\eta_k.$$

Hence, in the following, we use a notation η^j in stead of ξ^j.

The aggregate consisting of a (ϕ, ξ, η)-structure together with a Riemannian metric tensor g_{ji} which satisfies (1. 10) and (1. 11) is called a (ϕ, ξ, η, g) structure and g_{ji} is called a (ϕ, ξ, η, g)-metric.

On the other hand, let us consider a differentiable manifold with a contact structure [**1**] [**2**] and let $\eta = \eta_i dx^i$ be the 1-form which defines the contact structure. Then we have

$$\eta \wedge d\eta \underbrace{\wedge \cdots\cdots \wedge}_{m\text{-times}} d\eta \neq 0, \qquad (n = 2m + 1),$$

where operator $\wedge$ in the last equation means exterior multiplication. From the given contact structure, as is well known [**5**] we can find a (ϕ, ξ, η, g)-structure such that the vector field η_i is the one given by the coefficients of the 1-form η and

(1. 12)
$$2\, g_{ir}\phi_j{}^r = 2\,\phi_{ji} = \partial_j\eta_i - \partial_i\eta_j.$$

Such a metric g_{ji} is not determined uniquely, but in the following we shall confine ourselves to a fixed (ϕ, ξ, η, g)-metric. As was shown by S.Sasaki and Y. Hatakeyama [**5**], in our case, both N_{kj}, N_j are identically zero and the vanishing $N_{kj}{}^i$ implies $N_j{}^i = 0$. So, if we assume the vanishing of $N_{kj}{}^i$, the three other tensors defined by (1. 7), (1. 8) and (1. 9) vanish.

In the following, the author try to discuss the properties of the space with a contact structure whose $N_{kj}{}^i$ with respect to the (ϕ, ξ, η, g)-structure associated

2) This notation is slightly different to S. Sasaki's.

with the given contact structure in the above way is identically zero. From now on, for the convenience, we shall call such a contact structure a *normal* contact structure, and under the symbol M we always understand a *normal contact space* i. e. a space with a normal contact structure.

By S. Sasaki and Y. Hatakeyama, it has been shown that the following relations are always valid in M,

$$\nabla_j \eta_i = \phi_{ji}, \tag{1. 13}$$

$$\nabla_k \phi_{ji} = \eta_j g_{ik} - \eta_i g_{jk}, \tag{1. 14}$$

from which we have

$$\nabla^r \phi_{ir} = (n-1)\eta_i, \tag{1. 15}$$

where ∇_j means the covariant differentiation with respect to $\left\{ {h \atop j\,i} \right\}$ and $\nabla^r = g^{rj}\nabla_j$. These identities play the fundamental role in our discussions. Moreover, by definition, we find that the form $\phi = \phi_{ji} dx^j \wedge dx^i$ is closed.

2. Identities. Let $R_{kji}{}^h$ be the Riemannian curvature tensor, i. e.

$$R_{kji}{}^h = \partial_k \left\{ {h \atop j\,i} \right\} - \partial_j \left\{ {h \atop k\,i} \right\} + \left\{ {h \atop k\,r} \right\} \left\{ {r \atop j\,i} \right\} - \left\{ {h \atop j\,r} \right\} \left\{ {r \atop k\,i} \right\},$$

and put $R_{ji} = R_{rji}{}^r$, $R_{kjih} = R_{kji}{}^r g_{rh}$, $R = R_{ji} g^{ji}$ and

$$R^*_{ji} = \frac{1}{2} \phi^{rs} R_{rski} \phi_j{}^k, \tag{2. 1}$$

where $\phi^{rs} = g^{ri}\phi_i{}^s$.

Applying the Ricci's identity to η_j, we obtain the identity

$$\nabla_l \nabla_k \eta_j - \nabla_k \nabla_l \eta_j = - \eta_r R_{lkj}{}^r. \tag{2. 2}$$

If we substitute (1. 13) into the last equation, we get

$$\eta_r R_{lkj}{}^r = - \nabla_l \phi_{kj} + \nabla_k \phi_{lj}.$$

By means of (1. 14), this relation changes its form as

$$\eta_r R_{lkj}{}^r = \eta_l g_{jk} - \eta_k g_{jl}. \tag{2. 3}$$

Transvecting this with g^{jl}, we have

$$\eta_r R_k{}^r = (n-1)\eta_k \tag{2. 4}$$

from which we have

$$R_k{}^r \eta_r \eta^k = R_{kr}\eta^k \eta^r = n-1, \tag{2. 5}$$

where we have used (1. 1). If the space under consideration is an Einstein space, from (2. 4) we have the following

THEOREM 2.1. *If a normal contact space is an Einstein one, the scalar curvature R has a positive constant value $n(n-1)$.*

COROLLARY 1. *If a normal contact space is a space of constant curvature, the scalar curvature has a positive constant value $n(n-1)$.*

Next, for the later use, we shall prove the following

LEMMA 2.1. *The relation $R - R^* = (n-1)^2$ holds good, where $R^* = R^*_{ji} g^{ji}$.*

PROOF. Differentiating covariantly (1. 14) and using (1. 13), we get

$$\nabla_l \nabla_k \phi_{ji} = \phi_{lj} g_{ik} - \phi_{li} g_{jk}, \tag{2. 6}$$

from which we have

$$-\phi_{jr} R_{lki}{}^r - \phi_{ri} R_{lkj}{}^r = \phi_{lj} g_{ik} - \phi_{kj} g_{il} - \phi_{li} g_{jk} + \phi_{ki} g_{jl}. \tag{2. 7}$$

Transvecting this with g^{ki}, we find

$$-\phi_{jr} R_l{}^r + \phi^{rk} R_{lkrj} = (n-2)\phi_{lj}.$$

On taking account of skew-symmetric property of ϕ^{rk} we have

$$-\phi_{jr} R_l{}^r - \frac{1}{2}\phi^{rk} R_{rkjl} = -(n-2)\phi_{jl}, \tag{2. 8}$$

from which we obtain

$$-\phi_{jr}\phi_i{}^j R_l{}^r - R^*_{il} = -(n-2)\phi_{jl}\phi_i{}^j. \tag{2. 9}$$

Making use of (1. 5) and (2. 5), we can see

$$R_{il} - R^*_{il} = (n-2) g_{il} + \eta_i \eta_l. \tag{2. 10}$$

Regarding (1. 1) and (1. 6), we can complete the proof of the lemma.

It is necessary to bear in mind that R^*_{il} is symmetric and that $H_{jl} \overset{\text{def}}{=} \phi_{jr} R_l{}^r$ is skew-symmetric. As an application of this lemma, we can prove the

LEMMA 2.2. *If an n-dimensional normal contact space$(n > 3)$ is a conformally flat one, the scalar curvature has a positive constant value $n(n-1)$.*

PROOF. From the assumption, the curvature tensor of M has the following form:

$$R_{kjih} = \frac{1}{n-2}(g_{kh} R_{ji} - g_{jh} R_{ki} + g_{ji} R_{kh} - g_{ki} R_{jh}) - \frac{R}{(n-1)(n-2)}(g_{ji} g_{kh} - g_{ki} g_{hj}).$$

Hence we have

$$R^*_h = \frac{1}{n-2}(R_{ji}\phi^i{}_l \phi_h{}^j + R_{lh} - R_{rh}\eta^r \eta_l) - \frac{R}{(n-1)(n-2)}(g_{hl} - \eta_h \eta_l),$$

which implies

$$(n-2)R^* = R - 2(n-1). \tag{2. 11}$$

From the Lemma 2. 1, the last equation leads $R = n(n-1)$. This completes the proof.

Let us differentiate (2. 10) covariantly, then by means of (1. 13), we get

$$\nabla_k R_{il} - \nabla_k R^*_{il} = \phi_{ki}\eta_l + \eta_i \phi_{kl}.$$

Transvecting this with g^{ik} and using (1. 3), (1. 4) we have

$$\nabla^r R_{rl} = \nabla^r R^*_{rl}. \tag{2. 12}$$

On the other hand, from the Bianchi's identity, we know

$$\nabla_r R_{lkj}{}^r + \nabla_k R_{lj} - \nabla_l R_{kj} = 0,$$

from which we have

$$\nabla_r R = 2\nabla^h R_{hr}.$$

So, we obtain

$$\nabla_r R = 2\nabla^h R^*_{hr}. \tag{2. 13}$$

3. Some types of Riemannian spaces with normal contact structures. In this section we obtain some results on special kinds of Riemannian spaces. At first, let us consider a Riemannian space with a parallel Ricci tensor which admits a normal contact structure. Then, by Ricci's identity on the Ricci tensor, we have

$$R_{ir}R_{kjh}{}^r + R_{rh}R_{kji}{}^r = 0. \tag{3. 1}$$

Transvecting this equation with η^i and making use of (2. 3), (2. 4), we get

$$(n-1)(\eta_k g_{hj} - \eta_j g_{hk}) - R_h{}^i(\eta_k g_{ji} - \eta_j g_{ki}) = 0.$$

If we multiply the last equation by η^k, it follows that

$$R_{hj} = (n-1)g_{hj}.$$

Thus we have the

THEOREM 3. 1. *A normal contact space with parallel Ricci tensor is an Einstien space.*

Next, we consider a symmetric space which is characterized by $\nabla_l R_{kji}{}^h = 0$. By applying the Ricci's identity to the curvature tensor, we find

$$R_{kji}{}^r R_{mlr}{}^h - R_{kjr}{}^h R_{mli}{}^r - R_{kri}{}^h R_{mlj}{}^r - R_{rji}{}^h R_{mlk}{}^r = 0. \tag{3. 2}$$

On multiplying this by η_h and summing over h, we get

$$R_{kjil}\eta_m - R_{kjim}\eta_l + (\eta_m g_{lj} - \eta_l g_{mj})g_{ki} - (\eta_m g_{lk} - \eta_l g_{mk})g_{ji} = 0, \tag{3. 3}$$

by taking account of (2. 3).

Transvecting this with η^m and making use of (1. 1), (1. 10) and (2. 3), we have

$$R_{kjil} = g_{kl}g_{ji} - g_{jl}g_{ki}.$$

Thus we get the

THEOREM 3. 2. *If a normal contact space is a symmetric one, it is a space of constant curvature.*

COROLLARY 1. *A semi-simple Lie group can not admits a normal contact structure.*

In concluding this section, we shall prove the following

THEOREM 3. 3. *A conformally flat normal contact space is necessarily a space of constant curvature* $(n > 3)$.

PROOF. From our assumption, we can apply the Lemma 2.2 and we have

$$R_{kjih} = \frac{1}{n-2}(g_{kh}R_{ji} - g_{jh}R_{ki} + g_{ji}R_{kh} - g_{ki}R_{jh}) - \frac{n}{n-2}(g_{ji}g_{hk} - g_{ki}g_{hj}). \tag{3. 4}$$

Transvecting this with η^h and making use of (2. 3), (2. 4) we get

$$\eta_k R_{ji} - \eta_j R_{ki} = (n-1)(\eta_k g_{ji} - \eta_j g_{ki}).$$

This implies that

$$R_{ji} = (n-1)g_{ji}.$$

If we substitute thus obtained Ricci tensor into (3. 4), we have at last

$$R_{kjih} = g_{ji}g_{kh} - g_{ki}g_{jh}.$$

This completes the proof.

4. The form $\widehat{R}$. In this section, we introduce two forms $\widehat{R}$ and H, and show some properties of these two forms. In the first place, we define an anti-symmetric tensor $\widehat{R}_{ji}$ by

$$\widehat{R}_{kj} = \frac{1}{2}R_{kji}{}^{h}\phi_h{}^{i} = -\frac{1}{2}\phi^{hi}R_{hikj}{}^{3)}. \tag{4. 1}$$

Then, we have the following

LEMMA 4.1. *The differential form* $\widehat{R}$ *defined by* $\widehat{R} = \widehat{R}_{kj}dx^k \wedge dx^j$ *is closed.*

PROOF. Now, we define by $\mathfrak{S}\{a_{kji}\}$ the cyclic sum of a given tensor a_{kji}, namely

$$\mathfrak{S}\{a_{kji}\} = a_{kji} + a_{ikj} + a_{jik}.$$

3) The definition of $\hat{R}_{ji}$ differs from S. Tachibana's [6] [7] by a constant factor.

With this notation, we have

$$\mathfrak{S}\{\nabla_l \widehat{R}_{kj}\} = \frac{1}{2}\mathfrak{S}\{\nabla_l(R_{kjs}{}^r\phi_r{}^s)\}$$
$$= \frac{1}{2}[\mathfrak{S}\{(\nabla_l R_{kjs}{}^r)\phi_r{}^s\} + \mathfrak{S}\{R_{kjs}{}^r(\nabla_l\phi_r{}^s)\}].$$

Since, by Bianchi's identity, the first term of the right hand side of the last equation vanishes identically, we have

$$\mathfrak{S}\{\nabla_l \widehat{R}_{kj}\} = \frac{1}{2}\mathfrak{S}\{R_{kjs}{}^r(\eta_r\delta_l{}^s - \eta^s g_{lr})\}$$
$$= \frac{1}{2}[\mathfrak{S}\{\eta_r R_{kjl}{}^r\} - \mathfrak{S}\{\eta^s R_{kjsl}\}]$$
$$= 0. \qquad \text{Q. E. D.}$$

Next if we put $H = H_{jl}dx^j \wedge dx^l$, $H_{jl} = \phi_{jr}R_l{}^r$, we have the

LEMMA 4. 2. *The form H defined by the above relation is closed.*

From (1. 12), (2. 8) and Lemma 4. 1, we can easily verify the lemma.

If we transvect the equation (2. 8) with $\phi_i{}^j + \eta_i\eta^j$ and take account of (1. 3), we have

$$-\phi_{jr}\phi_i{}^j R_l{}^r - \frac{1}{2}\phi^{rk}R_{rkjl}(\phi_i{}^j + \eta_i\eta^j) = (n-2)\phi_{lj}\phi_i{}^j. \tag{4. 2}$$

Comparing (4. 2) with (2. 9), it is known that

$$R^*_{li} = \widehat{R}_{lj}(\phi_i{}^j + \eta_i\eta^j). \tag{4. 3}$$

On the other hand, by definition, we obtain

$$R^*_{li} = \widehat{R}_{lj}\phi_i{}^j. \tag{4. 4}$$

From (4. 3) and (4. 4), we find

$$\widehat{R}_{lj}\eta^j = 0. \tag{4. 5}$$

The last two equations shows us that

$$\widehat{R}_{rl} = R^*_{li}\phi_r{}^i. \tag{4. 6}$$

Now, we have three closed forms ϕ, $\widehat{R}$ and H in our space and they are dependent by accordance of (2. 8). It is naturally arised a problem that under what condition ϕ and $\widehat{R}$ or H are dependent. We discuss this problem as follows.

THEOREM 4. 1. *In order that two closed forms ϕ and $\widehat{R}$ are dependent, it is necessary and sufficient that*

$$R_{ji} = bg_{ji} + c\eta_j\eta_i \tag{4. 7}$$

valid where b and c mean some constants.

In the first place, we prove the

LEMMA 4. 3. *If M has the Ricci tensor R_{ji} of the form* (4. 7), *b and c must be constants* ($n>3$).

PROOF. From (2. 4) and (4. 7), we find

$$b + c = n - 1, \tag{4. 8}$$

hence, it is sufficient to show that $b = \text{const}$. Using (4. 7) and (4. 8) we have

$$\nabla_k R = (n - 1)\nabla_k b. \tag{4. 9}$$

On the other hand, taking account of (1. 3), (1. 4) and (1. 13), we get

$$\nabla_k R = 2\nabla^r R_{rk} = 2(\nabla_k b - \nabla_r b \cdot \eta^r \eta_k). \tag{4. 10}$$

Comparing (4. 9) and (4. 10), we have

$$2(\nabla_k b - \nabla_r b \cdot \eta^r \eta_k) = (n - 1)\nabla_k b$$

from which we obtain

$$(n - 1)\nabla_k b \cdot \eta^k = 0.$$

Substituting the last equation into the preceding one, we have $b = \text{const}$.

PROOF OF THE THEOREM. At first, we shall show the necessity of the condition. Let us assume that $\widehat{R}_{ji} = a\phi_{ji}$. Transvectng this with $\phi_l{}^j$ and using (4. 4), we get

$$R^*_{il} = a(g_{il} - \eta_i \eta_l).$$

If we recall (2. 10), we find

$$R_{il} = \{a + (n - 2)\}g_{il} + (1 - a)\eta_i \eta_l. \tag{4. 11}$$

Conversely, if our space has a Ricci tensor of the form (4. 7), by means of (2. 10) and (4. 6), we have easily

$$\widehat{R}_{rl} = \{b - (n - 2)\}\phi_{rl}.$$

This completes the proof.

COROLLARY 1. *In order that the differential form $\widehat{R} - a\phi$ be trivial, it is necessary and sufficient that* (4. 11) *be valid.*

If we operate $\nabla^l = g^{lr}\nabla_r$ to (4. 6), we have

$$\begin{aligned}\nabla^l \widehat{R}_{rl} &= (\nabla^l R^*_{il})\phi_r{}^i + R^*_{il}(\nabla^l \phi_r{}^i) \\ &= \frac{1}{2}(\nabla_i R)\phi_r{}^i + R^*_{il}(\eta_r g^{li} - \eta^i \delta_r{}^l),\end{aligned}$$

by virtue of (2. 13). By definition and symmetric property of R^*_{ji}, we have $R^*_{ji}\eta^j = R^*_{lj}\eta^j = 0$. Hence, it follows that

$$\nabla^l \widehat{R}_{rl} = \frac{1}{2}(\nabla_i R)\phi_r{}^i + R^* \eta_r. \tag{4. 12}$$

Accordingly, if the form $\widehat{R} - a\phi$ be harmonic, by virtue of (1. 15), we get

$$\nabla^l(\widehat{R}_{rl} - a\phi_{rl}) = \frac{1}{2}(\nabla_i R)\phi_r{}^i + R^* \eta_r - a(n-1)\eta_r = 0 \tag{4. 13}$$

which implies that

$$R = (n-1)(n-1+a). \tag{4. 14}$$

Conversely, if (4. 14) be valid, we have $R^* = a(n-1)$, hence $\widehat{R} - a\phi$ is harmonic. Thus we have the following

LEMMA 4. 4. *In order that the differential form* $\widehat{R} - a\phi$ *be harmonic, it is necessary and sufficient that* (4. 14) *holds good.*

If we take a differential form H instead of $\widehat{R}$, we can see that an analogous theorem to the Theorem 4.1 be valid.

As applications of our discussion in this section, we have the following two theorems.

THEOREM 4. 2. *In order that a normal contact space M be an Einstein space, it is necessary and sufficient that the differential form* $\widehat{R}$ *be idential with* ϕ.

PROOF. By virtue of corollary 1 of the Theorem 4. 1, the sufficiency is trivial. The necessity is verified by use of the Theorem 2. 1.

From the discussions in § 2, the space with $R = n(n-1)$ seems to be quite meaningful. From this point of view, we state the

THEOREM 4. 3. *In order that a normal contact space M have the scalar curvature $R = n(n-1)$ with respect to a fixed (ϕ, ξ, η, g)-metric, it is necessary and sufficient that the differential form* $\widehat{R} - \phi$ *be harmonic.*

COROLLARY 1. *If* $\widehat{R}$ *coincide with* ϕ, $R = n(n-1)$.

5. Some (ϕ, η, g)-connection. In the present section, we shall introduce a (ϕ, η, g)-connection, i. e. a connection with respect to which ϕ_{ji}, η_j and g_{ji} are all covariant constants. To introduce such a connection, there may be many ways and many different connections may be defined. In this paper, however, making use of the fundamental tensor ϕ_{ji} and vector η_j, we construct the connection in the following way.

Let us consider a tensor $T_{ji}{}^h$ defined by

$$T_{ji}{}^h = p\phi_j{}^h \eta_i + q\phi_{ji}\eta^h + r\phi_i{}^h \eta_j, \tag{5. 1}$$

where p, q and r are constants. Now we shall determine p, q and r so that the connection parameter defined by

$$\Gamma_{ji}{}^{h} = \left\{ \begin{matrix} h \\ j\ i \end{matrix} \right\} + T_{ji}{}^{h} \tag{5. 2}$$

may be a (ϕ, η, g)-connection.

Covariant differentiation of η^h with respect to $\Gamma_{ji}{}^h$ yields

$$\tilde{\nabla}_j \eta^h = \nabla_j \eta^h + T_{ji}{}^h \eta^i = \phi_j{}^h + p\phi_j{}^h = 0, \tag{5. 3}$$

where $\tilde{\nabla}_j$ means covariant differentiation with respect to $\Gamma_{ji}{}^h$.

From the assumption that $\Gamma_{ji}{}^h$ is a metric connection, it follows

$$\begin{aligned} \tilde{\nabla}_j g_{ik} &= \nabla_j g_{ik} - T_{ji}{}^h g_{hk} - T_{jk}{}^h g_{hi} \\ &= -(p\phi_{jk}\eta_i + q\phi_{ji}\eta_k + r\phi_{ik}\eta_j) \\ &\quad - (p\phi_{ji}\eta_k + q\phi_{jk}\eta_i + r\phi_{ki}\eta_j) \\ &= -(p+q)(\phi_{jk}\eta_i + \phi_{ji}\eta_k) = 0. \end{aligned} \tag{5. 4}$$

Since the second parenthesis of the last term of (5. 4) can not be zero, we have $p = -1$, $q = 1$ by means of (5. 3). Regarding this fact, we have the

THEOREM 5. 1 *For any constant r, put*

$$\Gamma_{ji}{}^h = \left\{ \begin{matrix} h \\ j\ i \end{matrix} \right\} + \phi_{ji}\eta^h - \phi_j{}^h\eta_i + r\phi_i{}^h\eta_j. \tag{5. 5}$$

Then, ϕ_{ji}, η_j and g_{ji} are all covariant constants with respect to the connection $\Gamma_{ji}{}^h$.

PROOF. From the above discussions, it is sufficient to show that ϕ_{ji} is covariant constant. We compute here $\tilde{\nabla}_j \phi_{ir}$.

$$\begin{aligned} \tilde{\nabla}_j \phi_{ir} &= \nabla_j \phi_{ir} - T_{ji}{}^h \phi_{hr} - T_{jr}{}^h \phi_{ih} \\ &= (\eta_i g_{jr} - \eta_r g_{ij}) - \phi_{hr}(\phi_{ji}\eta^h - \phi_j{}^h\eta_i + r\phi_i{}^h\eta_j) \\ &\qquad - \phi_{ih}(\phi_{jr}\eta^h - \phi_j{}^h\eta_r + r\phi_r{}^h\eta_j) \\ &= (\eta_i g_{jr} - \eta_r g_{ij}) - \{-\eta_i(\eta_j\eta_r - g_{jr}) + r\eta_j(\eta_i\eta_r - g_{ir})\} \\ &\qquad - \{-\eta_r(g_{ji} - \eta_j\eta_i) + r\eta_j(g_{ir} - \eta_i\eta_r)\} \\ &= 0. \end{aligned}$$

Q. E. D.

By further conditions, the constant r in (5. 5) may be determined. For example, in order to discover the connection whose symmetric part be identical with the Christoffel's symbol, it must be that $r = 1$.

The (ϕ, η, g)-connection introduced by S.Sasaki and Y. Hatakeyama [4] corresponds to the case where $r = 0$ in our connection.

Now, let us denote the curvature tensor with respect to the connection (5. 5) by $K_{kji}{}^h$, it follows that

$$K_{kji}{}^h = R_{kji}{}^h + \nabla_k T_{ji}{}^h - \nabla_j T_{ki}{}^h + T_{kr}{}^h T_{ji}{}^r - T_{jr}{}^h T_{ki}{}^r.$$

If we define an anti-symmetric tensor $\widehat{K}_{kj}$ by $\widehat{K}_{kj} = \frac{1}{2} K_{kji}{}^{h}\phi_{h}{}^{i}$, then we have after some calculations

$$\widehat{K}_{kj} = \widehat{R}_{kj} + \{(1-n)r+1\}\phi_{kj}. \tag{5. 6}$$

Thus, the form $\widehat{K} = \widehat{K}_{ji}dx^{j} \wedge dx^{i}$ can be expressible in a linear form of $\widehat{R}$ and ϕ.

REFERENCES

[1] W. M. BOOTHBY AND H. C. WANG, On contact manifolds. Ann. of Math. 68 (1958) 721–734.

[2] J. W. GRAY, Some global properties of contact structures, Ann. of Math. 69 (1959) 421–450.

[3] S. SASAKI, On differentiable manifolds with certain structures which are closely related to almost contact structure I, Tôhoku Math. Jour. 12 (1960) 459–476.

[4] S. SASAKI AND Y. HATAKEYAMA, On differentiable manifolds with certain structures which are closely related to almost contact structure II, Tôhoku Math. Jour. 13 (1961) 281–294.

[5] S. SASAKI AND Y. HATAKEYAMA, On differentiable manifolds with contact metric structures, to appear in Jour. of the Math. Soc. Japan.

[6] S. TACHIBANA, On automorphisms of certain compact almost-Hermitian spaces, Tôhoku Math. Jour. 13 (1961) 179–185.

[7] S. TACHIBANA, On infinitesimal conformal and projective transformations of compact K-spaces, Tôhoku Math. Jour. 13 (1961), 386–392.

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