SOME THEOREMS ON THE CROSSED PRODUCTS OF FINITE FACTORS

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1. Let **M** be a finite factor with the invariant $C = 1$ and G a group of automorphisms[1)] of **M**. If G is outer, that is all member of G except the unit element are outer automorphisms, the crossed product (**M**, G) is a finite factor (see [1]). The purpose of this note is to prove the following two theorems.

THEOREM 1. *Let* **M** *be a finite factor with the invariant* $C = 1$ *and* G *a group of outer automorphisms of* **M**. *Then any subfactor* **N** *of the crossed product* (**M**, G) *containing* $\widetilde{\mathbf{M}}$[2)] *is isomorphic to the crossed product* (**M**, G_0) *of* **M** *by a subgroup* G_0 *of* G.

When G is finite, this theorem is contained in [2 : Lemma 9].

THEOREM 2. *Let* **M** *be a finite factor with the invariant* $C = 1$, *and both* G *and* H *be groups of outer automorphisms of* **M**. *Then there exists an isomorphism between the crossed product* (**M**, G) *and* (**M**, H) *which leaves* $\widetilde{\mathbf{M}}$ *invariant if and only if the following conditions are true.*

(1) *There is an automorphism* η *of* **M** *such that the correspondence* $\alpha(\in G) \to \sigma(\in H)$ *defined by the relation* $\eta^{-1}\alpha\eta \equiv \sigma(\mathrm{mod}\ \mathbf{I})$, *where* **I** *is the set of all inner automorphisms of* **M**, *gives an isomorphism of* G *onto* H.

(2) *There exists a family* $\{W_\alpha\}_{\alpha\in G}$ *of unitary operators of* **M** *such that each* W_α *induces the automorphism* $\eta^{-1}\alpha\eta\sigma^{-1}$ *of* **M** *and for each* $\alpha, \beta \in G$, $W^\sigma_{\alpha\beta} = W^\sigma_\alpha W_\beta$.

2. Firstly we shall give a short explanation on the construction and a basic property of the crossed product of a finite factor.

Let **M** be a finite factor with the invariant $C = 1$ on a Hilbert space **H** and G a group of automorphisms of **M**. Let $\alpha \to U_\alpha$ be a faithful unitary representation of G such that $U^*_\alpha A U_\alpha = A^\alpha$ for all $A \in \mathbf{M}$. For each $A \in \mathbf{M}$, $\beta \in G$ we define the operators $\widetilde{A}$, $\widetilde{U}_\beta$ on $\mathbf{H} \otimes l_2(G)$ by

$$\widetilde{A}\left(\sum_{\alpha\in G}\varphi_\alpha \otimes \varepsilon_\alpha\right) = \sum_{\alpha\in G} A\,\varphi_\alpha \otimes \varepsilon_\alpha$$

$$\widetilde{U}_\beta\left(\sum_{\alpha\in G}\varphi_\alpha \otimes \varepsilon_\alpha\right) = \sum_{\alpha\in G} U_\beta\varphi_\alpha \otimes \varepsilon_{\beta\alpha}$$

1) An automorphism of a factor means a $*$-automorphism.

2) The definition of $\widetilde{\mathbf{M}}$ will be given in §2.

for all $\sum_{\alpha\in G}\varphi_\alpha \otimes \varepsilon_\alpha \in \mathbf{H}\otimes l_2(G)$. The set $\{\widetilde{A}\,; A \in \mathbf{M}\}$ will be denoted by $\widetilde{\mathbf{M}}$. Then, $\alpha \rightarrow \widetilde{U}_\alpha$ is a faithful unitary representation of G on $\mathbf{H}\otimes l_2(G)$ such as $\widetilde{U}_\alpha^* \widetilde{A}\widetilde{U}_\alpha = \widetilde{A^\alpha}$ for $A \in \mathbf{M}$. The W^*-algebra on $\mathbf{H}\otimes l_2(G)$ generated by $\{\widetilde{\mathbf{M}}, \widetilde{U}_\alpha;\alpha \in G\}$ is the crossed product of $\mathbf{M}$ by G in the sense of [1], and is denoted by $(\mathbf{M}, G)$. Each element $A \in (\mathbf{M}, G)$ is uniquely expressed in the form $A = \sum'_{\alpha\in G}\widetilde{A}_\alpha\widetilde{U}_\alpha$, where $A_\alpha \in \mathbf{M}$ and $\sum'$ is taken in the sense of the metrical convergence.

3. To prove Theorem 1 we shall provide two lemmas. The proof of Lemma 1 is seen in the proof of [1; Theorem 3], and so we omit the proof.

LEMMA 1. *Let* $\mathbf{M}$ *be a finite factor and* α *an outer automorphism of* $\mathbf{M}$. *If* $A \in \mathbf{M}$ *possesses the property that* $BA = AB^\alpha$ *for all* $B \in \mathbf{M}$, *then* $A = 0$.

Throughout the remainder of § 3 we shall use the symbols $\mathbf{M}, \mathbf{N}$ and G with the meaning attributed to them in the statement of Theorem 1 in § 1. Then each element $A \in \mathbf{N}$ can be uniquely expressed in the form $A = \sum'_{\alpha\in G}\widetilde{A}_\alpha\widetilde{U}_\alpha$ where $A_\alpha \in \mathbf{M}$ for each $\alpha \in G$. We denote the set of α-components[3] A_α of $A \in \mathbf{N}$ by $\mathbf{N}_\alpha$ for each $\alpha \in G$. Then it is easily seen that each $\mathbf{N}_\alpha$ is a two-sided ideal of $\mathbf{M}$. Since $\mathbf{M}$ is topologically, and so algebraically simple, either $\mathbf{N}_\alpha = (0)$ or $\mathbf{N}_\alpha = \mathbf{M}$ for each $\alpha \in G$. Let $G_0 = \{\alpha \in G\,;\, \mathbf{N}_\alpha \neq (0)\}$. Then G_0 is a subgroup of G and we have the following

LEMMA 2. *For each* $\alpha \in G_0$, $\widetilde{U}_\alpha \in \mathbf{N}$.

PROOF. Let ε be the conditional expectation of $(\mathbf{M}, G)$ relative to $\mathbf{N}$ in the sense of [3]. Fix an arbitrary $\alpha_0 \in G_0$. Since $\widetilde{U}^\varepsilon_{\alpha_0} \in \mathbf{N}$, $\widetilde{U}^\varepsilon_{\alpha_0}$ is uniquely expressed in the form

$$\widetilde{U}^\varepsilon_{\alpha_0} = \sum\nolimits'_{\alpha\in G_0}\widetilde{A}_\alpha\widetilde{U}_\alpha, \tag{*}$$

where $A_\alpha \in \mathbf{M}$ for every $\alpha \in G_0$. As $\widetilde{B}\widetilde{U}_{\alpha_0} = \widetilde{U}_{\alpha_0}\widetilde{B^{\alpha_0}}$ for any $B \in \mathbf{M}$, we have $\widetilde{B}\widetilde{U}^\varepsilon_{\alpha_0} = \widetilde{U}^\varepsilon_{\alpha_0}\widetilde{B^{\alpha_0}}$ for all $B \in \mathbf{M}$. Hence by (*) and the uniqueness of the expression

$$BA_\alpha = A_\alpha B^{\alpha_0\alpha^{-1}} \qquad \text{for all } B \in \mathbf{M} \text{ and } \alpha \in G_0. \tag{**}$$

Lemma 1 and (**) imply that $A_\alpha = 0$ for all $\alpha \in G_0$, $\alpha \neq \alpha_0$ and $\widetilde{U}^\varepsilon_{\alpha_0} = \widetilde{A}_{\alpha_0}\widetilde{U}_{\alpha_0}$. Further $BA_{\alpha_0} = A_{\alpha_0}B$ for all $B \in \mathbf{M}$ by (**), and so $A_{\alpha_0} = \lambda I$, $\widetilde{U}^\varepsilon_{\alpha_0} = \lambda\widetilde{U}_{\alpha_0}$ where λ is a scalar. Thus $(1-\lambda)\widetilde{U}^\varepsilon_{\alpha_0} = 0$, and either $\lambda = 1$ or $\widetilde{U}^\varepsilon_{\alpha_0} = 0$. This shows that either $\widetilde{U}^\varepsilon_{\alpha_0} = \widetilde{U}_{\alpha_0} \in \mathbf{N}$ or $\widetilde{U}^\varepsilon_{\alpha_0} = 0$. Suppose that $\widetilde{U}^\varepsilon_{\alpha_0} = 0$. Since $\mathbf{N}_{\alpha_0} = \mathbf{M}$,

3) For $A = \sum'_{\alpha\in G}\widetilde{A}_\alpha\widetilde{U}_\alpha \in (\mathbf{M}, G)$, $A_\alpha \in \mathbf{M}$ is called the α-component of A.

there exists an element $B = \sum'_{\alpha\in G_0} \widetilde{B}_\alpha \widetilde{U}_\alpha \in \mathbf{N}$ such as $B_{\alpha_0} \neq 0$. Then

$$\sum'_{\alpha\in G_0} \widetilde{B}_\alpha \widetilde{U}_\alpha = B = B^\varepsilon = \sum'_{\alpha\in G_0'} \widetilde{B}_\alpha \widetilde{U}_\alpha$$

where G_0' is a subset of G_0 which does not contain α_0. This contradicts the uniqueness of the expression of each element of $\mathbf{N}$. Hence $\widetilde{U}_\alpha \in \mathbf{N}$ for all $\alpha \in G$.

PROOF OF THEOREM 1. By Lemma 2, $R(\widetilde{\mathbf{M}}, \widetilde{U}_\alpha ; \alpha \in G_0) \subseteqq \mathbf{N}$. On the other hand, it is obvious that $R(\widetilde{\mathbf{M}}, \widetilde{U}_\alpha ; \alpha \in G_0) \supseteqq \mathbf{N}$ by the definition of G_0. Thus $\mathbf{N} = R(\widetilde{\mathbf{M}}, \widetilde{U}_\alpha ; \alpha \in G_0)$. By Corollary of Theorem 5 in [1], $(\mathbf{M}, G_0)$ is isomorphic to $R(\widetilde{\mathbf{M}}, \widetilde{U}_\alpha ; \alpha \in G_0)$ and the theorem is proved.

4. In this section we shall prove Theorem 2 and use the notations $\mathbf{M}$, G, H and $\mathbf{I}$ with the same meaning as those in the statement of Theorem 2.

Let $\alpha \to U_\alpha$ (resp. $\sigma \to V_\sigma$) be the unitary representation of G (resp. H) which appears in the construction of the crossed product $(\mathbf{M}, G)$ (resp. $(\mathbf{M}, H)$).

PROOF OF NECESSITY. Let Φ be an isomorphism between $(\mathbf{M}, G)$ and $(\mathbf{M}, H)$ which leaves $\widetilde{\mathbf{M}}$ invariant. Then Φ induces an automorphism η of $\widetilde{\mathbf{M}}$ defined by $\widetilde{A}^\eta = \Phi(\widetilde{A})$ for all $A \in \mathbf{M}$. For each $\alpha \in G$ and $A \in \mathbf{M}$, we have

$$\Phi(\widetilde{U}_\alpha)^* \widetilde{A} \Phi(\widetilde{U}_\alpha) = \Phi(\widetilde{U}_\alpha^*) \Phi(\widetilde{A}^{\eta^{-1}}) \Phi(\widetilde{U}_\alpha) = \Phi(\widetilde{U}_\alpha^* \widetilde{A}^{\eta^{-1}} \widetilde{U}_\alpha) = \widetilde{A}^{\eta^{-1}\alpha\eta}$$

and so $\Phi(\widetilde{U}_\alpha)$ induces an automorphism $\eta^{-1}\alpha\eta$ of $\widetilde{\mathbf{M}}$. Putting $\Phi(\widetilde{U}_\alpha) = \sum'_{\sigma\in H} \widetilde{A}_\sigma \widetilde{V}_\sigma \in (\mathbf{M}, H)$, we have

$$(\sum'_{\sigma\in H} \widetilde{A}_\sigma \widetilde{V}_\sigma)\, \widetilde{A}^{\eta^{-1}\alpha\eta} = \widetilde{A}(\sum'_{\sigma\in H} \widetilde{A}_\sigma \widetilde{V}_\sigma)$$

for all $A \in \mathbf{M}$. Thus we have for each $\sigma \in H$

$$A_\sigma A^{\eta^{-1}\alpha\eta\sigma^{-1}} = AA_\sigma \qquad \text{for all } A \in \mathbf{M}.$$

Hence, by Lemma 1, except for $\sigma \in H$ such as $\eta^{-1}\alpha\eta \equiv \sigma \pmod{\mathbf{I}}$ $A_\sigma = 0$. On the other hand, if there exist σ_1, σ_2 such as $\eta^{-1}\alpha\eta \equiv \sigma_1 \pmod{\mathbf{I}}$, $\eta^{-1}\alpha\eta \equiv \sigma_2 \pmod{\mathbf{I}}$, $\sigma_1\sigma_2^{-1} = (\eta^{-1}\alpha\eta\sigma_1^{-1})^{-1}(\eta^{-1}\alpha\eta\sigma_2^{-1}) \in H$ is inner, and so $\sigma_1 = \sigma_2$ by our assumption on H. If all $\eta^{-1}\alpha\eta\sigma^{-1}$ are outer, $A_\sigma = 0$ for all $\sigma \in H$ and $\Phi(\widetilde{U}_\alpha) = 0$, which is a contradiction. Therefore, for each $\alpha \in G$ there exists a unique $\sigma \in H$ and a unique unitary operator $W_\alpha \in \mathbf{M}$ such that $\eta^{-1}\alpha\eta \equiv \sigma \pmod{\mathbf{I}}$, $\Phi(\widetilde{U}_\alpha) = \widetilde{W}_\alpha \widetilde{V}_\sigma$ and W_α induces and inner automorphism $\eta^{-1}\alpha\eta\sigma^{-1}$ of $\mathbf{M}$. Next we shall prove that the mapping $\alpha \to \sigma$ defined above gives an isomorphism of G onto H. To prove that the mapping $\alpha \to \sigma$ is onto, let $\widetilde{V}_\sigma = \Phi\left(\sum'_{\alpha\in G} \widetilde{A}_\alpha \widetilde{U}_\alpha\right)$, then we have

$$\widetilde{A}\widetilde{V}_\sigma = \Phi(\widetilde{A}^{\eta^{-1}})\Phi\left(\sum'_{\alpha\in G}\widetilde{A}_\alpha\widetilde{U}_\alpha\right) = \Phi\left(\sum'_{\alpha\in G}\widetilde{A}^{\eta^{-1}}\widetilde{A}_\alpha\widetilde{U}_\alpha\right)$$
$$= \widetilde{V}_\sigma\widetilde{A}^\sigma = \Phi\left(\sum'_{\alpha\in G}\widetilde{A}_\alpha\widetilde{U}_\alpha\right)\Phi(\widetilde{A}^{\sigma\eta^{-1}})$$
$$= \Phi\left(\sum'_{\alpha\in G}\widetilde{A}_\alpha\widetilde{A}^{\eta^{-1}\alpha^{-1}}\widetilde{U}_\alpha\right),$$

for all $A \in \mathbf{M}$, and for each $\alpha \in G, AA_\alpha^\eta = A_\alpha^\eta A^{\sigma\eta^{-1}\alpha^{-1}\eta}$ for all $A \in \mathbf{M}$. By the same reason as before, there is a unique $\alpha \in G$ such as $\eta^{-1}\alpha\eta \equiv \sigma \pmod{\mathbf{I}}$. Thus the mapping is onto. Since $\Phi(\widetilde{U}_\alpha) = \widetilde{W}_\alpha\widetilde{V}_\sigma$, $\Phi(\widetilde{U}_\beta) = \widetilde{W}_\beta\widetilde{V}_\tau$ for each pair $\alpha,\beta \in G$.

$$\Phi(\widetilde{U}_{\alpha\beta}) = \Phi(\widetilde{U}_\alpha)\Phi(\widetilde{U}_\beta) = (\widetilde{W}_\alpha\widetilde{V}_\sigma)(\widetilde{W}_\beta\,\widetilde{V}_\tau) = \widetilde{W}_\alpha\widetilde{W}_\beta^{\sigma^{-1}}\widetilde{V}_{\sigma\tau}$$
$$= \widetilde{W}_{\alpha\beta}\widetilde{V}_\omega, \text{ where } \eta^{-1}\beta\eta \equiv \tau \pmod{\mathbf{I}},\ \eta^{-1}\alpha\beta\eta \equiv \omega \pmod{\mathbf{I}},$$

and $\widetilde{W}_{\alpha\beta}^*\widetilde{W}_\alpha\widetilde{W}_\beta^{\sigma^{-1}} = \widetilde{V}_\omega\widetilde{V}_{\sigma\tau}^* \in \mathbf{M}$. Thus $\omega = \sigma\tau$ and $W_{\alpha\beta}^\sigma = W_\alpha^\sigma W_\beta$ for each α, $\beta \in G$. Hence the necessity of (1), (2) is proved.

PROOF OF SUFFICIENCY. Suppose that the conditions (1), (2) are satisfied. We define a mapping Φ of $(\mathbf{M}, G)$ into $(\mathbf{M}, H)$ by

$$\Phi\left(\sum'_{\alpha\in G}\widetilde{A}_\alpha\widetilde{U}_\alpha\right) = \sum'_{\sigma\in H}\widetilde{B}_\sigma\,\widetilde{V}_\sigma \text{ for each } \sum'_{\alpha\in G}\widetilde{A}_\alpha\widetilde{U}_\alpha \in (\mathbf{M}, G),$$

where $\sigma \equiv \eta^{-1}\alpha\eta \pmod{\mathbf{I}}$ and $\widetilde{B}_\sigma = \widetilde{A}_\alpha^\eta\,\widetilde{W}_\alpha$ for $\alpha \in G$.

If $\Phi\left(\sum'_{\alpha\in G}\widetilde{A}_\alpha\widetilde{U}_\alpha\right) = 0$, we have $A_\alpha^\eta W_\alpha = 0$ for each $\alpha \in G$, and $A_\alpha = 0$ for all $\alpha \in G$. Thus the mapping Φ is one-to-one. Further, for any $\sum'_{\sigma\in H}\widetilde{A}_\sigma\,\widetilde{V}_\sigma$ $\in (\mathbf{M}, H)$

$$\Phi\left(\sum'_{\alpha\in G}\widetilde{B}_\alpha^{\eta^{-1}}\widetilde{W}_\alpha^{*\eta^{-1}}\widetilde{U}_\alpha\right) = \sum'_{\sigma\in H}\widetilde{A}_\sigma\widetilde{V}_\sigma,$$

where $\eta^{-1}\alpha\eta \equiv \sigma \pmod{\mathbf{I}}$ and $B_\alpha = A_\sigma$ for all $\alpha \in G$, and thus the mapping Φ is onto.

To complete the proof we need only to prove that

$$\Phi\left(\left(\sum'_{\alpha\in G}\widetilde{A}_\alpha\widetilde{U}_\alpha\right)^*\right) = \left(\Phi\left(\sum'_{\alpha\in G}\widetilde{A}_\alpha\widetilde{U}_\alpha\right)\right)^*$$
$$\Phi\left(\left(\sum'_{\alpha\in G}\widetilde{A}_\alpha\widetilde{U}_\alpha\right)\left(\sum'_{\alpha\in G}\widetilde{B}_\alpha\widetilde{U}_\alpha\right)\right) = \Phi\left(\sum'_{\alpha\in G}\widetilde{A}_\alpha\widetilde{U}_\alpha\right)\Phi\left(\sum'_{\alpha\in G}\widetilde{B}_\alpha\widetilde{U}_\alpha\right),$$

and $\left[\left[\Phi\left(\sum'_{\alpha\in G}\widetilde{A}_\alpha\widetilde{U}_\alpha\right)\right]\right] = \left[\left[\sum'_{\alpha\in G}\widetilde{A}_\alpha\widetilde{U}_\alpha\right]\right]$

for any $\sum'_{\alpha\in G}\widetilde{A}_\alpha\widetilde{U}_\alpha$, $\sum'_{\alpha\in G}\widetilde{B}_\alpha\widetilde{U}_\alpha \in (\mathbf{M}, G)$. Since we have $W_\alpha^* = W_{\alpha^{-1}}$ by condition (2),

$$\Phi\left(\left(\sum'_{\alpha\in G}\widetilde{A}_\alpha\widetilde{U}_\alpha\right)^*\right) = \Phi\left(\sum'_{\alpha\in G}\widetilde{A}_\alpha^{*\alpha}\widetilde{U}_{\alpha^{-1}}\right) = \sum'_{\sigma\in H}\widetilde{B}_{\sigma^{-1}}\widetilde{V}_{\sigma^{-1}},$$

where $\sigma \equiv \eta^{-1}\alpha\eta (\text{mod } \mathbf{I})$ and $\widetilde{B}_{\sigma^{-1}} = \widetilde{A}_\alpha^{*\alpha\eta}\widetilde{W}_{\alpha^{-1}}$, and so

$$\widetilde{B}_{\sigma^{-1}}\widetilde{V}_{\sigma^{-1}} = \widetilde{V}_{\sigma^{-1}}(\widetilde{A}_\alpha^{*\alpha\eta}\widetilde{W}_\alpha^{*\sigma})^{\sigma^{-1}} = \widetilde{V}_{\sigma^{-1}}(\widetilde{W}_\alpha^{\sigma}\widetilde{A}_\alpha^{*\alpha\eta\eta^{-1}\alpha^{-1}\eta\sigma})^{\sigma^{-1}}$$
$$= \widetilde{V}_{\sigma^{-1}}\widetilde{W}_\alpha^{*}\widetilde{A}_\alpha^{*\eta}.$$

Thus, $\Phi\left(\left(\sum'_{\alpha\in G}\widetilde{A}_\alpha\widetilde{U}_\alpha\right)^*\right) = \left(\Phi\left(\sum'_{\alpha\in G}\widetilde{A}_\alpha\widetilde{U}_\alpha\right)\right)^*.$

Moreover,

$$\Phi\left(\sum'_{\alpha\in G}\widetilde{A}_\alpha\widetilde{U}_\alpha\right)\Phi\left(\sum'_{\alpha\in G}\widetilde{B}_\alpha\widetilde{U}_\alpha\right) = \sum'_{\sigma,\tau\in H}\widetilde{C}_\sigma\widetilde{D}_\tau^{\sigma^{-1}}\widetilde{V}_{\sigma\tau}$$

where $\sigma \equiv \eta^{-1}\alpha\eta (\text{mod } \mathbf{I})$, $\tau \equiv \eta^{-1}\beta\eta (\text{mod } \mathbf{I})$, $\widetilde{C}_\sigma = \widetilde{A}_\alpha^{\eta}\widetilde{W}_\alpha$ and $\widetilde{D}_\tau = \widetilde{B}_\beta^{\eta}\widetilde{W}_\beta$, and so

$$\widetilde{C}_\sigma\widetilde{D}_\tau^{\sigma^{-1}}\widetilde{V}_{\sigma\tau} = (\widetilde{A}_\alpha^{\eta}\widetilde{W}_\alpha)(\widetilde{B}_\beta^{\eta}\widetilde{W}_\beta)^{\sigma^{-1}}\widetilde{V}_{\sigma\tau} = \widetilde{A}_\alpha^{\eta}\widetilde{B}_\beta^{\sigma^{-1}\eta}\widetilde{W}_\alpha\widetilde{W}_\beta^{\sigma^{-1}}\widetilde{V}_{\sigma\tau}$$
$$= \widetilde{A}_\alpha^{\eta}\widetilde{B}_\beta^{\alpha^{-1}\eta}\widetilde{W}_{\alpha\beta}\widetilde{V}_{\sigma\tau}.$$

Hence,

$$\Phi\left(\sum'_{\alpha\in G}\widetilde{A}_\alpha\widetilde{U}_\alpha\right)\Phi\left(\sum'_{\alpha\in G}\widetilde{B}_\alpha\widetilde{U}_\alpha\right)$$
$$= \Phi\left(\sum'_{\alpha,\beta\in G}\widetilde{A}_\alpha\widetilde{B}_\beta^{\alpha^{-1}}\widetilde{U}_{\alpha\beta}\right)$$
$$= \Phi\left(\left(\sum'_{\alpha\in G}\widetilde{A}_\alpha\widetilde{U}_\alpha\right)\left(\sum'_{\alpha\in G}\widetilde{B}_\alpha\widetilde{U}_\alpha\right)\right).$$

Finally,

$$\left[\left[\Phi\left(\sum'_{\alpha\in G}\widetilde{A}_\alpha\widetilde{U}_\alpha\right)\right]\right]^2 = \left[\left[\sum'_{\sigma\in H}\widetilde{B}_\sigma\widetilde{V}_\sigma\right]\right]^2 = \sum_{\alpha\in G}[[A_\alpha^{\eta}W_\alpha]]^2$$
$$= \sum_{\alpha\in G}[[A_\alpha]]^2 = \left[\left[\sum'_{\alpha\in G}\widetilde{A}_\alpha\widetilde{U}_\alpha\right]\right]^2,$$

where $\sigma \equiv \eta^{-1}\alpha\eta (\text{mod } \mathbf{I})$ and $\widetilde{B}_\sigma = \widetilde{A}_\alpha^{\eta}\widetilde{W}_\alpha$, and the proof of sufficiency is completed.

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