ON SIMPLE GROUPS ASSOCIATED WITH THE REAL SIMPLE LIE ALGEBRAS

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Introduction. In this paper we shall construct the simple groups analogous to the non compact real forms of the simple Lie groups and extend to them the structural properties of the groups of Chevalley [2]. The family of the groups obtained here contain some simple groups constructed by R.Steinberg [4]. Further it seems that the simple groups obtained from the Lie algebras of classical types are identified with the simple groups obtained from the unitary or orthogonal groups of non zero indices.

Whether the infinite groups constructed are new or not has not been settled yet, it seems that there are no new simple finite groups among them.

Section 1 contains preliminaries from the theory of real simple Lie algebras and we shall introduce the restricted root systems (cf. [3]) which play an important role throughout the paper. Section 3 contains brief description of Chevalley groups constructed in [2] from an arbitrary field and a complex semi-simple Lie algebra. In section 4, making use of the involution defined in section 2, we define the groups analogous to the real forms of the semi-simple Lie groups and also we shall have some structural properties of the groups. In section 5, we shall introduce some maximal subgroups of the groups which are analogous to those of [1]. Excepting few cases, the proof of simplicity of the groups obtained from the simple Lie algebras is given in section 6.

1. Restricted root systems.

1.1. Let $\mathfrak{g}$ be a semi-simple Lie algebra over the field R of real numbers and let $\mathfrak{k}$ be a maximal compact subalgebra of $\mathfrak{g}$, i,e., a subalgebra of $\mathfrak{g}$ corresponding to a maximal compact subgroup of the adjoint group of $\mathfrak{g}$. Let $\mathfrak{g}_C$ be the complexification of $\mathfrak{g}$. Then there exists a uniquely determined compact form $\mathfrak{g}_u$ of $\mathfrak{g}_C$ such that $\mathfrak{g} \cap \mathfrak{g}_u = \mathfrak{k}$ and that, denoting by $\mathfrak{p}$ the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to the Killing form, we have

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p} \ , \ \mathfrak{g}_u = \mathfrak{k} + \sqrt{-1}\,\mathfrak{p}.$$

Let $\mathfrak{h}^-$ be a maximal abelian subalgebra in $\mathfrak{p}$; it can be extended to a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, i.e., a maximal subalgebra of $\mathfrak{g}$ such that the adjoint representation of any $H \in \mathfrak{h}$ is semi-simple. Then we have

$$\mathfrak{h} = \mathfrak{h}^+ + \mathfrak{h}^-, \qquad \mathfrak{h}^+ = \mathfrak{h} \cap \mathfrak{k}, \qquad \mathfrak{h}^- = \mathfrak{h} \cap \mathfrak{p}.$$

Let $\mathfrak{h}_C$ be the complexification of $\mathfrak{h}$ and let

$$\mathfrak{g}_C = \mathfrak{h}_C + \sum_{r \in \Delta} \mathfrak{g}_r$$

be the corresponding Cartan decomposition of $\mathfrak{g}_C$ where Δ denotes the root system of $\mathfrak{g}_C$ relative to $\mathfrak{h}_C$. Let further $\mathfrak{h}_0$ be the subspace of $\mathfrak{h}_C$ over R consisting of all $H \in \mathfrak{h}_C$ such that $r(H)$ is real for all $r \in \Delta$, then $\mathfrak{h}_0 = \sqrt{-1}\,\mathfrak{h}^+ + \mathfrak{h}^-$; $\mathfrak{h}_0$ becomes a real euclidean space with respect to the Killing form, so that we can consider Δ as a subset of $\mathfrak{h}_0$(i. e. we identify $r \in \Delta$ with the uniquely determined element H'_r of $\mathfrak{h}_0$ such that $(H'_r, H) = r(H)$ for all $H \in \mathfrak{h}_0$, ( ) denoting the Killing form).

Let σ be the conjugation of $\mathfrak{g}_C$ with respect to the real form $\mathfrak{g}$. σ induces an orthogonal transformation of $\mathfrak{h}_0$ which leaves Δ invariant. We denote $\sigma(r)$ by $\bar{r}$ for $r \in \Delta$ and put

$$\Delta_0 = \{r \in \Delta\,;\, \bar{r} = -r\}$$

Δ_0 becomes a subsystem of Δ; we denote the ranks of Δ, Δ_0 by l, l_0 respectively.

In Δ we can define a linear order satisfying the following property which we call a σ-order:

If $r \notin \Delta_0$ and $r > 0$, then $\bar{r} > 0$.

(For instance, take a lexicographical order with respect to a base $(H_1, \cdots\cdots, H_l)$ of $\mathfrak{h}_0$ such that $(H_1, \cdots\cdots, H_p)$ forms a base of $\mathfrak{h}^-$.) We call a fundamental system of Δ corresponding to a σ-order a σ-fundamental system.[1] If $\Pi = \{a_1, \cdots\cdots, a_l\}$ is a σ-fundamental system of Δ, then $\Pi_0 = \Pi \cap \Delta_0$ is a fundamental system of Δ_0. For any root $r = \sum_{i=1}^{l} c_i a_i$, $h(r) = \sum_{i=1}^{l} c_i$ is called the height of the root r. From now on we shall assume that $\mathfrak{g}$ is not compact, i.e., $\mathfrak{h}^- \neq (0)$ what is the same $\Delta - \Delta_0 \neq (0)$.

1. 2. Consider now the set $\Delta - \Delta_0$. For any vector v in $\mathfrak{h}_0$ we denote by v^* the projection of v to $\mathfrak{h}^-$; for any two roots r, s in $\Delta - \Delta_0$, we call r, s are equivalent if and only if $r^* = ks^*$ for some positive rational number k and denote it by $r \sim s$; for any root $r \in \Delta - \Delta_0$, we denote by Σ_r the class of equivalent elements containing r. Then Σ_r satisfies the following properties:

(1) For $r_1, r_2 \in \Sigma_r$ if $r_1 + r_2 \in \Delta$, then $r_1 + r_2 \in \Sigma_r$,

(2) For $r_1, r_2 \in \Sigma_r$ such that $r_1 - r_2 \in \Delta$, if r_1, r_2 and $r_1 - r_2$ are all positive or all negative relative to a σ-order, then $r_1 - r_2 \in \Sigma_r$,

1) cf. Satake [3] p. 80.

(3) Σ_r is σ-invariant, where we call a subset Δ' of Δ σ-invariant if σ transforms the elements of Δ' into itself.

On the other hand, the roots r of $\Delta - \Delta_0$ have one and only one property of the following:

I $r = \bar{r}$ and can not be written as a sum of a conjugate pair of roots.

II $r = \bar{r}$ and r is a sum of a conjugate pair of roots.

III $r \neq \bar{r}$ and $r + \bar{r}$ is not a root.

IV $r \neq \bar{r}$ and $r + \bar{r}$ is a root.

We call a root $r \in \Delta - \Delta_0$ is of type I, II, III or IV respectively if it satisfies the corresponding property. Then we have the following:

LEMMA 1. *If* $r \in \Delta - \Delta_0$ *is of type* III, *then* r *and* $\bar{r}$ *are orthogonal.*

For the proof, see Satake [3], Appendix, proof of Lemma 2, p.107.

LEMMA 2. *If* $r \in \Delta - \Delta_0$ *is of type* IV, $r + \bar{r}$ *is the only root which can be represented by a linear combination of* r *and* $\bar{r}$ *with positive coefficients.*

PROOF. Let Δ_r be the σ-invariant subsystem of Δ generated by r. Since r is of type IV, the rank of Δ_r is 2. The linear order of Δ_r induced by a σ-order of Δ is also called σ-order of Δ_r. If Δ_r is of type A_2 the lemma is obvious. Suppose Δ_r is of type B_2. Then $r - \bar{r}$ is a root. In fact, if $r - \bar{r}$ is not a root, then $r, \bar{r}$ is a fundamental root system of Δ_r with respect to its σ-order. Thus $r + 2\bar{r}$ or $2r + \bar{r}$ is a root and then we have both of them are roots in Δ_r. This is a contradiction. Therefore $r - \bar{r}$ is a root. We put $s = r - \bar{r}$, then $s \in \Delta_0$ and we may assume that s is positive with respect to σ-order. Then (r, s) is a σ-fundamental system of Δ_r and we have our assertion. If Δ_r is of type G_2, then $\Delta = \Delta_r$, this contradicts to the fact that r is of type IV. Thus we complete the proof of the lemma.

For any roots s of Σ_r, s^* are vectors in $\mathfrak{h}^-$ which have a same direction. Among the vectors s^* there exists a uniquely determined vector r^* such that the absolute value of the length of r^* is minimum. We call r^* the representative vector of Σ_r and denote by Δ^* the set of representative vectors. From the definition, there exists a one to one correspondence between the set of the classes of equivalent elements Σ_r and the set Δ^*. Therefore we denote hereafter Σ_r by Σ_{r^*}. For any root r in Σ_{r^*}, we denote by w_r^* the restriction to $\mathfrak{h}^-$ of w_r (resp. $w_r\ w_{\bar{r}}$ or $w_{r+\bar{r}}$) if r is of type I or II (resp. of type III or of type IV). Then w_r^* is an orthogonal transformation of $\mathfrak{h}^-$. Moreover we can easily see that

(4) if $s_1 \sim s_2$, then we have $w_r^*(s_1) \sim w_r^*(s_2)$,

(5) if $r_1 \sim r_2$, then we have $w_{r_1}^*(s) \sim w_{r_2}^*(s)$.

Therefore w_r^* is independent of the choice of r in Σ_{r^*} and uniquely determined by a class Σ_{r^*}, i.e., by an element r^* of Δ^* and we denote it by $w_{r^*}^*$.

We denote by W^* the finite group generated by $w_{r^*}^*$ for all $r^* \in \Delta^*$. Then we have

PROPOSITION 1. *Δ^* is a root system in $\mathfrak{h}^-$ and W^* is the Weyl group of Δ^*, i.e., Δ^* and W^* satisfy the following conditions*:

(6) *Δ^* is a finite subset of $\mathfrak{h}^-$.*

(7) *The vector space $\mathfrak{h}^-$ is generated by Δ^*.*

(8) *If $r^* \in \Delta^*$, then $-r^* \in \Delta^*$ and $\pm kr^* \notin \Delta^*$ for any positive integer $k > 1$.*

(9) *If $r^* \in \Delta^*$, then $w^*_{r^*}$ transforms Δ^* into itself and $w^*_{r^*}(r^*) = -r^*$.*

The proof is easily obtained from the definitions of Δ^* and W^*. We call Δ^* the restricted root system of $\mathfrak{g}$ and the rank of Δ^* the restricted rank of $\mathfrak{g}$.

1.3. Now let

(10) $\Pi = \{a_1, \cdots\cdots, a_{l-l_0}, a_{l-l_0+1}, \cdots\cdots, a_l\}$

be a σ-fundamental system of Δ such that $\Pi_0 = \{a_{l-l_0+1}, \cdots\cdots, a_l\}$ is a fundamental system of Δ_0. The σ-order of Δ induces on Δ^* a linear order which we call a σ-order of Δ^*. We can construct a fundamental system Π^* of Δ^* with respect to the σ-order in the following way which we call a restricted fundamental system relative to $\mathfrak{h}^-$.[2] Namely, there exists a permutation $i \to i'$ of order 2 of the set of indices $\{1, \cdots\cdots, l-l_0\}$ such that

$$\bar{a}_i = a_{i'} + \sum_{j=l-l_0+1}^{l} c_j^{(i)} a_j,\ c_j^{(i)} \geqq 0 \text{ for } 1 \leqq i \leqq l - l_0.^{3)}$$

Thus we can put $l - l_0 = p_1 + 2p_2$, $p_1 + p_2 = p$ and

$$(11)\qquad i' = \begin{cases} i & \text{for } 1 \leqq i \leqq p_1 \\ i + p_2 & \text{for } p_1 + 1 \leqq i \leqq p_1 + p_2 \\ i - p_2 & \text{for } p_1 + p_2 + 1 \leqq i \leqq p_1 + 2p_2. \end{cases}$$

Now we put $\Pi^* = \{a_1^*, \cdots\cdots, a_p^*\}$, then we have

PROPOSITION 2. *Π^* is a fundamental system of Δ^* relative to a σ-order of Δ^*, i.e., it satisfies the following conditions*:

(12) *a_i^* is an element of Δ^*.*

(13) *$a_1^*, \cdots\cdots, a_p^*$ are linearly indenpendent.*

(14) *If $i \neq j$, then $-2(a_i^*, a_j^*)/(a_i^*, a_i^*)$ is a non-negative integer.*

(15) *For any positive root $r^* \in \Delta^*$, r^* can be expressed by $r^* = \sum_i^p c_i a_i^*$, where c_i are non negative integers.*

PROOF. As for the proof of (13) and (14), see Satake [3]. If $r = \sum_{i=1}^{l} c_i a_i$ is

2) cf. Satake [3], Lemma 1, p. 80.

3) cf. ibid.

a root in Δ, then we have $r^* = \sum_{i=1}^{p_1} c_i a_i^* + 2\sum_{i=p_1+1}^{p} c_i a_i^*$. Therefore we have (15). We shall prove (12). Assume that there exists a root r^* such that $|a_i^*| \geqq |r^*|$. Then $a_i^* = k\left(\sum_{i=1}^{p_1} c_i a_i^* + 2\sum_{i=p_1+1}^{p} c_i a_i^*\right)$, where $r = \sum_{i=1}^{l} c_i a_i$ and $k \geqq 1$ is a rational number. Since $a_1^*, \ldots\ldots, a_p^*$ are linearly independent and c_i are non negative integers, $c_j = 0$ if $j \neq i$ and $c_i = k = 1$. Therefore a_i^* is an element of Δ^*.

We denote by W the Weyl group of Δ. Let W_σ be the subgroup of W consisting of all $w \in W$ such that $w\sigma = \sigma w$, what is the same $w\mathfrak{h}^- = \mathfrak{h}^-$; let W_0 be the subgroup of W consisting of all $v \in \mathfrak{h}^-$ such that $w(v) = v$ for all $v \in \mathfrak{h}^-$. Then W_0 is a normal subgroup of W_σ and the factor group W_σ / W_0 is isomorphic to W^*[4].

Let ρ be the natural homomorphism of W_σ onto W^* and we put $\rho(w) = w^*$ for $w \in W_\sigma$. Then we have that w transforms Σ_{r^*} onto $\Sigma_{w^*(r^*)}$, for, if $s \in \Sigma_{r^*}$ and $w \in W_\sigma$, we have $w(s)^* = w^*(s^*)$.

2. Construction of an involution.

2.1. The notations being as in previous section, let $(H_1, \ldots\ldots, H_l, X_r, r \in \Delta)$ be a base of $\mathfrak{g}_C$ which satisfies the following conditions and we call it a canonical base of $\mathfrak{g}_C$:

$H_i \in \mathfrak{h}_C$ $(1 \leqq i \leqq l)$, $X_r \in \mathfrak{g}_r$ $(r \in \Delta)$

$[H, X_r] = r(H)X_r$ for $H \in \mathfrak{h}_C$, $r(H_i)$ is an integer for $1 \leqq i \leqq l$.

$[X_r, X_{-r}] = H_r$ where $H_r = 2H_r' / r(H_r')$

$$[X_r, X_s] = N_{r,s} X_{r+s} \quad \text{if } r+s \neq 0 \text{ is a root,}$$
$$= 0 \quad \text{if } r+s \neq 0 \text{ is not a root,}$$

where $N_{r,s} = \pm(p+1)$, p being the greatest integer $i \geqq 0$ such that $s - ir$ is a root.[5]

We shall define an automorphism σ_C of $\mathfrak{g}_c$ such that

$$\sigma_C H_r = H_{\bar{r}} \quad \text{and} \quad \sigma_C X_r = \mu_r X_{\bar{r}}$$

where μ_r are integers which shall be determined as follows.

In order that σ_C is an automorphism, it is sufficient that μ_r satisfy the following relations:

$$N_{r,s}\mu_{r+s} = N_{\bar{r},\bar{s}}\, \mu_r \mu_s, \tag{16}$$

$$\mu_r \mu_{-r} = 1. \tag{17}$$

So that we shall define first the numbers μ_r for the roots of Π. Namely we put

$\mu_{a_i} = -1$ for $a_i \in \Pi_0$ and $\mu_{a_i} = 1$ for $a_i \in \Pi - \Pi_0$.

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4) cf. Satake [3] Proposition A, p. 108.

5) cf. Chevalley [2], I, Th. 1.

For any positive root r, there exists a series of roots $(r_1,\cdots\cdots r_N)$ such that $r_1 = a_{i_1}$, $r_j = r_{j-1} + a_{i_j}$ $(2 \leqq j \leqq N,\ 1 \leqq i_1,\cdots\cdots,i_N \leqq l)$, $r_N = r$.[6] Therefore we put

$$\mu_r = \varepsilon\mu_{a_{i_1}}\mu_{a_{i_2}}\cdots\cdots\mu_{a_{i_N}}$$

$$(18) \qquad \varepsilon = \frac{N_{\bar{a}_{i_1},\bar{a}_{i_2}}\, N_{\bar{r}_2,\bar{a}_{i_3}}\cdots\cdots N_{\bar{r}_{N-1},\bar{a}_{i_N}}}{N_{a_{i_1},a_{i_2}}\, N_{r_2,a_{i_3}}\cdots\cdots N_{r_{N-1},a_{i_N}}}\ .$$

Then we can easily see that μ_r is independent of the choice of the series $(r_1,\cdots\cdots,r_N)$. Thus μ_r is uniquely determined by $\mu_{a_1},\cdots\cdots,\ \mu_{a_l}$. For any negative root $-r$, we put $\mu_{-r} = \mu_r^{-1}$. In this way we can define the numbers μ_r for all root $r \in \Delta$ which satisfy the relations (16) and (17). From definition, we have $\mu_r = \pm 1$ for any root $r \in \Delta$. Moreover, we have $\mu_r = -1$ for all root $r \in \Delta_0$ and $\mu_r = 1$ for all root r of type III. The automorphism σ_C defined here is of order 2 or 4.

2. 2. Now let K be an arbitrary field with involutive automorphism $\xi \to \bar{\xi}$, $\xi \in K$. Let K_0 be the subfield of K consisting of all the elements ξ of K such that $\xi = \bar{\xi}$. Then we have $K = K_0(\theta)$ for some element θ of K, and $\theta^2 = \alpha$ is an element of K_0. Let $\mathfrak{g}_K$ be the Lie algebra over K generated by $H_i \otimes 1_K$ $(1 \leqq i \leqq l)$ and $X_r \otimes 1_K$ $(r \in \Delta)$ where $\otimes$ is the tensor product and 1_K is the unit element of K. We denote them again by $H_i (1 \leqq i \leqq l)$, $X_r (r \in \Delta)$ and they form a base of $\mathfrak{g}_K$ which we call a canonical base of $\mathfrak{g}_K$.

We shall define a semi-linear automorphism σ_K of $\mathfrak{g}_K$ of order 2 such that

$$\sigma_K \xi H_r = \bar{\xi}\, H_{\bar{r}}, \qquad \sigma_K\, \xi\, X_r = \bar{\xi}\, \nu_r\, X_{\bar{r}}$$

where $\xi \in K$ and $\nu_r \in K$ which shall be determined as follows.

Let $\Pi = \{a_1,\cdots\cdots,a_l\}$ be a σ-fundamental system of Δ defined in 1.3. We put

$$\nu_{a_i} = -1 \text{ for } l - l_0 + 1 \leqq i \leqq l,\ \nu_{a_i} = 1 \text{ for } 1 \leqq i \leqq p_1$$
$$\nu_{a_i} = \nu_{a_i'} = 1 \text{ for } p_1 + 1 \leqq i \leqq p_1 + p_2 \text{ if } \mu_{a_i}\mu_{\bar{a}_i} = 1$$
$$\nu_{a_i} = \theta, \nu_{a_i'} = \alpha^{-1}\theta \text{ for } p_1 + 1 \leqq i \leqq p_1 + p_2 \text{ if } \mu_{a_i}\mu_{\bar{a}_i} = -1.$$

For any positive root r, we put

$$\nu_r = \varepsilon\nu_{a_{i_1}}\cdots\cdots\nu_{a_{i_N}}$$

where $\varepsilon = \pm 1$ defined by (18). For any negative root $-r$, we put $\nu_{-r} = \nu_r^{-1}$. Then $\nu_r \in K$ satisfy the following conditions which show that σ_K is a semi-linear automorphism of $\mathfrak{g}_K$ of order 2.

$$N_{r,s}\ \nu_{r+s} = N_{\bar{r},\bar{s}}\,\nu_r\nu_s,\ \nu_r\nu_{-r} = 1,\ \bar{\nu}_r\nu_{\bar{r}} = 1$$

In fact, the first two relations are obvious from the definition. The third relation is true for the roots a_i $(1 \leqq i \leqq l)$. For any positive root r, we see that $\bar{\nu}_r = \varepsilon\bar{\nu}_{a_{i_1}}\cdots\cdots\bar{\nu}_{a_{i_N}}$ and $\nu_{\bar{r}} = \varepsilon^{-1}\nu_{\bar{a}_{i_1}}\cdots\cdots\nu_{\bar{a}_{i_N}}$. Therefore we have $\bar{\nu}_r\nu_{\bar{r}} = 1$.

6) cf. Abe [1], Lemma 1.

3. Chevalley groups.

3.1. Let $\mathfrak{g}_C$ be a semi-simple Lie algebra over the complex number field C and $\mathfrak{h}_C$ be a Cartan subalgebra of $\mathfrak{g}_C$. $\mathfrak{g}_K$ being as in 2.2, let $(H_1, \cdots, H_l, X_r, r \in \Delta)$ be a canonical base of $\mathfrak{g}_K$. We denote by P the additive group generated by the weights (with respect to $\mathfrak{h}_C$) of all representations of $\mathfrak{g}_C$ and by $\overline{X}$ the group of all homomorphisms of P into the multiplicative group K^* of non zero elements of K. For any element χ of $\overline{X}$, there exists an automorphism of $\mathfrak{g}_K$ such that $H_i \to H_i$, $X_r \to \chi(r)\ X_r$ which we denote by $h(\chi)$; we denote by $\overline{\mathfrak{H}}$ the group of all automorphisms $h(\chi)$ of $\mathfrak{g}_K$ for $\chi \in \overline{X}$.

For any $r \in \Delta$ and any $\zeta \in K^*$, we denote by $\chi_{r,\zeta}$ the element of $\overline{X}$ such that $\chi_{r,\zeta}(s) = \zeta^{s(H_r)}$; we denote by $\overline{\mathfrak{H}}'$ the subgroup of $\overline{\mathfrak{H}}$ generated by $h(\chi_{r,\zeta})$ for all $r \in \Delta$, $\zeta \in K^*$.

For any root $r \in \Delta$, we put $x_r(\xi) = \exp\xi(\mathrm{ad}\ X_r)$ for $\xi \in C$. Then there is a matrix $A_r(T)$ whose coefficients are polynomials of T with integer coefficients such that the matrix of $x_r(\xi)$ with respect to the canonical base of $\mathfrak{g}_C$ is $A_r(\xi)$. Making use of the matrix $A_r(T)$ and of the canonical base of $\mathfrak{g}_K$, we can define an automorphism of $\mathfrak{g}_K$ which we denote also by $x_r(\xi)$, $\xi \in K$. Denote by $\overline{\mathfrak{U}}$ (resp. $\overline{\mathfrak{V}}$) the group of automorphisms of $\mathfrak{g}_K$ generated by $x_r(\xi)$, $\xi \in K$, where r runs over all the positive (resp. negative) roots with respect to the linear order of Δ defined by a σ-fundamental system Π.

Now we denote by $\overline{G}$ the group of automorphisms of $\mathfrak{g}_K$ generated by $\overline{\mathfrak{H}}, \overline{\mathfrak{U}}$ and $\overline{\mathfrak{V}}$ and by $\overline{G}'$ the subgroup of $\overline{G}$ generated by $\overline{\mathfrak{U}}$ and $\overline{\mathfrak{V}}$. Then we have $\overline{\mathfrak{H}}' \subset \overline{G}'$ and $\overline{G}'$ is a normal subgroup of $\overline{G}$ and

$$\overline{G} / \overline{G}' \simeq \overline{\mathfrak{H}} / \overline{\mathfrak{H}}', \qquad \overline{G}' \cap \overline{\mathfrak{H}} = \overline{\mathfrak{H}}'.$$

3.2. For any two positive roots r,s in Δ, we have

$$x_r(\xi)x_s(\eta)\ x_r(-\xi) = x_s(\eta)\ \Pi\ x_{ir+js}(C_{i,j;r,s}\xi^i\eta^j)$$

where the product is taken over all pairs (i,j) of integers such that $ir+js$ is a root, the pairs being arranged such that the roots $ir+js$ form an increasing sequence, and where $C_{i,j;r,s}$ are integral constants depending only on Δ.[7]

For any root r, there is a homomorphism ϕ_r of $SL_2(K)$ onto a subgroup of $\overline{G}$ such that

$$\phi_r\begin{pmatrix}1 & \\ \xi & 1\end{pmatrix} = x_{-r}(\xi),\ \phi_r\begin{pmatrix}1 & \xi \\ & 1\end{pmatrix} = x_r(\xi),\ \phi_r\begin{pmatrix}\zeta & \\ & \zeta^{-1}\end{pmatrix} = h(\chi_{r,\zeta}).$$

We denote by $\omega_r = \phi_r\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$, then we have

$$\omega_r x_s(\xi)\omega_r^{-1} = x_{w_r(s)}\ (\eta_{r,s}\xi)$$

7) cf. Chevalley [3] p. 36.

where $\eta_{r,s} = \pm 1$ depending only on Δ.

For any $\chi \in \overline{X}$ and $r \in \Delta$, we have

$$h(\chi)x_r(\xi)\, h(\chi)^{-1} = x_r(\chi(r)\xi).$$

We denote by $\overline{\mathfrak{W}}$ the subgroup of $\overline{G}$ generated by $\overline{\mathfrak{H}}$ and the elements ω_r for all root $r \in \Delta$. Then there exists one and only one homomorphism τ of $\overline{\mathfrak{W}}$ onto the Weyl group W which has the following conditions: If $\omega \in \overline{\mathfrak{W}}$ is mapped by τ onto an element w of W, we have

$$\omega h(\chi)\omega^{-1} = h(\chi')$$

where $\chi' \in \overline{X}$ is defined by $\chi'(s) = \chi(w^{-1}(s))$; the kernel of τ is $\overline{\mathfrak{H}}$.

For each $w \in W$, we denote by $\overline{\mathfrak{U}}_w$ the subgroup of $\overline{\mathfrak{U}}$ generated by all $x_r(\xi)$, $\xi \in K$, such that $r > 0$ and $w(r) < 0$. We choose a representative system of the classes of $\overline{\mathfrak{W}}$ modulo $\overline{\mathfrak{H}}$ and denote the elements of the system by $\omega(w)$, $w \in W$. Then we have that every element in $\overline{G}$ (resp. $\overline{G}'$) is written uniquely in the form $uh\omega(w)u'$ where $u \in \overline{\mathfrak{U}}$, $h \in \overline{\mathfrak{H}}$(resp. $h \in \overline{\mathfrak{H}}'$), $w \in W$ and $u' \in \overline{\mathfrak{U}}_w$.[8)]

If $\mathfrak{g}_\sigma$ is simple, then excepting the following cases, $\overline{G}'$ is the commutator group of $\overline{G}$ and is simple.

a) K is a finite field with 2 elements and $\mathfrak{g}_\sigma$ is one of the types (A_1), (B_2) and (G_2)

b) K is a finite field with 3 elements and $\mathfrak{g}_\sigma$ is of types (A_1).[9)]

4. Construction and structure of the groups.

4. 1. Let $\mathfrak{g}$ be a real semi-simple Lie algebra and the notations are same as in previous sections. Let $\overline{G}$ be the Chevalley group defined by $\mathfrak{g}_\sigma$ and K. We put

$$G = \{x \in \overline{G}\,;\ \sigma_K x \sigma_K^{-1} = x\}.$$

We denote by G'(resp. $\mathfrak{H}, \mathfrak{U}$ and $\mathfrak{V}$) the intersection of $\overline{G}'$(resp. $\overline{\mathfrak{H}}, \overline{\mathfrak{U}}$ and $\overline{\mathfrak{V}}$) and G.

We denote by Σ (resp. Σ_0, Σ_1) the set of all positive roots in Δ (resp. Δ_0, $\Delta - \Delta_0$); we denote by $-\Sigma_1$ the set of the roots $-r$ for $r \in \Sigma_1$. Then Σ is the sum of Σ_0 and Σ_1 and further Σ_1 is the sum of the sets Σ_{r^*} for all $r^* > 0$ in Δ^*. Moreover each subset is closed under the addition of roots and σ-invariant. Therefore, denoting by $\overline{\mathfrak{U}}_0$(resp. $\overline{\mathfrak{U}}_1$, $\overline{\mathfrak{U}}_{r^*}$ and $\overline{\mathfrak{V}}_1$) the subgroup of $\overline{\mathfrak{U}}$ generated by $x_r(\xi)$ for all the roots $r \in \Sigma_0$(resp. Σ_1, Σ_{r^*} and $-\Sigma_1$) and $\xi \in K$, we have

$$\overline{\mathfrak{U}} = \overline{\mathfrak{U}}_0\overline{\mathfrak{U}}_1, \quad \overline{\mathfrak{U}}_1 = \overline{\mathfrak{U}}_{r_1^*}\cdots\cdots\overline{\mathfrak{U}}_{r_N^*}$$

8) cf. Chevalley [2], III Th. 2.
9) cf. Chevaley [2], IV Th. 3.
10) cf. Chevalley [2], Lemma 11, p. 41.

where $r_1^*, \cdots, r_N^*$ are the positive roots of Δ^*.[10] Further we have

$$\sigma_K \overline{\mathfrak{U}}_0 \sigma_K^{-1} \subset \mathfrak{B}, \sigma_K \overline{\mathfrak{U}}_{r_i^*} \sigma_K^{-1} = \overline{\mathfrak{U}}_{r_i^*} (1 \leqq i \leqq N).$$

Therefore we have $\sigma_K \overline{\mathfrak{U}} \sigma_K^{-1} \cap \overline{\mathfrak{U}} = \overline{\mathfrak{U}}_1$. Now we put $\mathfrak{U}_{r^*} = \overline{\mathfrak{U}}_{r^*} \cap \mathfrak{U}$, then

$$\mathfrak{U} = \mathfrak{U}_{r_1^*} \cdots \cdots \mathfrak{U}_{r_N^*}$$

and any element $u \in \mathfrak{U}$ can be represented by $u = u_1 \cdots\cdots u_N$ uniquely where $u_i \in \mathfrak{U}_{r_i^*}$.

For any root $r^* \in \Delta^*$, $\mathfrak{U}_{r^*}$ is generated by the following where s runs over all the roots in Σ_{r^*}.

(19) $x_s^*(\xi) = x_s(\xi)$ where $\xi \in K$ and $\xi = \nu_s \bar{\xi}$, for s of type I

(20) $x_{s,\bar{s}}^*(\xi) = x_s(\xi) x_{\bar{s}}(\nu_{\bar{s}} \bar{\xi})$ where $\xi \in K$, for s of type III

(21) $x_{s,\bar{s},s+\bar{s}}^*(\xi, \eta) = x_s(\xi)\ x_{\bar{s}}(\nu_s \bar{\xi})\ x_{s+\bar{s}}(\eta)$

where $\eta - \nu_{s+\bar{s}} \bar{\eta} = N_{s,\bar{s}} \nu_s \xi \bar{\xi}$, for s of type IV.

We denote by X the subgroup of $\overline{X}$ consisting of the elements such that $\chi(\bar{r}) = \overline{\chi(r)}$ for all $r \in \Delta$; denote by $\mathfrak{H}$ the subgroup of $\overline{\mathfrak{H}}$ consisting of the elements $h(\chi)$ for all $\chi \in X$. For any $\chi \in X$, we denote by χ_0(resp. χ_1) the homomorphism of P into K^* such that $\chi_0 \equiv \chi$ on $\sqrt{-1}\ \mathfrak{h}^+$ (resp. $\chi_1 \equiv \chi$ on $\mathfrak{h}^-$) and $\chi_0 \equiv 1$ on $\mathfrak{h}^-$(resp. $\chi_1 \equiv 1$ on $\sqrt{-1}\ \mathfrak{h}^+$). Then χ_0 and χ_1 are elements of X and $\chi = \chi_0 \chi_1$. Thus $h(\chi) = h(\chi_0) h(\chi_1)$. Therefore, denoting by $\mathfrak{H}_0$(resp. $\mathfrak{H}_1$) the subgroup of $\mathfrak{H}$ consisting of all the elements $h(\chi)$ such that $\chi \equiv 1$ on $\mathfrak{h}^-$ (resp. on $\sqrt{-1}\ \mathfrak{h}^+$), we have

$$\mathfrak{H} = \mathfrak{H}_0 \mathfrak{H}_1, \qquad \mathfrak{H}_0 \cap \mathfrak{H}_1 = (1).$$

4.2. We denote by $\overline{G}(\Delta_0)$ the subgroup of $\overline{G}'$ generated by $x_r(\xi)$ for all $r \in \Delta_0$ and all $\xi \in K$; denote by $G(\Delta_0) = \overline{G}(\Delta_0) \cap G'$. Now we denote by $\mathfrak{Z}$ (resp. $\mathfrak{Z}'$) the subgroup of G generated by $\mathfrak{H}$ (resp. $\mathfrak{H}'$) and $G(\Delta_0)$. Then we have

$$G / G' \simeq \mathfrak{Z} / \mathfrak{Z}' \simeq \mathfrak{H} / \mathfrak{H}'.$$

Moreover we have

LEMMA 3. $\mathfrak{Z}\mathfrak{U} = \mathfrak{U}\mathfrak{Z}$, $\mathfrak{U} \cap \mathfrak{Z} = (1)$ *and* $\mathfrak{U}$ *is a normal subgroup of* $\mathfrak{U}\mathfrak{Z}$.

PROOF. If $r \in \Delta - \Delta_0$ and $s \in \Delta_0$, then we have $(r+s)^* = r^*$. Therefore the elements $x_s(\xi)\ u_{r^*} x_s(\xi)^{-1}$ for $x_s(\xi) \in \overline{\mathfrak{U}}_0$, $\omega(w) u_{r^*} \omega(w)^{-1}$ for $w \in W_0$ and $h u_{r^*} h^{-1}$ for $h \in \overline{\mathfrak{H}}$ are contained in $\overline{\mathfrak{U}}_{r^*}$. Thus $z u_{r^*} z^{-1} \in \overline{\mathfrak{U}}_{r^*} \cap \mathfrak{U} = \mathfrak{U}_{r^*}$. On the other hand $\overline{\mathfrak{U}} \cap \overline{G}(\Delta_0) = \overline{\mathfrak{U}}_0$, $\overline{\mathfrak{U}}_0 \cap \mathfrak{U} = (1)$ and $\mathfrak{H} \cap \mathfrak{U} = (1)$. Therefore we have $\mathfrak{U} \cap \mathfrak{Z} = (1)$.

LEMMA 4. *There exists a representative system* $\{\omega(w)\}$ *such that* $\omega(w)$ *is* *an element of G' if and only if $w \in W_\sigma$.*

PROOF. If $r \in \Delta_0$, then $\sigma_K x_r(\xi)\sigma_K^{-1} = x_{-r}(\bar{\xi})$. Therefore if we denote by σ the involution of $SL_2(K)$ defined by

$$\sigma\begin{pmatrix}1 & \xi \\ & 1\end{pmatrix} = \begin{pmatrix}1 & \\ -\bar{\xi} & 1\end{pmatrix}, \sigma\begin{pmatrix}1 & \\ \xi & 1\end{pmatrix} = \begin{pmatrix}1 & -\bar{\xi} \\ & 1\end{pmatrix}$$

then we have $\phi_r\sigma(x) = \sigma_K(\phi_r(x))\sigma_K^{-1}$. On the other hand

$$\sigma\begin{pmatrix} & 1 \\ -1 & \end{pmatrix} = \sigma(\begin{pmatrix}1 & 1 \\ & 1\end{pmatrix}\begin{pmatrix}1 & \\ -1 & 1\end{pmatrix}\begin{pmatrix}1 & 1 \\ & 1\end{pmatrix}) = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}.$$

Thus $\omega_r = \sigma_K\omega_r\sigma_K^{-1}$ and we may take $\omega_{r_*} = \omega_r$ itself for $\omega(w_r)$. If $r \in \Delta - \Delta_0$ and is of type I or II, $\sigma_K x_r(\xi)\sigma_K^{-1} = x_{\bar{r}}(\nu_r\bar{\xi})$ where $\nu_r = \pm 1$. We can easily see that if $\nu_r = 1$ (resp. $\nu_r = -1$), setting $\omega_{r^*} = \omega_r$ (resp. $\omega_{r^*} = h(\chi_{r,\theta})\omega_r$), we have ω_{r^*} is an element of G'. If r is of type III, there exists a homomorphism ϕ_{r^*} of $SL_2(K)$ into G' such that

$$\phi_{r^*}\begin{pmatrix}1 & \xi \\ & 1\end{pmatrix} = x_r(\xi)x_{\bar{r}}(\nu_r\bar{\xi}),\ \phi_{r^*}\begin{pmatrix}1 & \\ \xi & 1\end{pmatrix} = x_{-r}(\xi)\,x_{-\bar{r}}(\nu_r^{-1}\bar{\xi}).$$

Hence, setting $\omega_{r^*} = \phi_{r^*}\begin{pmatrix} & 1 \\ -1 & \end{pmatrix} = h(\chi_{\bar{r},\nu_r})\omega_r\omega_{\bar{r}}$, we have ω_{r^*} is an element of G'. Finally, if r is of type IV, in the same way as the case of type I or II, setting $\omega_{r^*} = \omega_{r+\bar{r}}$ or $\omega_{r^*} = h(\chi_{r+\bar{r},\theta})\omega_{r+\bar{r}}$, we have ω_{r^*} is an element of G'. Since W_σ is generated by w_r for the roots $r \in \Delta_0$ and $r \in \Delta - \Delta_0$ of type I or II, $w_r w_{\bar{r}}$ for the roots $r \in \Delta - \Delta_0$ of type III and $w_{r+\bar{r}}$ for the roots $r \in \Delta - \Delta_0$ of type IV, we can choose $\omega(w)$ such that $\omega(w) \in G'$ for any $w \in W_\sigma$. It is easy to see that, if $w \notin W_\sigma$, we cannot choose $\omega(w)$ such that $\omega(w) \in G'$. Thus we have the lemma.

Hereafter we shall choose once for all a representative system $\omega(w)$, $w \in W$, such that $\omega(w) \in G'$ for $w \in W_\sigma$.

Let $w \in W_\sigma$. For any $r \in \Delta_0$, we have $w(r) \in \Delta_0$, for $\overline{w(r)} = w(\bar{r}) = -w(r)$. Therefore $\omega(w)\ \bar{G}(\Delta_0)\mathfrak{H}\omega(w)^{-1} = \bar{G}(\Delta_0)\mathfrak{H}$. Thus we have the following:

LEMMA 5. $\omega(w)\mathfrak{Z}\omega(w)^{-1} = \mathfrak{Z}$ *for* $w \in W_\sigma$.

We denote by $\mathfrak{W}^*$ the subgroup of G generated by $\mathfrak{Z}$ and the elements $\omega(w)$ for all $w \in W_\sigma$. We denote w^* the image of w by the natural homomorphism of W_σ onto W^* and we define a homomorphism τ^* of $\mathfrak{W}^*$ onto W^* by

$$z\omega(w) \rightarrow w^* \text{ for } z \in \mathfrak{Z} \text{ and } w \in W_\sigma.$$

The kernel of τ^* is $\mathfrak{Z}$. Therefore we have $\mathfrak{W}^* / \mathfrak{Z} \simeq W^*$.

We shall choose once for all a representative system of the classes of $\mathfrak{W}^*$ modulo $\mathfrak{Z}$ and we denote $\omega(w^*)$, $w^* \in W^*$. (It is sufficient to set $\omega(w^*) = \omega(w)$ where w is an element of the class of W_σ modulo W_0 corresponding to w^*.) Then we have

$$\omega(w^*)\,\mathfrak{U}_{r^*}\omega(w^*)^{-1} = \mathfrak{U}_{w^*(r^*)}.$$

4.3. We shall now consider some subgroups of G of restricted rank 1. Let r be a root in $\Delta - \Delta_0$. We denote by Δ_r the σ-invariant subsystem of Δ generated by r; we denote by $\overline{G}(\Delta_r)$ the subgroup of $\overline{G}$ generated by $x_s(\xi)$ for all $s \in \Delta_r$, $\xi \in K$; we set $G(\Delta_r) = \overline{G}(\Delta_r) \cap G$. Then we have

LEMMA 6. *Any element of $G(\Delta_r)$ is expressed uniquely by the following form*:

$$x = uz \text{ or } uz\omega_{r^*}u' \text{ where } u,u' \in \mathfrak{U} \cap G(\Delta_r),\ z \in \mathfrak{Z} \cap G(\Delta_r).$$

PROOF. If r is of type I,II or III, $G(\Delta_r)$ is a subgroup of type A_1, and there exists a homomorphism of $SL_2(K)$ onto $G(\Delta_r)$ (cf. 4.2). Therefore the lemma is true. If r is of type IV, then $G(\Delta_r)$ is a subgroup of type A_2 or B_2. Let (a, b) be a σ-fundamental system of Δ_r. We may assume that $a = r$, $a + b = \bar{r}$. For any element $x \in G(\Delta_r)$, we have

$$x = uh\omega(w)u',\ \omega(w) = \omega(w_0)\ \omega(w^*) \bmod \mathfrak{Z},\ w_0 \in W_0.$$

Therefore we can easily see that $x = u_1 z\omega(w^*)u_1'$ where $u_1, u_1' \in \overline{\mathfrak{U}}_1$, $z \in \overline{\mathfrak{Z}}$. If we set $\omega(w^*)\ u_1'\ \omega(w^*)^{-1} = v_1$, then $\omega(w^*)\sigma_K\ u_1'\sigma_K^{-1}\omega(w^*)^{-1} = \sigma_K\ v_1\sigma_K^{-1}$ and $u_1 z v_1 = \sigma_K u_1 z v_1 \sigma_K^{-1}$. Since $\overline{\mathfrak{U}}_1\overline{\mathfrak{Z}} \cap \overline{\mathfrak{B}}_1 = (1)$, we have that v_1 and $u_1 z$ are elements of G and also, since $\overline{\mathfrak{U}}_1 \cap \overline{\mathfrak{Z}} = (1)$, we have that u_1 and z are elements of G. Therefore u_1,z and u_1' are elements of $G(\Delta_r)$. Thus the lemma is proved.

Now making use of the lemma 6, we can prove the following proposition in the same way as in Lemma 10, p.40, of Chevalley [2].

PROPOSITION 3. $G = \mathfrak{U}\mathfrak{W}^*\mathfrak{U}$.

Thus we have the following structure theorem on G analogous to that of Chevalley groups.

THEOREM 1. *The group G is the union of the sets $\mathfrak{U}\mathfrak{Z}\omega(w^*)\mathfrak{U}_{w^*}$ where w^* runs over all the elements of the restricted Weyl group W^* and $\mathfrak{U}_{w^*}$ is the subgroup of $\mathfrak{U}$ generated by the subgroups $\mathfrak{U}_{r^*}$ such that $r^* > 0$ and $w^*(r^*) < 0$. The sets are disjoint each other and an element x of $\mathfrak{U}\mathfrak{Z}\omega(w^*)\mathfrak{U}_{w^*}$ is expressed uniquely by the form*:

$$uz\omega(w^*)u' \text{ where } u \in \mathfrak{U},\ z \in \mathfrak{Z} \text{ and } u' \in \mathfrak{U}_{w^*}.$$

COROLLARY 1. *The group G' is the union of the sets $\mathfrak{U}\mathfrak{Z}'\omega(w^*)\mathfrak{U}_{w^*}$ where w^* runs over all the elements of the group W^*. The sets are disjoint each* *other.*

COROLLARY 2. $\mathfrak{U}\mathfrak{Z}$ *is the normalizer of* $\mathfrak{U}$.

PROOF. We have proved that, the normalizer of $\mathfrak{U}$ contains $\mathfrak{U}\mathfrak{Z}$. Let x be an element of the normalizer of $\mathfrak{U}$; let $x = uz\omega(w^*)u'$, $u \in \mathfrak{U}$, $z \in \mathfrak{Z}$, $u' \in \mathfrak{U}_{w^*}$. Then $\omega(w^*)$ is an element of the normalizer of $\mathfrak{U}$. Therefore $\omega(w^*)\mathfrak{U}_{r^*}\omega(w^*)^{-1} = \mathfrak{U}_{w^*(r^*)} \subset \mathfrak{U}$ for all positive root r^*. The operator w^* which transforms any positive root to a positive root is the identity. Thus $\omega(w^*) \in \mathfrak{Z}$ and we have $x \in \mathfrak{U}\mathfrak{Z}$.

COROLLARY 3. $\mathfrak{W}^*$ *is the normalizer of* $\mathfrak{H}_1$.

PROOF. We have proved that the normalizer of $\mathfrak{H}_1$ contains $\mathfrak{W}^*$. Let x be an element of the normalizer of $\mathfrak{H}_1$ and let $x = uz\omega(w^*)u'$, $u \in \mathfrak{U}$, $z \in \mathfrak{Z}$, $u' \in \mathfrak{U}_{w^*}$. For an element $h \in \mathfrak{H}_1$, we set $h' = xhx^{-1} \in \mathfrak{H}_1$. Then

$$u\, z\omega(w^*)u'h = uz(\omega(w^*)h\omega(w^*)^{-1})\ \omega(w^*)(h^{-1}u'h) = (h'uh'^{-1})h'z\omega(w^*)u'.$$

Therefore we have $u = h'uh'^{-1}$ and $u' = h^{-1}u'h$. Since h is an arbitrary element of $\mathfrak{H}_1$, we have $u = 1$ and also $u' = 1$. Thus we have $x = z\omega(w^*) \in W^*$.

COROLLARY 4. $\mathfrak{Z}$ *is the centralizer of* $\mathfrak{H}_1$.

PROOF. We have proved that the centralizer of $\mathfrak{H}_1$ contains $\mathfrak{Z}$. Let $x = z\omega(w^*) \in W^*$ be an element of the centralizer of $\mathfrak{H}_1$ with $z \in \mathfrak{Z}$. Then $h(\chi) = \omega(w^*)h\omega(w^*)^{-1} = h(\chi')$ for all $h(\chi) \in \mathfrak{H}_1$, i. e., $\chi \equiv \chi'$ on $\mathfrak{h}^-$. Therefore we have $w^* = 1$ and $x = z \in \mathfrak{Z}$.

5. Some maximal subgroups.

5.1. Let Δ be a root system of a semi-simple Lie algebra over C; let $\Pi = \{a_1,\cdots\cdots,a_l\}$ be a fundamental root system of Δ. For any integer $i, 1 \leqq i \leqq l$, we denote by $\Gamma_{(i)}$ (resp. $\Delta_{(i)}$ and $\Sigma_{(i)}$) the subset of Δ consisting of all roots $r = \sum_{k=1}^{l} c_k a_k$ such that $c_i \geqq 0$ (resp. $c_i = 0$ and $c_i > 0$). From definition, $\Sigma_{(i)}$ consists of positive roots and $\Delta_{(i)}$ is a subsystem of Δ. Further $\Gamma_{(i)}$ is the sum of $\Delta_{(i)}$ and $\Sigma_{(i)}$, and Δ is the sum of $\Gamma_{(i)}$ and $-\Sigma_{(i)}$, where $-\Sigma_{(i)}$ is the set of the roots $-r$ for $r \in \Sigma_{(i)}$.

We denote by $W_{(i)}$ the subgroup of W generated by w_r for all $r \in \Delta_{(i)}$, and we set $A_{s,r} = 2(s,r)/(r,r)$ for any two roots r,s in Δ.

LEMMA 7. *For any element* $w \in W$, *we have* $w \in W_{(i)}$ *if and only if* w *transforms* $\Delta_{(i)}$ *and* $\Sigma_{(i)}$ *onto themselves.*

PROOF. If $w \in W_{(i)}$, it is easy to see that $w(\Delta_{(i)}) = \Delta_{(i)}$ and $w(\Sigma_{(i)}) = \Sigma_{(i)}$. Suppose $w(\Delta_{(i)}) = \Delta_{(i)}$ and $w(\Sigma_{(i)}) = \Sigma_{(i)}$. Since $w(\Pi \cap \Delta_{(i)}) = \Pi_1$ is a fundamental root system of $\Delta_{(i)}$, there exists an element $w_1 \in W_{(i)}$ such that $w_1 w(\Pi \cap \Delta_{(i)})$

$= \Pi \cap \Delta_{(i)}$. On the other hand $w_1 w(\Sigma_{(i)}) = \Sigma_{(i)}$. Therefore $w_1 w$ transforms any positive root onto a positive root and we have $w_1 w = 1$,i.e., $w = w_1^{-1} \in W_{(i)}$.

LEMMA 8. *For any element $w \in W$ such that $w \notin W_{(i)}$, there exists a positive root s of $\Gamma_{(i)}$ such that $w(s)$ or $w^{-1}(s)$ is a root in $-\Sigma_{(i)}$.*

PROOF. Since $w \in W_{(i)}$, from lemma 7, we have $w(\Delta_{(i)}) \neq \Delta_{(i)}$ or $w(\Sigma_{(i)}) \neq \Sigma_{(i)}$, i.e., there exists a root $r \in \Delta_{(i)}$ such that $w(r) \in (\pm \Sigma_{(i)})$ or there exists a root $r \in \Delta_{(i)}$ such that $w(r) \in (-\Sigma_{(i)})$ or $\Delta_{(i)}$. In the first case, since $r \in \Delta_{(i)}$, there exists a root $r \in \Delta_{(i)}$ such that $w(r) \in (-\Sigma_{(i)})$. In the second case, if $w(r) \in \Delta_{(i)}$for $r \in \Sigma_{(i)}$, then $-r = w^{-1}\ w(-r) \in (-\Sigma_{(i)})$ where $w(-r) \in \Delta_{(i)}$. This completes the proof.

LEMMA 9. *For any root r in $\Sigma_{(i)}$, there exist a positive root s and an element w_1 of $W_{(i)}$ such that $w_1 w_r w_1^{-1}(s) = -a_i$.*

PROOF. First we assume that $A_{r,a_k} \leqq 0$ for any root $a_k, k \neq i$. Since $A_{r,a_k} A_{a_k,r} \geqq 0$, we have also $A_{a_k,r} \leqq 0$. Therefore the root $w_r(a_k) = a_k - A_{a_k,r} r$ is positive, for r is a positive root. Since $w_r \neq 1$, we have $w_r(a_i) < 0$. In fact, if $w_r(a_i) > 0$, then w_r transforms all positive roots onto positive roots and $w_r = 1$. This is a contradiction. Therefore, setting $s = -w_r(a_i)$, we have $w_r(s) = -a_i$.

If there exists a root $a_k, k \neq i$, such that $A_{r,a_k} > 0$, then we have $s = w_{a_k}(r) = r - A_{r,a_k} a_k$ is a root in $\Sigma_{(i)}$. Then $w_{a_k} w_r w_{a_k}^{-1} = w_s$ where $w_{a_k} \in W_{(i)}$. For root s, if there exists a root a_k, $k \neq i$, such that $A_{s,a_k} > 0$, repeating this process, we may reduce this case to the first case.

5. 2. Let $\Delta^*, \Pi^* = (a_1^*, \cdots\cdots, a_p^*)$ be a restricted root system and its σ-fundamental system respectively. For any integer $i, 1 \leqq i \leqq p$, we define the subset $\Gamma^*_{(i)}$, $\Delta^*_{(i)}$ and $\Sigma^*_{(i)}$ as in 5. 1. We denote by $G_{(i)}$ (resp. $G'_{(i)}$) the subgroup of G (resp. G') generated by $\mathfrak{Z}$(resp. $\mathfrak{Z}'$) and $\mathfrak{U}_{r^*}$ for all roots $r^* \in \Gamma^*_{(i)}$. We shall prove that the group $G_{(i)}$ (resp. $G'_{(i)}$) is a maximal subgroup of G (resp. G').

LEMMA 10. *G is generated by the subgroups $\mathfrak{Z}$ and $\mathfrak{U}_{\pm a_k^*}$, $1 \leqq k \leqq p$, and G' is generated by the subgroups $\mathfrak{U}_{\pm a_k^*}$, $1 \leqq k \leqq p$.*

PROOF Let G_0 be a subgroup of G generated by $\mathfrak{Z}$ and $\mathfrak{U}_{\pm a_k^*}$, $1 \leqq k \leqq p$. Since W^* is generated by $w_{a_k^*}$, $1 \leqq k \leqq p$, G_0 contains $\mathfrak{W}^*$. For any root r^* of Δ^*, there exists an element $w^* \in W^*$ such that $w^*(r^*) = a_k^*$ or $-a_k^*$ for some k. Therefore we have $\omega(w^*)^{-1}\mathfrak{U}_{\pm a_k^*}\,\omega(w^*) = \mathfrak{U}_{r^*} \subset G_0$. Therefore we have $G_0 = G$. Similarly, we can prove the second assertion.

PROPOSITION 4. *The group $G_{(i)}$ (resp. $G'_{(i)}$) is a maximal subgroup of G (resp. G'). The intersection of the groups $G_{(i)}$ (resp. $G'_{(i)}$), $1 \leqq i \leqq p$, is $\mathfrak{U}\mathfrak{Z}$ (resp. $\mathfrak{U}\mathfrak{Z}'$).*

PROOF. Let H be a subgroup of G such that $H \supsetneqq G_{(i)}$; let $x \neq 1$ be an element of H such that $x \notin G_{(i)}$. Then $x = uz\omega(w^*)u'$, where $u,u' \in \mathfrak{U}$ and $z \in \mathfrak{Z}$. Since u,u' and z are the elements of $G_{(i)}$, we have $\omega(w^*) \notin G_{(i)}$ and

$w^* \neq 1$. Therefore $w^* \notin W^*_{(i)}$. From lemma 8, there exists a root $s^* \in \Gamma^*_{(i)}$ such that $w^*(s^*)$ or $w^{*-1}(s^*)$ is a root of $-\Sigma^*_{(i)}$. Since $\omega(w^*)$ and $\mathfrak{U}_{s^*}$ are contained in H, $\omega(w^*)\mathfrak{U}_{s^*}\omega(w^*)^{-1}$ and $\omega(w^*)^{-1}\mathfrak{U}_{s^*}\omega(w^*)$ is contained in H. Therefore we have $\mathfrak{U}_{\pm r^*} \subset H$ for a root $r^* \in \Sigma_{(i)}$. Thus we have $\omega(w^*_{r^*}) \in H$. We have also $\omega(w^*_1 w^*_{r^*} w^{*-1}_1) \in H$ for all $w^*_1 \in W^*_{(i)}$, for $\omega(w^*_1) \in H$. Therefore, from lemma 9, we have $\mathfrak{U}_{-a^*_i} \subset H$. Thus $\mathfrak{U}_{\pm a^*_k}$ $(1 \leqq k \leqq p)$ and $\mathfrak{Z}$ are contained in H. From lemma 10, we have $H = G$. This proves that $G_{(i)}$ is a maximal subgroup. Similarly, we can prove that the group $G'_{(i)}$ is a maximal subgroup of G'.

Since $G_{(i)}$ contains $\mathfrak{U}\mathfrak{Z}$, we have that the intersection of the groups $G_{(i)}$ contains $\mathfrak{U}\mathfrak{Z}$. Suppose that an element $x = uz\omega(w^*)u'$ of G contained in the intersection of the groups $G_{(i)}$. Since $w^* \in W^*_{(i)}$, from lemma 7, $w^*(a^*_i) \in \Sigma^*_{(i)}$ for $1 \leqq i \leqq p$. Therefore w^* transforms any positive root onto a positive root. Thus $w^* = 1$ and we have that x is an element of $\mathfrak{U}\mathfrak{Z}$. Similarly, we can prove that the intersection of the groups $G'_{(i)}$ is $\mathfrak{U}\mathfrak{Z}'$[11].

6. Proof of simplicity.

6.1. In this section, we assume that G is a group of restricted rank 1. From theorem 1, we have that G is the union of two sets $\mathfrak{U}_{r^*}\mathfrak{Z}$ and $\mathfrak{U}_{r^*}\mathfrak{Z}\omega_{r^*}\mathfrak{U}_{r^*}$ where r^* is a restricted σ-fundamental root. Therefore $M = \mathfrak{U}_{r^*}\mathfrak{Z}$ (resp. $M' = \mathfrak{U}_{r^*}\mathfrak{Z}'$) is a maximal subgroup of G (resp. G').

LEMMA 11. *Let N be a normal subgroup of G', $\neq (1)$, then $M'N = G'$.*

PROOF We assume that $N \subset M'$. Let $x = uz$ be an element $\neq 1$ of M'. If $u \neq 1$, then $\omega_{r^*}\, x\, \omega_{r^*}^{-1} = \omega_{r^*}\, u\, \omega_{r^*}^{-1}\, \omega_{r^*}\, z\omega_{r^*}^{-1} = v\, z'$ where $v \neq 1$ is an element of $\mathfrak{U}_{-r^*}$ and $z' \in \mathfrak{Z}'$. Therefore we have $\omega_{r^*}\, x\, \omega_{r^*}^{-1} \in M'$ and this contradicts to the assumption. If $u = 1$ for all $x \in N$, i.e., $N \subset \mathfrak{Z}'$, being z an element $\neq 1$ of N, there exists an element u of $\mathfrak{U}_{r^*}$ such that $u^{-1}zuz^{-1} = u^{-1}u' \neq 1$. Since it is an element of N, this contradicts to the assumption. Thus we have $N \not\subset M'$. Since M' is a maximal subgroup, we have $N\, M' = G'$.

LEMMA 12. *If $K_0 \neq F_2$ and F_3, then G' is its own commutator group, where F_q is a finite field with q elements.*

PROOF. It is sufficient to see that the elements of G' defined by (19), (20) and (21) in 4.1. are commutators of elements of the group G'. We denote by $(x, y) = x y x^{-1} y^{-1}$ for any elements x, y of G'. Then we have

$$(h(\chi_{r,\zeta}),\ x^*_r(\xi)) = x^*_r((\zeta^2 - 1)\xi) \qquad \text{for the root } r \text{ of type I or II}$$

$$(h(\chi_{r,\zeta}\, \chi_{\bar{r},\bar{\zeta}}),\ x^*_{r,\bar{r}}(\xi)) = x^*_{r,\bar{r}}((\zeta^2 - 1)\xi) \qquad \text{for the root } r \text{ of type III}$$

$$(h(\chi_{r+\bar{r},\zeta}),\ x^*_{r,\bar{r},r+\bar{r}}(\xi, \eta)) = x^*_{r,\bar{r},r+\bar{r}}((\zeta^n - 1)\xi,\ N_{r,\bar{r}}\nu_r((\zeta^n - 1)\xi\bar{\xi} + (\zeta^2 - 1)\eta)$$

11) The lemma 3 in [2] is not true if the characteristic of the field is 2 or 3 and $\mathfrak{g}_C$ is of type B_l, C_l, F_4 or G_2. But the proposition 3 in [2] is true for all cases which can be proved by the same way as the proof of this proposition.

for the root r of type IV, where $n = \pm 1$ or ± 2.
Since there exists an element ζ of K_0 such that $\zeta^2 \neq 1$, we have the lemma.

PROPOSITION 5. *If $K_0 \neq F_2$ and F_3 and if Δ^* is of restricted rank* 1, *then G' is a simple group.*

PROOF. Let N be a normal subgroup $\neq (1)$ of G'. From lemma 11, we have $N M' = G'$. Since $\mathfrak{U}_{r^*}$ is a normal subgroup of M', $N\mathfrak{U}_{r^*}$ is a normal subgroup of G'. Therefore $\omega_{r^*} \mathfrak{U}_{r^*} \omega_{r^*}^{-1} = \mathfrak{U}_{-r^*}$ is contained in $N \mathfrak{U}_{r^*}$. Therefore we have $N \mathfrak{U}_{r^*} = G'$. Then we have

$$G' / N = N \mathfrak{U}_{r^*} / N \simeq \mathfrak{U}_{r^*} / \mathfrak{U}_{r^*} \cap N.$$

Since G' is its own commutator group and $\mathfrak{U}_{r^*}$ is a nilpotent group, we have $G' = N$. Therefore G' is a simple group.

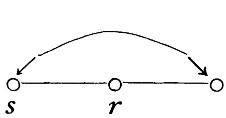
6. 2. To prove simplicity for general case, we shall first have some relations between the root system and the restricted root system.

LEMMA 13. *Let r^*, s^* be two roots of a σ-fundamental system Π^* and assume that $r^* + s^*$ is a root of Δ^*. Let r be a root of Σ_{r^*} and s be a root of Σ_{s^*} and assume $s \neq \bar{s}$. As for the roots of Δ which are expressed by linear combinations of $r, \bar{r}$ and $s, \bar{s}$ with positive coefficients, the only following cases are possible.*

0) $r, \bar{r}$; $s, \bar{s}$; $r + \bar{s}$, $\bar{r} + s$
1) $r (= \bar{r})$; $s, \bar{s}$; $r + s$, $r + \bar{s}$; $r + s + \bar{s}$
2) $r, \bar{r}$; $s, \bar{s}$; $r + \bar{s}$, $\bar{r} + s$; $r + 2\bar{s}$, $\bar{r} + 2s$
3) $r (= \bar{r})$; $s, \bar{s}$, $s + \bar{s}$; $r + s$, $r + \bar{s}$, $2r + s + \bar{s}$; $r + s + \bar{s}$
4) $r, \bar{r}$; $s, \bar{s}$, $s + \bar{s}$; $r + s$, $\bar{r} + \bar{s}$, $r + \bar{r} + s + \bar{s}$; $r + s + \bar{s}$, $\bar{r} + s + \bar{s}$.

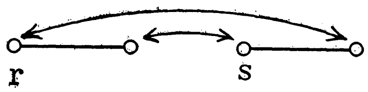
PROOF. We denote by $\Delta_{r,s}$ the σ-invariant subsystem of Δ generated by r and s. The involution σ induces on $\Delta_{r,s}$ an involution which we denote also by σ and σ-order of Δ induces a linear order of $\Delta_{r,s}$ which we call also a σ-order of $\Delta_{r,s}$. Since σ is not the identity on $\Delta_{r,s}$, if the rank of $\Delta_{r,s}$ is 2, then its restricted root system $\Delta_{r,s}^*$ is of rank 1 and this contradicts to the assumption that $r^* + s^*$ is a root of Δ^*. Therefore the rank of $\Delta_{r,s}$ is 3 or 4.

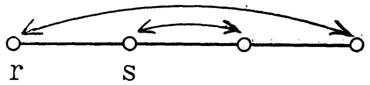
I) The case that the rank of $\Delta_{r,s}$ is 3.
We have only a following case:

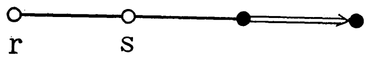


This is the case 1) of the lemma.

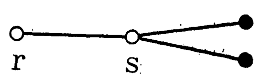
II) The case that the rank of $\Delta_{r,s}$ is 4.
We have only the following cases:

i)  This is the case 0) of the lemma.

ii)  This is the case 4) of the lemma.

iii)  This is the case 3) of the lemma.

iv)  This is the case 2) of the lemma.

v)  This is the case 1) of the lemma.[12]

6.3 Hereafter we consider the groups G, G' constructed from a real simple Lie algebra. Therefore the restricted root system Δ^* is simple. Then we have the following lemma.

LEMMA 14. *Let N be a normal subgroup of G'. If there exists a root $r^* \in \Delta^*$ such that $\mathfrak{U}_{r^*} \subset N$, then we have $N = G'$, except the case that Δ^* is of type G_2 and $K_0 \neq F_2$.*

PROOF. If all the roots in Δ^* have the same length, then any two roots in Δ^* is transitive by an operation of the Weyl group. Therefore we have $\mathfrak{U}_{s^*}$ contained in N for all roots $s^* \in \Delta^*$, for $\omega(w^*)\mathfrak{U}_{r^*}\omega(w^*)^{-1} = \mathfrak{U}_{w^*(r^*)}$ is contained in N. Since G' is generated by $\mathfrak{U}_{s^*} \in \Delta^*$, for all $s^* \in \Delta^*$, we have $N=G'$. Suppose that Δ^* contains two roots whose lengths are different each other. If Δ^* is of type G_2, then $\Delta^* = \Delta$, and G' is a Chevalley's group. This case has been proved by Chevalley [2], p.63. Therefore we suppose that Δ^* is of not type G_2. Let r^*, s^* be two roots in Δ^* such that the lengths of them are different and that $r^* + 2s^*$ is a root. Then we have that the lengths of r^* and $r^* + 2s^*$, the lengths of s^* and $r^* + s^*$ are equal respectively. Since any root in $\Sigma_{r^*+s^*}$ or $\Sigma_{r^*+2s^*}$ can be expressed by a linear combination of a root in Σ_{r^*} and roots in Σ_{s^*} with integer coefficients, we shall consider each cases separately.

If $r = \bar{r}$ and $s = \bar{s}$, then we have

---

12) In the root diagram, black vertices represent elements of Δ_0; white vertices connected by an arrow ←——→ are those which are corresponding to each other by the permutation $i \to i'$ of (11); $\underset{r}{\bigcirc}\Rightarrow\underset{s}{\bigcirc}$ means $(r, r) = 2(s, s)$. cf. Tables in Satake [3], p. 109, and Sugiura [5], p. 113.

$$x_r(\xi)x_s(\eta)x_r(\xi)^{-1}=x_s(\eta)x_{r+s}(N_{r,s}\xi\eta)x_{r+2s}\left(\frac{1}{2}N_{r,s}N_{s,r+s}\xi\eta^2\right)$$

$$x_s(\eta)x_r(\xi)x_s(\eta)^{-1}=x_r(\xi)x_{r+s}(N_{s,r}\eta\xi)x_{r+2s}\left(\frac{1}{2}(N_{s,r}N_{s,r+s}\xi\eta^2\right)$$

where $N_{r,s}=\pm 1, \frac{1}{2}N_{s,r}N_{s,r+s}=\pm 1$.

If $r\neq\bar{r}$ or $s\neq\bar{s}$, then we have the cases 1) ~ 4) in lemma 13. Notations being same as (19), (20) and (21), we have the following relations.

Case 1)

(23) $$x^*_r(\xi)x^*_{s,\bar{s}}(\eta)x^*_r(\xi)^{-1}=x^*_{s,\bar{s}}(\eta)x^*_{r+s,r+\bar{s}}(N_{r,s}\xi\eta)x^*_{r+s+\bar{s}}(N_{r,s}N_{r+s,\bar{s}}\nu_s\xi\eta\bar{\eta})$$

$$x^*_{s,\bar{s}}(\eta)x^*_r(\xi)x^*_{s,\bar{s}}(\eta)^{-1}=x^*_r(\xi)x^*_{r+s,r+\bar{s}}(N_{s,r}\xi\eta)x^*_{r+s+\bar{s}}(N_{\bar{s},r}N_{s,\bar{s}+r}\nu_s\xi\eta\bar{\eta})$$

where $N_{s,r}=\pm 1,\ N_{s,r+s}=\pm 1$.

Case 2)

(24) $$x^*_{r,\bar{r}}(\xi)x^*_{s,\bar{s}}(\eta)x^*_{r,\bar{r}}(\xi)^{-1}$$

$$=x^*_{s,\bar{s}}(\eta)x^*_{r+\bar{s},\bar{r}+s}(N_{r,\bar{s}}\nu_s\xi\bar{\eta})x^*_{r+2\bar{s},\bar{r}+2s}\left(\frac{1}{2}N_{r,\bar{s}}N_{r+\bar{s},\bar{s}}\nu_s^2\xi\bar{\eta}^2\right)$$

$$x^*_{s,\bar{s}}(\eta)x^*_{r,\bar{r}}(\xi)x^*_{s,\bar{s}}(\eta)^{-1}$$

$$=x^*_{r,\bar{r}}(\xi)x^*_{r+\bar{s},\bar{r}+s}(N_{\bar{s},r}\nu_s\xi\bar{\eta})x^*_{r+2\bar{s},\bar{r}+2s}\left(-\frac{1}{2}N_{\bar{s},r}N_{r+\bar{s},\bar{s}}\nu_s^2\xi\bar{\eta}^2\right)$$

where $N_{s,r}=\pm 1, \frac{1}{2}N_{s,r}N_{r+s,s}=\pm 1$.

Case 3)

(25) $$x^*_r(\xi)x^*_{s,\bar{s},s+\bar{s}}(\eta,\zeta)x^*_r(\xi)^{-1}$$

$$=x^*_{s,\bar{s},s+\bar{s}}(\eta,\zeta)x^*_{r+s,r+\bar{s},2r+s+\bar{s}}\left(N_{r,s}\xi\eta,\ -\frac{1}{2}N_{s+\bar{s},r}N_{r+s+\bar{s},r}\xi^2\zeta\right)$$

$$x^*_{r+s+\bar{s}}(N_{r+s,\bar{s}}N_{r,s}\nu_s\xi\eta\bar{\eta}+N_{r,s+\bar{s}}\xi\zeta)x^*_{s,\bar{s},s+\bar{s}}(\eta,\zeta)x^*_r(\xi)x^*_{s,\bar{s},s+\bar{s}}(\eta,\zeta)^{-1}$$

$$=x^*_r(\xi)x^*_{r+s,r+\bar{s},2r+s+\bar{s}}\left(N_{s,r}\xi\eta,\frac{1}{2}N_{r,s+\bar{s}}N_{r,r+s+\bar{s}}\xi^2\zeta\right)$$

$$x^*_{r+s+\bar{s}}(N_{s+\bar{s},r}\xi\zeta+N_{s,r+\bar{s}}N_{\bar{s},r}\nu_s\xi\eta\bar{\eta})$$

where $N_{s,r}, \frac{1}{2}N_{r,s+\bar{s}}N_{r,r+s+\bar{s}}$, $N_{r,s+\bar{s}}$ and $N_{s,r+\bar{s}}N_{\bar{s},r}$ are ± 1.

Case 4)

(26) $$x^*_{r,\bar{r}}(\xi)x^*_{s,\bar{s},s+\bar{s}}(\eta,\zeta)x^*_{r,\bar{r}}(\xi)^{-1}$$
$$= x^*_{s,\bar{s},s+\bar{s}}(\eta,\zeta)x^*_{r+s,\bar{r}+\bar{s},r+s+\bar{r}+\bar{s}}(N_{r,s}\xi\eta,\ N_{r,\bar{r}+s+\bar{s}}N_{r,s+\bar{s}}\xi^2\zeta)x^*_{r+s+\bar{s},\bar{r}+s+\bar{s}}(N_{r,s+\bar{s}}\xi\zeta)$$
$$x^*_{s,\bar{s},s+\bar{s}}(\eta,\zeta)x^*_{r,\bar{r}}(\xi)x^*_{s,\bar{s},s+\bar{s}}(\eta,\zeta)^{-1}$$
$$= x^*_{r,\bar{r}}(\xi)x^*_{r+s,\bar{r}+\bar{s},r+s+\bar{r}+\bar{s}}(N_{s,r}\xi\eta,\ N_{s+\bar{s}+\bar{r},r}N_{s+\bar{s},\bar{r}}\nu_r\xi^2\zeta)x^*_{r+s+\bar{s},r+s+\bar{s}}(N_{\bar{r},s+\bar{s}}\nu_r\xi\zeta)$$

where $N_{r,s}$, $N_{r,r+s+\bar{s}}$, etc. are all ± 1.
Therefore if $\mathfrak{U}_{r^*}$ and $\mathfrak{U}_{r^*+2s^*}$ are contained in N, then $\mathfrak{U}_{r^*+s^*}$ is also contained in N. Further if $\mathfrak{U}_{s^*}$ and $\mathfrak{U}_{r^*+s^*}$ are contained in N, then $\mathfrak{U}_{r^*+2s^*}$ is also contained in N. Thus we have the lemma.

PROPOSITION 6. *If G' is a group of restricted rank $\geqq 2$, then G' is its own commutator group except the cases that Δ^* is of type B_2 and K_0 is a field with 2 or 3 elements.*

PROOF. Since the commutator group G'' of G' is a normal subgroup of G', from lemma 14, it is sufficient to see that there exists a root $r^* \in \Delta^*$ such that $\mathfrak{U}_{r^*}$ is contained in G''. If Δ^* is of not type B_2, there exist two roots r^*, s^* of Δ^* such that (r^*, s^*) forms a σ-fundamental system of type A_2. The roots in $\Sigma_{r^*+s^*}$ can be expressed by sum of a root r in Σ_{r^*} and a root s in Σ_{s^*}. If $r=\bar{r}$ and $s=\bar{s}$, we have

$$(x_r(\xi),\ x_s(\eta)) = x_{r+s}(N_{r,s}\xi\eta) \qquad \text{where } N_{r,s} = \pm 1.$$

If $s \neq \bar{s}$ or $r \neq \bar{r}$, then we have the case 0) of lemma 13. Thus

$$(x^*_{r,\bar{r}}(\xi),\ x^*_{s,\bar{s}}(\eta)) = x_{r+\bar{s},\bar{r}+s}(N_{r,\bar{s}}\nu_s\xi\eta)$$

where $N_{r,s} = \pm 1$. Therefore, we have $\mathfrak{U}_{r^*+s^*} \subset G''$ and $G'' = G'$. If Δ^* is of type B_2, denoting by (r^*, s^*) a σ-fundamental system of Δ^*, as for the possible relations between roots of Σ_{r^*} and Σ_{s^*}, we have the cases 1) ~ 4) of lemma 13. We assume that the field $K_0 \neq F_2$ and F_3. So there exists an element λ of K_0 such that $\lambda^2 \neq 1$.

Case 1) From (23), we have

$$x = (x^*_r(1),\ x^*_{s,\bar{s}}(\eta)) = x^*_{r+s,r+\bar{s}}(\pm\eta)x^*_{r+s+\bar{s}}(\pm\nu_s\eta\bar{\eta})$$
$$(h(\chi_{r,\lambda}), x) = x^*_{r+s,r+\bar{s}}(\pm(\lambda-1)\eta)$$

Case 2) From (24), we have

$$x = (x^*_{s,\bar{s}}(1),\ x^*_r(\xi)) = x^*_{r+s,r+\bar{s}}(\pm\nu_s\xi)x^*_{r+2\bar{s},\bar{r}+2s}(\pm\nu_s^2\xi)$$
$$(h(\chi_{r,\lambda}\chi_{\bar{r},\bar{\lambda}}), x) = x^*_{r+s,r+\bar{s}}(\pm(\lambda-1)\nu_s\xi)$$

Case 3) From (25), we have

$$x = (x^*_r(1),\ x^*_{s,\bar{s},s+\bar{s}}(\eta,\zeta)) = x^*_{r+s,r+\bar{s},2r+s+\bar{s}}(\pm\eta,\pm\zeta)x^*_{r+s+\bar{s}}(\pm\zeta)$$
$$(h(\chi_{r,\lambda}), x) = x^*_{r+s,r+\bar{s},2r+s+\bar{s}}(\pm(\lambda-1)\eta,\ \pm\nu_{r+s}(\lambda-1)\eta\bar{\eta}+(\lambda^2-1)\zeta)$$

Case 4) From (26), we have

$$
\begin{aligned}
x &= (x^*_{r,\bar{r}}(1),\ x^*_{s,\bar{s},s+\bar{s}}(\eta, \zeta)) \\
&= x^*_{r+s,\bar{r}+\bar{s},r+s+\bar{r}+\bar{s}}(\pm \eta, \pm \zeta) x^*_{r+s+\bar{s},r+s+\bar{s}}(\pm \zeta \pm \nu_s \eta\eta) \\
&\quad (h(\chi_{r,\lambda}\chi_{\bar{r},\bar{\lambda}}), x) \\
&= x^*_{r+s,\bar{r}+\bar{s},r+s+\bar{r}+\bar{s}}(\pm (\lambda - 1)\eta,\ \pm \nu_{r+s}(\lambda - 1)\eta\bar{\eta} + (\lambda^2 - 1)\zeta).
\end{aligned}
$$

Therefore we have $\mathfrak{U}_{r^*+s^*}$ is contained in G'' and $G'' = G'$.

Now we have the following theorem.

THEOREM 2. *Let G' be a group defined by a real simple Lie algebra $\mathfrak{g}$ and a field $K = K_0(\theta)$ as in* 4.1. *Then G' is simple except the following cases:*

a) K_0 is a finite field with 2 elements and Δ^* is of type A_1, B_2 or G_2.

b) K_0 is a finite field with 3 elements and Δ^* is of type A_1 or B_2.

PROOF. Let N be a normal subgroup $\neq (1)$ of G'. Then in the same way as the proof of lemma 9, we have that there exists a maximal subgroup $G'_{(i)}$ such that $NG'_{(i)} = G'$. Let $\mathfrak{U}_{(i)}$ be a subgroup of $G'_{(i)}$ generated by the subgroups $\mathfrak{U}_{r^*}$ for all $r^* \in \Sigma_{(i)}$, then it is normal in $G'_{(i)}$. Since $N\mathfrak{U}_{(i)}$ is a normal subgroup of G' and contains $\mathfrak{U}_{\alpha_i^*}$, we have $N\mathfrak{U}_{(i)} = G'$. Then

$$G' / N = N\mathfrak{U}_{(i)} / N \simeq \mathfrak{U}_{(i)} / \mathfrak{U}_{(i)} \cap N.$$

Since G' / N is its own commutator group and $\mathfrak{U}_{(i)}$ is nilpotent, we have $G' / N = (1)$, i.e., $G' = N$. This completes the proof of the theorem.

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