ON THE DISTRIBUTION OF VALUES OF THE TYPE $\Sigma f(q^k t)$

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1. Let $f(t)$ be a measurable function satisfying the conditions;

$$f(t+1)=f(t), \quad \int_0^1 f(t)dt = 0 \quad \text{and} \quad \int_0^1 f^2(t)dt < +\infty. \tag{1. 1}$$

In [1] M.Kac proved that if $f(t)$ is a function of Lip α, $\alpha > 1/2$, or of bounded variation, then it is seen that, for $-\infty < \omega < +\infty$,

$$\lim_{n\to\infty} \left| \left\{ t\,;0 \leqq t \leqq 1, \frac{1}{\sigma\sqrt{n}} \sum_{k=0}^{n-1} f(2^k t) \leqq \omega \right\} \right| = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\omega} e^{-u^2/2} du, \tag{1. 2}$$

provided that the following limit is positive;

$$\sigma^2 = \lim_{n\to\infty} \int_0^1 \left\{ \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} f(2^k t) \right\}^2 dt.$$

At the end of that paper he proposed the problem to replace the sequence $\{2^k\}$ in (1. 2) by a sequence of real numbers satisfying the Hadamard's gap condition. In this direction R.Salem and A.Zygmund proved the central limit theorem of lacunary trigonometric series (c.f. [2]). Also they showed that if $f(t) = \cos 2\pi t + \cos 4\pi t$ and $n_k = 2^k - 1$, $k = 1, 2, \ldots\ldots$, then

$$\lim_{n\to\infty} \left| \left\{ t\,;0 \leqq t \leqq 1, \frac{1}{\sqrt{n}} \sum_{k=1}^{n} f(n_k t) \leqq \omega \right\} \right| = \frac{1}{\sqrt{\pi}} \int_0^1 dx \int_{-\infty}^{\omega/2|\cos \pi x|} e^{-u^2/2}\, du.$$

In this note we consider the sequence $\{f(q^k t)\}$, where q is any real number greater than 1. To state our result we need some definitions. For any measurable set A in $(-\infty, \infty)$ we define its relative measure $\mu_R\{A\}$ as follows;

$$\mu_R\{A\} = \lim_{T\to\infty} \frac{1}{2T} |A \cap (-T, T)|,$$

and for any measurable function $g(t)$ defined on $(-\infty, \infty)$ its relative mean $M\{g(t)\}$ as follows;

$$M\{g(t)\} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} g(t)dt,$$

provided the two limits exist (cf. [4]). It is easily seen that if $g(t)$ is periodic with period 1 and integrable on the interval (0, 1), then

$M\{g(t)\} = \int_0 g(t)dt$, and that if $f(t)$ satisfies the condition (1.1), then for each n the set $\{ t ; \sum_{k=0}^{n} f(q^k t) \leqq \omega\}$ has the relative measure for any q and ω.

The purpose of the present note is to prove the following

THEOREM. *Let q be any real number greater than* 1 *and $f(t)$ satisfy the condition* (1. 1) *and, for some $\varepsilon > 0$,*

$$\left[\int_0^1 |f(t) - S_n(t)|^2 dt\right]^{1/2} = O\left[(\log n)^{-(1+\epsilon)}\right], \qquad \text{as } n \to +\infty, \tag{1. 3}$$

where $S_n(t)$ denotes the n-th partial sum of the Fourier series of $f(t)$. Then the following limit

$$\sigma^2 = \lim_{n\to\infty} M\left\{ \left| \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} f(q^k t)\right|^2 \right\}^{*)}$$

exists and if σ^2 is positive, we have for any ω,

$$\lim_{n\to\infty} \mu_R \left\{ t ; \frac{1}{\sigma\sqrt{n}} \sum_{k=0}^{n-1} f(q^k t) \leqq \omega\right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\omega} e^{-u^2/2}\, du.$$

REMARK 1. If q^k is an irrational number for any positive integer k, then we have $\sigma^2 = \int_0^1 \{f(t)\}^2 dt$ (cf. the proof of Lemma 1).

REMARK 2. If $q = 2$, then we have, for each n,

$$\mu_R \left\{ t ; \frac{1}{\sigma\sqrt{n}} \sum_{k=0}^{n-1} f(q^k t) \leqq \omega\right\} = \left| \left\{ t ;\ 0 \leqq t \leqq 1, \frac{1}{\sigma\sqrt{n}} \sum_{k=0}^{n-1} f(q^k t) \leqq \omega\right\} \right|.$$

Hence if $\sigma^2 > 0$, then (1. 2) holds under the condition (1. 3) which is weaker than that of M.Kac.

To prove (1. 2) Kac approximated $\Sigma f(2^k t)$ by sums of independent functions using the system of Rademacher functions. To prove our theorem we approximate $\Sigma f(q^k t)$ by sums of gap sequences with infinite gaps (cf. [3]).

2. From now on let $f(t)$ and q satisfy the conditions of the theorm. Further without loss of generality we may assume that the Fourier series of $f(t)$ contains cosine terms only. This assumption is introduced solely for the purpose of shortening the formulas. Let us put

$$f(t) \sim \sum_{k=1}^{\infty} a_k \cos 2\pi k t, \qquad \text{and} \qquad S_n(t) = \sum_{k=1}^{n} a_k \cos 2\pi k t.$$

From (1. 3) it is seen that

*) σ denotes a non-negative number.

(2. 1) $$\left[\int_0^1 |f(t) - S_n(t)|^2 dt\right]^{1/2} = \left(\frac{1}{2}\sum_{k>n} a_k^2\right)^{1/2} \leqq A(\log n)^{-(1+\epsilon)}.^{*)}$$

Further let us put, for $k = 0, 1, \ldots\ldots, n$ and $n = 1, 2, \ldots\ldots$,

(2. 2) $$N_{k,n} = k[n^\beta],\ N'_{k,n} = N_{k+1,n} - [\log^2 n],$$

(2. 3) $$T_{k,n}(t) = \sum_{l=N_{k,n}}^{N'_{k,n}} g_n(q^l t), \quad \text{and} \quad R_{k,n}(t) = \sum_{N'_{k,n}<l<N_{k+1,n}} g_n(q^l t),$$

where

(2. 4) $$g_n(t) = S_{[n^{\beta/2}]}(t),$$

and β is a constant such that

(2. 5) $$0 < \beta < 1/3.$$

Then we have

(2. 6) $$|g_n(t)| \leqq \sum_{k=1}^{n^{\beta/2}} |a_k| \leqq \left(\sum_{k=1}^{\infty} a_k^2\right)^{1/2} n^{\beta/4} \leqq An^{\beta/4}.$$

LEMMA 1. *The following limit exists;*

$$\sigma^2 = \lim_{n\to\infty} M\left\{\left|\frac{1}{\sqrt{n}}\sum_{k=0}^{n-1} f(q^k t)\right|^2\right\}.$$

PROOF. We have

$$M\left\{\left|\frac{1}{\sqrt{n}}\sum_{k=0}^{n-1} f(q^k t)\right|^2\right\} = M\{f^2(t)\} + \frac{2}{n}\sum_{r=1}^{n-1}\sum_{k=0}^{n-1-r} M\{f(q^k t)f(q^{k+r}t)\}$$
$$= \int_0^1 f^2(t)dt + 2\sum_{r=1}^{n-1}\left(1 - \frac{r}{n}\right) M\{f(t)\, f(q^r t)\}.$$

By (2. 1), we have

$$|M\{f(t)f(q^r t)\}| = \frac{1}{2}\left|\sum_{m=kq^r} a_m a_k\right| \leqq \left(\sum_{k=1}^{\infty} a_k^2\right)^{1/2} \left(\sum_{m\geqq q^r} a_m^2\right)^{1/2} \leqq Ar^{-(1+\epsilon)}.$$

Hence $\sum_r |M\{f(t)\, f(q^r t)\}| < +\infty$, and this proves the lemma.

LEMMA 2. *We have*

$$\lim_{n\to\infty} M\left\{\left|\frac{1}{\sqrt{N_{n,n}}}\sum_{k=0}^{N_{n,n}-1} f(q^k t) - \frac{1}{\sqrt{N_{n,n}}}\sum_{k=0}^{n-1} T_{k,n}(t)\right|^2\right\} = 0,$$

and

*) Now and later A will denote a constant not necessarily the same.

$$\lim_{n\to\infty} M\left\{\frac{1}{N_{n,n}}\sum_{k=0}^{n-1} T_{k,n}^2(t)\right\} = \sigma^2.$$

PROOF. We have

$$M\left[\left|\frac{1}{\sqrt{N_{n,n}}}\sum_{k=0}^{N_{n,n}-1}\{f(q^k t) - g_n(q^k t)\}\right|^2\right]$$

$$\leqq \int_0^1 |f(t) - g_n(t)|^2\, dt + 2\sum_{r=1}^{N_{n,n}-1} |M[\{f(t) - g_n(t)\}\{f(q^r t) - g_n(q^r t)\}]|.$$

By (2. 1) and (2. 4), we have

$$\int_0^1 |f(t) - g_n(t)|^2 dt = \frac{1}{2}\sum_{k>n^{\beta/2}} a_k^2,$$

and

$$|M[\{f(t) - g_n(t)\}\{f(q^r t) - g_n(q^r t)\}]| = \left|\frac{1}{2}\sum_{\substack{k>n^{\beta/2}\\ m=kq^r}} a_m a_k\right| \leqq A\Big(\sum_{k>n^{\beta/2}} a_k^2\Big)^{1/2} r^{-(1+\varepsilon)}.$$

Since $\sum_{k>n^{\beta/2}} a_k^2 \to 0$ as $n\to +\infty$, it follows that

(2. 7) $$\lim_{n\to\infty} M\left[\left|\frac{1}{\sqrt{N_{n,n}}}\sum_{k=0}^{N_{n,n}-1}\{f(q^k t) - g_n(q^k t)\}\right|^2\right] = 0.$$

On the other hand from (2. 3), we have

(2. 8) $$\sum_{k=0}^{N_{n,n}-1} g_n(q^k t) - \sum_{k=0}^{n-1} T_{k,n}(t) = \sum_{k=0}^{n-1} R_{k,n}(t).$$

The maximum frequency of cosine terms of $R_{k,n}(t)$ is $q^{N_{k+1,n}-1}[n^{\beta/2}]$ and the minimum frequency of terms of $R_{k+1,n}(t)$ is $q^{N'_{k+1,n}-1}$, and by (2. 2), $q^{N'_{k+1,n}+1} > q^{N_{k+1,n}-1}[n^{\beta/2}]$ if $n > n_0$. Therefore the sequence $\{R_{k,n}(t)\}$, $k = 0, 1, \cdots\cdots, n-1$, is orthogonal on $(-\infty, +\infty)$ with respect to the relative measure if $n > n_0$.*) Further we have, by (2. 3), (2. 6) and (2. 2),

$$R_{k,n}^2(t) \leqq A(N_{k+1,n} - N'_{k,n})^2\, n^{\beta/2} \leqq A n^{\beta/2}\log^4 n.$$

Hence we have, by (2. 5),

(2. 9) $$M\left\{\left|\frac{1}{\sqrt{N_{n,n}}}\sum_{k=0}^{n-1} R_{k,n}(t)\right|^2\right\} = \frac{1}{N_{n,n}}\sum_{k=0}^{n-1} M\{R_{k,n}^2(t)\}$$

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*) We say that $f(t)$ and $g(t)$ are orthogonal on $(-\infty, \infty)$ with respect to the relative measure if $M\{g(t)f(t)\} = 0$.

$$\leqq A \frac{n^{1+\beta/2}}{n^{1+\beta}} (\log^4 n) = o(1), \qquad \text{as } n \to +\infty.$$

By (2. 7), (2. 8), (2. 9) and the Minkowski's inequality, we can prove the first part of the lemma. By Lemma 1 and the relation just proved it is seen that

$$\lim_{n\to\infty} M\left\{ \left| \frac{1}{\sqrt{N_{n,n}}} \sum_{k=0}^{n-1} T_{k,n}(t) \right|^2 \right\} = \sigma^2.$$

In the same way as $\{R_{k,n}(t)\}$ we can see that $\{T_{k,n}(t)\}$ $k = 0, 1, \cdots\cdots, n-1$, is orthogonal on the interval $(-\infty, \infty)$ with respect to the relative measure if $n > n_0$. Hence we have

$$\lim_{n\to\infty} M\left\{ \left| \frac{1}{\sqrt{N_{n,n}}} \sum_{k=0}^{n-1} T_{k,n}(t) \right|^2 \right\} = \lim_{n\to\infty} \frac{1}{N_{n,n}} \sum_{k=0}^{n-1} M\{T_{k,n}^2(t)\} = \sigma^2.$$

This is the second part of the lemma.

3. LEMMA 3. *We have*

$$\lim_{n\to\infty} M\left\{ \left| \frac{1}{N_{n,n}} \sum_{k=0}^{n-1} T_{k,n}^2(t) - \sigma^2 \right|^2 \right\} = 0.$$

PROOF. We have, by (2. 3) and (2. 4),

$$T_{k,n}^2(t) = \sum_{l=N_{k,n}}^{N'_{k,n}} g_n(q^l t) + 2 \sum_{r=1}^{N'_{0,n}} \sum_{l=N_{k,n}}^{N'_{k,n}-r} g_n(q^l t) g_n(q^{l+r} t),$$

$$g_n^2(q^l t) = \frac{1}{2} \sum_{s=1}^{[n^{\beta/2}]} a_s^2 \{1 + \cos 4\pi s q^l t\}$$

$$+ \sum_{0<s<m\leqq n^{\beta/2}} a_m a_s \{\cos 2\pi q^l (m-s)t + \cos 2\pi q^l (m+s)t\},$$

and

$$g_n(q^l t) g_n(q^{l+r} t) = \frac{1}{2} \sum_{\substack{m=sq^r \\ 0<m,\, s\leqq n^{\beta/2}}} a_m a_s \{1 + \cos 4\pi q^l m t\}$$

$$+ \frac{1}{2} \sum_{\substack{0<m,\, s\leqq n^{\beta/2} \\ 0<|m-sq^r|<1}} a_m a_s \{\cos 2\pi q^l (m - sq^r)t + \cos 2\pi q^l (m + sq^r)t\}$$

$$+ \frac{1}{2} \sum_{\substack{0<m,\, s\leqq n^{\beta/2} \\ |m-sq^r|\geqq 1}} a_m a_s \{\cos 2\pi q^l (m - sq^r)t + \cos 2\pi q^l (m + sq^r)t\},$$

and then we can write $T_{k,n}^2(t)$ in the following form

$$(3.\ 1) \qquad T_{k,n}^2(t) = M\{T_{k,n}^2(t)\} + U_{k,n}(t) + V_{k,n}(t),$$

where

$$(3.\ 2)\qquad V_{k,n}(t)=\sum_{r=1}^{N'_{0,n}}\sum_{l=N_{k,n}}^{N'_{k,n}-r}\sum_{\substack{0<m,s\leqq n^{\beta/2}\\ 0<|m-sq^r|<1}} a_m a_s\cos 2\pi q^l(m-sq^r)t,$$

and $U_{k,n}(t)$ is the sum of cosine terms whose frequencies are not less than $q^{N_{k,n}}$ and not greater than $2q^{N'_{k,n}}[n^{\beta/2}]$. Therefore $\{U_{k,n}(t)\}$, $k=0,1,2,\cdots\cdots,n-1$, is orthogonal on $(-\infty,+\infty)$ with respect to the relative measure if $n>n_0$. On the other hand from the definition of $U_{k,n}(t)$ and (2.3) (2.4), we have

$$|U_{k,n}(t)|\leqq(N'_{k,n}-N_{k,n})^2\left(\sum_{k=1}^{n^{\beta/2}}|a_k|\right)^2\leqq n^{5\beta/2}\left(\sum_{k=1}^{n^{\beta/2}}a_k^2\right)$$

Since $\{U_{k,n}(t)\}$ is orthogonal, we have ,by (2.5) and the above relation,

$$(3.\ 3)\qquad M\left\{\left|\frac{1}{N_{n,n}}\sum_{k=0}^{n-1}U_{k,n}(t)\right|^2\right\}=\frac{1}{N_{n,n}^2}\sum_{k=0}^{n-1}M\{U_{k,n}^2(t)\}$$

$$\leqq A\frac{n^{1+5\beta}}{n^{2+5\beta}}=o(1),\qquad \text{as } n\to+\infty.$$

In the same way we have, for any fixed θ and r such that $\theta\neq 0$ and $0<r<N'_{0,n}$,

$$M\left\{\left|\frac{1}{N_{n,n}}\sum_{k=0}^{n-1}\sum_{l=N_{k,n}}^{N'_{k,n}-r}\cos 2\pi q^l\theta t\right|^2\right\}=\frac{1}{N_{n,n}^2}\sum_{k=0}^{n-1}M\left\{\left|\sum_{l=N_{k,n}}^{N'_{k,n}-r}\cos 2\pi q^l\theta t\right|^2\right\}$$

$$<A\frac{n(N'_{1,n})}{n^{2+2\beta}}<A\,n^{-(1+\beta)},\qquad \text{if } n>n_0.$$

Changing the order of summation and apply the Minkowski's inequality to (3. 2), we have, by (2. 1) and the above relation,

$$(3.\ 4)\qquad M\left\{\left|\frac{1}{N_{n,n}}\sum_{k=0}^{n-1}V_{k,n}(t)\right|^2\right\}^{1/2}$$

$$\leqq\sum_{r=1}^{N'_{0,n}}\sum_{\substack{0<m,s\leqq n^{\beta/2}\\ 0<|m-sq^r|<1}}|a_m a_s|\ M\left\{\left|\frac{1}{N_{n,n}}\sum_{k=0}^{n-1}\sum_{l=N_{k,n}}^{N'_{k,n}-r}\cos 2\pi q^l(m-sq^r)t\right|^2\right\}^{1/2}$$

$$\leqq An^{-(1+\beta)/2}\sum_{r=1}^{\infty}\sum_{\substack{0<m,s\leqq n^{\beta/2}\\ 0<|m-sq^r|<1}}|a_m a_s|\leqq An^{-(1+\beta)/2}\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}|a_s|\{|a_{[sq^r]}|+|a_{[sq^r]+1}|\}$$

$$\leqq An^{-(1+\beta)/2}\sum_{r=1}^{\infty}\left(\sum_{s=1}^{\infty}a_s^2\right)^{1/2}\left(\sum_{m\geqq[q^r]}a_m^2\right)^{1/2}\leqq An^{-(1+\beta)/2}\sum_{r=1}^{\infty}r^{-(1+\varepsilon)}=o(1),$$

as $n\to+\infty$.

From (3. 1), (3. 3) and (3. 4), it is seen that

$$\lim_{n\to\infty}M\left[\left|\frac{1}{N_{n,n}}\sum_{k=0}^{n-1}T_{k,n}^2(t)-\frac{1}{N_{n,n}}\sum_{k=0}^{n-1}M\{T_{k,n}^2(t)\}\right|^2\right]=0.$$

Thus by Lemma 2, we can prove the lemma.

Let us put for any real number λ,

$$P_n(t,\lambda) = \prod_{k=0}^{n-1} \left\{1 + \lambda \frac{T_{k,n}(t)}{\sqrt{N_{n,n}}}\right\}. \tag{3. 5}$$

Then we have the following

LEMMA 4. *There exists an integer n_0 depending only on q such that $n > n_0$ implies*

$$M\{|P_n(t,\lambda)|^2\} \leqq e^{\lambda^2 A}, \textit{ and } \quad M\{P_n(t,\lambda)\} = 1.$$

PROOF. By the definition of $T_{k,n}(t)$ and Lemma 2, we have

$$M\left\{\frac{T^2_{k,n}(t)}{[n^\beta]}\right\} = M\left\{\frac{1}{N_{n,n}} \sum_{k=0}^{n-1} T^2_{k,n}(t)\right\} \to \sigma^2, \qquad \text{as } n \to +\infty.$$

Further by (2. 1) and (3. 2), we have

$$|V_{k,n}(t)| \leqq \sum_{n=1}^{N'_{0,k}} \sum_{l=N_{k,n}}^{N'_{k,n}-r} \sum_{\substack{0<m,s\leqq n^{\beta/2} \\ 0<|m-sq^r|<1}} |a_m a_s|$$

$$\leqq An^\beta \sum_{r=1}^{\infty} \left\{\sum_{s=1}^{\infty} a_s^2\right\}^{1/2} \left\{\sum_{m>q^r-1} a_m^2\right\}^{1/2} \leqq An^\beta.$$

Hence we have, by (3. 1) and Lemma 2 and the above relations,

$$\frac{T^2_{k,n}(t)}{N_{n,n}} \leqq \frac{A}{n} + \frac{U_{k,n}(t)}{N_{n,n}}.$$

This implies, by (3. 5),

$$|P_n(t,\lambda)|^2 \leqq \prod_{k=0}^{n-1} \left\{1 + \frac{\lambda^2 A}{n} + \frac{\lambda^2 U_{k,n}(t)}{N_{n,n}}\right\}. \tag{3. 6}$$

Now let $d_j \cos 2\pi u_j t$ be a term of $U_{j,n}(t)$, then $q^{N_{j,n}} \leqq u_j \leqq 2n^{\beta/2} q^{N'_{j,n}}$. Therefore by (2. 2), it follows that for any $k < n$,

$$u_k - \sum_{j=0}^{k-1} u_j \geqq q^{N_{k,n}} - 2n^{\beta/2} \sum_{j=0}^{k-1} q^{N'_{j,n}}$$

$$\geqq q^{N_{k,n}}\left(1 - 2n^{\beta/2}\, q^{-[\log^2 n]} \sum_{j=0}^{k-1} q^{-(k-1-j)[n^\beta]}\right) > 0, \qquad \text{if } n > n_0.$$

This implies that for any $0 \leqq j_0 < j_1 < \cdots\cdots < j_l < n$, we have

$$M\left\{\prod_{m=0}^{l} \cos 2\pi u_{j_m} t\right\} = 0, \qquad \text{for } n > n_0.$$

Thus we have

$$M\{|P_n(t,\lambda)|^2\} \leqq M\left[\prod_{k=0}^{n-1}\left\{1+\lambda^2\frac{A}{n}+\lambda^2\frac{U_{k,n}(t)}{N_{n,n}}\right\}\right]$$

$$=\left(1+\lambda^2\frac{A}{n}\right)^n \leqq e^{\lambda^2 A}, \qquad \text{for } n>n_0.$$

In the same way we can prove the second assertion of the lemma.

4. LEMMA 5. *If $\sigma^2>0$, then we have for any fixed λ,*

$$\lim_{n\to\infty} M\left[\exp\left\{\frac{i\lambda}{\sigma\sqrt{N_{n,n}}}\sum_{k=0}^{n-1}T_{k,n}(t)\right\}\right]=e^{-\lambda^2/2}.$$

PROOF. If we put

$$E_n=\left\{t;\left|\frac{1}{N_{n,n}}\sum_{k=0}^{n-1}T_{k,n}^2(t)-\sigma^2\right|<1\right\},$$

then by Lemma 3 and the Tchebyschev's inequality, it follows that

$$\lim_{n\to\infty}\mu_R\{E_n\}=1. \tag{4. 1}$$

By (2. 3), (2. 5) and (2. 6), we have

$$\max_{0\leqq k<n}\left|\frac{T_{k,n}(t)}{\sqrt{N_{n,n}}}\right| \leqq An^{-1/2+3\beta/4}=o(1), \qquad \text{as } n\to+\infty. \tag{4. 2}$$

Therefore if $t\in E_n$, then it is seen that

$$\sum_{k=0}^{n-1}\left|\frac{T_{k,n}(t)}{\sqrt{N_{n,n}}}\right|^3 \leqq A\max_{0\leqq k<n}\left|\frac{T_{k,n}^2(t)}{\sqrt{N_{n,n}}}\right|=C_n=o(1), \qquad \text{as } n\to+\infty, \tag{4. 3}$$

and

$$\left|P_n\left(t,\frac{\lambda}{\sigma}\right)\right|^2 \leqq \prod_{k=0}^{n-1}\left(1+\lambda^2\frac{T_{k,n}^2(t)}{\sigma^2 N_{n,n}}\right)\leqq e^{\lambda^2(1+\sigma^2)/\sigma^2}. \tag{4. 4}$$

We have by (4. 1) and the fact that the integrand is less than one,

$$\left|M\left[\exp\left\{\frac{i\lambda}{\sigma\sqrt{N_{n,n}}}\sum_{k=0}^{n-1}T_{k,n}(t)\right\}\right]\right.$$

$$\left.-\lim_{T\to\infty}\frac{1}{2T}\int_{(-T,T)\cap E_n}\exp\left\{\frac{i\lambda}{\sigma\sqrt{N_{n,n}}}\sum_{k=0}^{n-1}T_{k,n}(t)\right\}dt\right|\leqq\mu_R(E_n'),$$

where $E_n'=(-\infty,\infty)-E_n$ and $\mu_R(E_n')\to 0$, as $n\to+\infty$.
Using the relation $\exp z=(1+z)\exp\{z^2/2+O(|z|^3)\}$ as $|z|\to 0$, and (4. 2), (4. 3) and (4. 4), we have

$$\lim_{T\to\infty}\frac{1}{2T}\int_{(-T,T)\cap E_n}\exp\left\{\frac{i\lambda}{\sigma\sqrt{N_{n,n}}}\sum_{k=0}^{n-1}T_{k,n}(t)\right\}dt$$

$$= \lim_{T\to\infty} \frac{1}{2T} \int_{(-T,T)\cap E_n} P_n\left(t, \frac{\lambda}{\sigma}\right) \exp\left\{\frac{-\lambda^2}{2\sigma^2 N_{n,n}} \sum_{k=0}^{n-1} T_{k,n}^2(t)\right\} dt + o(1),$$

as $n \to +\infty$.

By (4. 4) and (4. 1), it is seen that if $t \in E_n$, then

$$\left| P_n\left(t, \frac{\lambda}{\sigma}\right)\left[\exp\left\{\frac{-\lambda^2}{2\sigma^2 N_{n,n}} \sum_{k=0}^{n-1} T_{k,n}^2(t)\right\} - e^{-\lambda^2/2}\right]\right| \leqq B_\lambda \left| \frac{1}{\sigma^2 N_{n,n}} \sum_{k=0}^{n-1} T_{k,n}^2(t) - 1 \right|,$$

where B_λ is a constant depending on λ.

By Lemma 3, the relative mean of the right hand side of the above formula tends to zero as $n \to +\infty$. Hence for the proof of lemma it is sufficient to show that

$$\lim_{T\to\infty} \frac{1}{2T} \int_{(-T,T)\cap E_n} P_n\left(t, \frac{\lambda}{\sigma}\right) dt = 1 + o(1), \qquad \text{as } n \to +\infty,$$

and by the second assertion of Lemma 4, the above relation reduces to

$$\lim_{T\to\infty} \frac{1}{2T} \int_{(-T,T)\cap E_n'} P_n\left(t, \frac{\lambda}{\sigma}\right) dt = o(1), \qquad \text{as } n \to +\infty.$$

By (4. 1) and the first part part of Lemma 4, we have

$$\left| \lim_{T\to\infty} \frac{1}{2T} \int_{(-T,T)\cap E_n'} P_n\left(t, \frac{\lambda}{\sigma}\right) dt \right| \leqq \left[M\left\{ \left| P_n\left(t, \frac{\lambda}{\sigma}\right)\right|^2 \right\} \mu_R\{E_n'\} \right]^{1/2} = o(1),$$

as $n \to +\infty$.

LEMMA 6. *If $\sigma^2 > 0$, then we have for any ω,*

$$\lim_{n\to\infty} \mu_R\left\{t; \frac{1}{\sigma\sqrt{N_{n,n}}} \sum_{k=0}^{n-1} T_{k,n}(t) \leqq \omega\right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\omega} e^{-u^2/2}\, du.$$

PROOF. Let us put

$$Q_n(t) = \frac{1}{\sigma\sqrt{N_{n,n}}} \sum_{k=0}^{n-1} T_{k,n}(t).$$

Further let $\varphi_\varepsilon^+(t)$ (or $\varphi_\varepsilon^-(t)$) be the familar trapezoidal function equal to 1 in the interval (ω_1, ω_2) (or $(\omega_1 + \varepsilon, \omega_2 - \varepsilon)$) vanishing outside the interval $(\omega_1 - \varepsilon, \omega_2 + \varepsilon)$ (or (ω_1, ω_2)) and linear elsewhere, where ε is a real number such that $0 < 2\varepsilon < \omega_2 - \omega_1$. Then we have

(4. 5) $$M\{\varphi_\varepsilon^-(Q_n(t))\} \leqq \mu_R\{t; \omega_1 \leqq Q_n(t) \leqq \omega_2\} \leqq M\{\varphi_\varepsilon^+(Q_n(t))\} \text{ *)}.$$

If we put

*) Since $\varphi_\varepsilon^\pm(Q_n(t))$ are uniformly almost periodic, $M\{\varphi_\varepsilon^\pm(Q_n(t))\}$ exist.

$$\Phi_\varepsilon^{\pm}(\xi) = \int_{-\infty}^{\infty} \varphi_\varepsilon^{\pm}(t)e^{-it\xi}\,dt,$$

then $\Phi_\varepsilon^{\pm}(\xi)$ are absolutely integrable on $(-\infty, \infty)$. Therefore we have

$$M\{\varphi_\varepsilon^{\pm}(Q_n(t))\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi_\varepsilon^{\pm}(\xi) M[\exp\{i\xi Q_n(t)\}]\,d\xi. \tag{4. 6}$$

Since $\Phi_\varepsilon^{\pm}(\xi)$ are absolutely integrable and $M[\exp\{i\xi Q_n(t)\}]$ converges bounbedly to $e^{-\xi^2/2}$ as $n\rightarrow +\infty$, we have by (4. 5), (4. 6) and the Prancherel's relation,

$$\begin{aligned}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \varphi_\varepsilon^{-}(t)e^{-t^2/2}\,dt &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi_\varepsilon^{-}(\xi)e^{-\xi^2/2}\,d\xi \\ &\leqq \varliminf_{n\to\infty}\ \mu_R\{t\,;\omega_1 \leqq Q_n(t) \leqq \omega_2\} \leqq \varlimsup_{n\to\infty}\ \mu_R\{t\,;\omega_1 \leqq Q_n(t) \leqq \omega_2\} \\ &\leqq \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi_\varepsilon^{+}(\xi)e^{-\xi^2/2}d\xi = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \varphi_\varepsilon^{+}(t)e^{-t^2/2}dt.\end{aligned}$$

Since ε is arbitrary we can prove the lemma.

5. Proof of the Theorem. By Lemma 1, we can prove the first part of the theorem. By the first assertion of Lemma 2 and Lemma 6, we obtain

$$\lim_{n\to\infty}\mu_R\left\{t\,; \frac{1}{\sigma\sqrt{N_{n,n}}}\sum_{k=0}^{N_{n,n}-1} f(q^k t) \leqq \omega\right\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\omega} e^{-u^2/2}du. \tag{5. 1}$$

On the other hand we have, by (2. 2),

$$\lim_{n\to\infty}\frac{N_{n+1,n+1}}{N_{n,n}} = 1.$$

By the above relation and Lemma 1, we have for any m such that $N_{n,n} < m \leqq N_{n+1,n+1}$

$$\begin{aligned}M\left\{\left|\frac{1}{\sqrt{N_{n,n}}}\sum_{k=N_{n,n}}^{m} f(q^k t)\right|^2\right\} &= M\left\{\left|\frac{1}{\sqrt{N_{n,n}}}\sum_{k=0}^{m-N_{n,n}} f(q^k t)\right|^2\right\} \\ &\leqq A\,\frac{N_{n+1,n+1}-N_{n,n}}{N_{n,n}} \rightarrow 0, \qquad \text{as } m\rightarrow +\infty,\end{aligned}$$

and

$$M\left\{\left|\left(\frac{1}{\sqrt{N_{n,n}}}-\frac{1}{\sqrt{m}}\right)\sum_{k=0}^{m-1} f(q^k t)\right|^2\right\} \leqq A\left|\frac{1}{\sqrt{N_{n,n}}}-\frac{1}{\sqrt{m}}\right|^2 m = o(1), \qquad \text{as } m\rightarrow +\infty.$$

Hence we have

$$M\left\{\left|\frac{1}{\sqrt{m}}\sum_{k=0}^{m-1} f(q^k t) - \frac{1}{\sqrt{N_{n,n}}}\sum_{k=0}^{N_{n,n}-1} f(q^k t)\right|^2\right\} = o(1), \qquad \text{as } m \to +\infty.$$

By the above relation and (5. 1), we can prove the theorem.

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