SOME PROPERTIES OF MANIFOLDS WITH CONTACT METRIC STRUCTURE

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Introduction. Recently S.Sasaki [2][1] defined the notion of (ϕ, ξ, η, g)-structure of a differentiable manifold and showed that the structure is closely related to almost contact structure defined by J.W.Gray [1]. Further he and one of the authors [3] defined four tensors $N^i{}_{jk}$, $N^i{}_j$, N_{jk} and N_j associated with this structure and enumerated relations connecting these tensors. Especially N_{jk} and N_j vanish identically when the structure is the one associated to contact structure, or so-called contact metric structure. And it was shown that the vanishment of $N^i{}_{jk}$ implies the vanishment of all other tensors $N^i{}_j$, N_{jk} and N_j, and that in the case of contact metric structure the vanishment of $N^i{}_j$ is equivalent to the fact that the vector field ξ^i is a Killing vector field.

In this note we call contact metric structure with vanishing $N^i{}_j$ or $N^i{}_{jk}$ K-contact metric structure or normal contact metric structure respectively, and we shall study some conditions for a manifold with almost contact metric structure or a Riemannian manifold to admit such structure.

1. Conditions for manifolds to admit *K*-contact metric structure. In this section, we shall study the case of K-contact metric structure, i. e., the case such that the associated vector field ξ^i is a Killing vector field. We shall begin with the following

LEMMA 1. *Suppose ξ^i be a Killing vector field on an m-dimensional Riemannian manifold M^m, then the relations*

$$\xi^i{}_{,j,k} = R^i{}_{jhk}\xi^h \tag{1. 1}$$

hold good, where commas mean the covariant differentiation with respect to the Riemannian connection and $R^i{}_{jhk}$ is the curvature tensor.

PROOF. Since ξ^i is a Killing vector field, we have

$$\pounds(\xi)g_{ij} = 0,$$

where $\pounds(\xi)$ means the Lie derivation with respect to the infinitesimal transformation ξ^i, which implies

1) Numbers in brackets refer to the bibliography at the end of the paper.

$$\pounds(\xi)\left\{ {i \atop jk} \right\} = \xi^i_{,j,k} - R^i{}_{jhk}\xi^h = 0.$$

Hence we get the relations (1. 1). Q. E. D.

COROLLARY. *If M^{2n+1} is a manifold with K-contact metric structure, then the tensor fields ξ^i and ϕ^i_j satisfy the following relations*

$$\xi^i_{,j} = -\frac{1}{2}\phi^i_j, \tag{1. 2}$$

$$\phi^i_{j,k} = -2R^i{}_{jhk}\xi^h. \tag{1. 3}$$

PROOF. By definition, we have

$$\phi_{ij} = \eta_{j,i} - \eta_{i,j}.$$

Since $\eta_{i,j}$ is skew symmetric with respect to the indices i and j, we have

$$\phi_{ij} = -2\eta_{i,j}.$$

Transvecting these by g^{ki}, we get (1. 2). The second part follows immediately from (1. 2) and Lemma 1. Q. E. D.

THEOREM 1. *If M^{2n+1} is a manifold with K-contact metric structure, then the sectional curvatures for planes containing the vector ξ^i are always equal to $\frac{1}{4}$ at every point of M^{2n+1}.*

PROOF. From the above corollary, we have

$$\begin{aligned} R_{ijhk}\xi^i\xi^h &= R^i{}_{jhk}\xi^h\eta_i \\ &= -\frac{1}{2}\phi^i_{j,k}\,\eta_i = \frac{1}{2}\phi^i_j\eta_{i,k} \\ &= -\frac{1}{4}\,\phi^i_j\phi_{ik} = -\frac{1}{4}(g_{jk} - \eta_j\eta_k). \end{aligned}$$

So, if v^i is a unit vector orthogonal to ξ^i, and K is a sectional curvature for a plane spanned by ξ^i and v^i, we get

$$K = \frac{R_{ijhk}\xi^i v^j\xi^h v^k}{(g_{ij}g_{hk} - g_{ih}g_{jk})\,\xi^i v^j\xi^h v^k} = \frac{-\frac{1}{4}\,(g_{jk} - \eta_j\eta_k)v^j v^k}{-1} = \frac{1}{4}.$$

Q. E. D.

Now we consider the converse of this theorem, which characterizes in some sense a manifold with K-contact metric structure.

THEOREM 2. *Suppose that a Riemannian manifold M satisfies the following two conditions*:

(i) *M admits a unit Killing vector field ξ^i*,

(ii) *the sectional curvatures for planes containing ξ^i are equal to $\frac{1}{4}$ at every point of M.*

Then M admits K-contact metric structure defined by $\eta_i = g_{ij}\xi^j$.

PROOF. Since ξ^i is a unit vector field, we have

$$\eta_i\xi^i = g_{ij}\xi^j\xi^i = 1. \tag{1}$$

Next, if we put

$$\phi_j^i = -2\xi^i_{,j},$$

then we can easily verify the relations

$$\phi_j^i\xi^j = 0 \tag{2}$$

hold good. Next, as ξ^i is a Killing vector field, by virtue of Lemma 1, we get

$$\phi^i_{j,k} = -2\xi^i_{,j,k} = -2R^i{}_{jhk}\xi^h.$$

So making use of (2), we get

$$\begin{aligned}\phi^i_j\phi^j_k &= -2\phi^i_j\xi^j_{,k} = 2\phi^i_{j,k}\xi^j \\ &= -4R^i{}_{jhk}\xi^h\xi^j.\end{aligned}$$

On the other hand, the condition (ii) gives the relations

$$R_{ijhk}\xi^i\xi^h = -\frac{1}{4}(g_{jk} - \eta_j\eta_k),$$

because both sides of these equations are symmetric with respect to the indices j and k, and these imply

$$\phi^i_j\phi^j_k = -\delta^i{}_k + \xi^i\eta_k. \tag{3}$$

And from the definition of ϕ^i_j, we get

$$\phi_{ij} = -2\eta_{i,j} = \eta_{j,i} - \eta_{i,j}, \tag{4}$$

$$g_{ij}\phi^i_k\phi^j_h = g_{kh} - \eta_k\eta_h. \tag{5}$$

Therefore, the tensors ϕ^i_j, ξ^i, η_j and g_{ij} give a K-contact metric structure to the manifold in consideration. Q. E. D.

REMARK 1. In this theorem, the existence of a unit Killing vector field in condition (i) can be replaced by the existence of a Killing autoparallel vector field, as can be easily verified.

REMARK 2. The value of sectional curvatures in condition (ii) need not be necessarily equal to $\frac{1}{4}$, but it may be equal to any positive constant K. In this case, it is sufficient to replace the fundamental metric tensor g_{ij} by $\frac{1}{4K}g_{ij}$ and ξ^i by $2\sqrt{K}\xi^i$ respectively.

2. Conditions equivalent to normality of contact metric structure.

In this section, we shall study the conditions equivalent to normality of contact metric structure. To begin with, we propose a lemma for later use.

LEMMA 2. *In a manifold with almost contact metric structure, we suppose that the following relations are valid:*

$$2\phi_{ij,k} = \eta_i g_{jk} - \eta_j g_{ik}. \tag{2. 1}$$

Then ξ^i is a Killing vector field and the almost contact metric structure is the one induced by contact structure defined by η_i.

PROOF. Contracting (2. 1) with ξ^i, we get

$$2\phi_{ij,k}\xi^i = g_{jk} - \eta_j\eta_k$$

from which we have

$$-2\phi_{ij}\xi^i_{,k} = \phi_{ij}\phi^i{}_k,$$

or equivalently

$$\phi^i{}_j(\phi_{ik} + 2\eta_{i,k}) = 0. \tag{2. 2}$$

On the other hand, it is clear that the relations

$$\xi^i\eta_j(\phi_{ik} + 2\eta_{i,k}) = 0 \tag{2. 3}$$

hold good. From (2. 2) and (2. 3), it follows that $\eta_{i,k} = -\frac{1}{2}\phi_{ik}$ and ξ^i is a Killing vector field, and this shows that the almost contact metric structure is induced by contact structure defined by η_i. Q. E. D.

THEOREM 3. *In a manifold with normal contact metric structure, we have*

$$2\phi_{ij,k} = \eta_i g_{jk} - \eta_j g_{ik}. \tag{2. 4}$$

Conversely, in a manifold M with almost contact metric structure, if the relations (2. 4) *hold good, then M has normal contact metric structure defined by η_i.*

PROOF. Since the first part of Theorem 3 is proved in [4], we have only to give the proof for converse. By Lemma 2 we see that η_i defines contact metric structure in M and that the relations

$$\eta_{i,j} = -\frac{1}{2}\phi_{ij} \tag{2. 5}$$

hold good. Putting (2. 4) and (2. 5) into $g_{ir}N^r{}_{jk}$, and making use of the relations

$$\phi_{ij} + \phi_{ji} = 0, \ \phi_{ij,k} + \phi_{jk,i} + \phi_{ki,j} = 0,$$

we find that $N^i{}_{jk}$ vanishes. Q. E. D.

THEOREM 4. *If M is a manifold with normal contact metric structure, then the following relations hold good:*

$$\eta_h R^h{}_{kij} = -\frac{1}{4}(\eta_i g_{jk} - \eta_j g_{ik}). \tag{2. 6}$$

Conversely, if a Riemannian manifold M admits a unit Killing vector field ξ^i and the covector $\eta_i = g_{ij}\xi^j$ satisfies the above relations, then M admits normal contact structure defined by η_i.

PROOF. By the normality of the structure, ξ^i is a Killing vector field. Hence we get the relations

$$\phi_{ij,k} = -2\eta_h R^h{}_{kij} \tag{2. 7}$$

by virtue of the corollary of Lemma 1. So (2. 6) follows from Lemma 2.

Conversely, if $\eta_i = g_{ij}\xi^j$ satisfies the relations (2. 6), we can easily verify that sectional curvatures for planes containing ξ^i are always equal to $\frac{1}{4}$. So, by virtue of Theorem 2, we see that M admits K-contact metric structure defined by η_i. Moreover, since ξ^i is a Killing vector field, we have the relations (2. 7) and the normality of the structure follows immediately by Lemma 2.

LEMMA 3. *In a manifold with contact metric structure, we suppose that ξ^i is a Killing vector field. If, for any two vectors X^i and Y^i which are orthogonal to the vector field ξ^i with respect to the metric g, we have the relations*

$$\phi_{ij,k}X^iY^j = 0,$$

then $N^i{}_{jk}$ vanishes.

PROOF. For any vectors W^i and Z^i, $(\delta^i{}_l - \xi^i\eta_l)W^l$ and $(\delta^j{}_m - \xi^j\eta_m)Z^m$ are orthogonal to ξ^i, and hence

$$\phi_{ij,k}(\delta^i{}_l - \xi^i\eta_l)(\delta^j{}_m - \xi^j\eta_m)W^lZ^m = 0,$$

from which it follows

$$\phi_{lm,k} - \phi_{lj,k}\xi^j\eta_m - \phi_{im,k}\xi^i\eta_l = 0.$$

On the other hand, making use of (2. 2), we get

$$\begin{aligned}\phi_{lm,k} &- \phi_{lj,k}\xi^j\eta_m - \phi_{im,k}\xi^i\eta_l \\ &= \phi_{lm,k} + \phi_{lj}\xi^j{}_{,k}\eta_m + \phi_{im}\xi^i{}_{,k}\eta_l \\ &= \phi_{lm,k} - \frac{1}{2}\phi_{lj}\phi^j{}_k\eta_m - \frac{1}{2}\phi_{im}\phi^i{}_k\eta_l \\ &= \phi_{lm,k} + \frac{1}{2}(\eta_m g_{lk} - \eta_l g_{mk}).\end{aligned}$$

Therefore, Lemma 2 follows from Theorem 3. Q. E. D.

THEOREM 5. *Let M be a manifold with contact metric structure such that ξ^i is a Killing vector field, then the vanishment of $N^i{}_{jk}$ is equivalent to the following condition for any vector X^i and Y^i orthogonal to ξ^i:*

$$\eta_h R^h{}_{kij} X^i Y^j = 0.$$

PROOF. As we have seen, (2. 7) is valid. So Theorem 5 is a consequence of Lemma 3.

REMARK. $N^i{}_{jk} = 0$ is also equivalent to the condition that ξ^i is a Killing vector field and $\phi_{ij,k}$ is hybrid with respect to i and j in contact manifold (see [5]).

By virtue of Theorems 1, 2 and 5, we get the following

THEOREM 6. *In order that a Riemannian manifold M admits normal contact metric structure, it is necessary and sufficient that M satisfies the following three conditions:*

(i) *M admit a unit Killing vector field ξ^i,*

(ii) *the sectional curvatures for planes containing ξ^i are equal to $\frac{1}{4}$ at every point of M,*

(iii) *if X^i and Y^i are vectors orthogonal to ξ^i, then the relations*

$$R_{ijkh}\xi^i X^k Y^h = 0$$

hold good.

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