

BEST APPROXIMATION BY WALSH POLYNOMIALS

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1. Let Ψ be the system of Walsh functions, in the sense of R. E. A. C. Paley [3]; it is a complete orthonormal system over the unit interval $[0,1]$ and may be considered as the set of the characters of the "dyadic group" of N.J.Fine [1]. Further, G. Morgenthaler [2] introduced the notion of (W) -continuity, i.e. the image in the unit interval of the continuity in the dyadic group, which describes the behavior of the Walsh-Fourier expansion of a function more precisely than the ordinary one does. For example, there is a very simple and complete parallelism between the order of the best approximation by Walsh polynomials and the smoothness (in the sense of (W) -continuity) of a function. However, this parallelism seems to have escaped attention, and since it has interesting applications, we study it in details.

In what follows we shall confine our considerations on the dyadic group; once this is done, the case of the unit interval will then follow without difficulty. Let us list some preliminary definitions and notations; the reader is referred to Fine [1].

The dyadic group G is the set of the sequences $x = (x_n), n = 1, 2, \dots$, consisting of 0's and 1's, with termwise addition modulo 2, denoted by $+$. The topology of G is defined by the system of neighborhoods of identity $V_n = \{x \in G; x_1 = \dots = x_n = 0\} (n = 0, 1, 2, \dots)$ (with a convenience $V_0 = G$), or equivalently by the distance $\lambda(x) = \sum_{n=1}^{\infty} 2^{-n} x_n$. The total measure of G is normalized to be equal to 1.

The Rademacher functions are the "elementary characters" of G :

$$\phi_n(x) = (-1)^{x_{n+1}} \quad (n = 0, 1, \dots)$$

and the Walsh functions, the characters of G , are given by

$$\psi_n(x) \equiv 1, \quad \psi_n(x) = \phi_{n_1}(x) \cdot \dots \cdot \phi_{n_r}(x) \quad \text{for } n = 2^{n_1} + \dots + 2^{n_r} \\ (n_1 > \dots > n_r \geq 0 \text{ integers}).$$

A (Walsh) polynomial is a finite linear combination (over complex num-

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bers) of Walsh functions : thus $\sum_{\nu=0}^{n-1} c_{\nu} \psi_{\nu}(x)$ is a polynomial of degree n if $c_{n-1} \neq 0$.

The set of the polynomials with degree not greater than n is denoted by \mathfrak{P}_n .

Finally let us write, for the modulus of continuity and the best approximation respectively,

$$\begin{aligned} \omega^{(p)}(2^{-n}, f) &= \sup \{ \|f(x+h) - f(x)\|_p ; h \in V_{n-1} \} \\ E_m^{(p)}(f) &= \inf \{ \|f(x) - P(x)\|_p ; P \in \mathfrak{P}_m \} \end{aligned}$$

where the norm is given by

$$\|g\|_p = \left(\int_G |g(x)|^p dx \right)^{1/p} \quad 1 \leq p < \infty$$

and

$$\|g\|_{\infty} = \sup \{ |g(x)| ; x \in G \}.$$

2 . We begin with some elementary lemmas :

LEMMA 1. *Let $D_m(u)$ be the Dirichlet kernel of order m , i.e.*

$$D_m(u) = \sum_{\nu=0}^{m-1} \psi_{\nu}(u)$$

then

$$D_{2^n}(u) = 2^n \quad (u \in V_{n-1}), \quad = 0 \quad (u \notin V_{n-1}).$$

This is well known, see for example [3] or [1].

LEMMA 2. *Let $s_m(x) = s_m(x; f)$ be the m -th partial sum of the Walsh-Fourier series of a function $f(x) \in L^p(G)$, $1 \leq p \leq \infty$, then*

$$\|s_{2^n}(x; f)\|_p \leq \|f\|_p.$$

This Lemma is also known. A combination of Lemma 1 and the generalized Minkowski inequality will readily give the assertion.

$$\text{LEMMA 3. } E_{2^n}^{(p)}(f) \leq \|f(x) - s_{2^n}(x)\|_p \leq 2E_{2^n}^{(p)}(f).$$

The first half is trivial. The second is implicitly proved in B. Sz.-Nagy [5], where the result is stated in terms of Haar functions. We give here a proof for the sake of completeness. Let $P(x) \in \mathfrak{P}_{2^n}$ be the polynomial yielding the best approximation, then

$$\|f(x) - s_{2^n}(x; f)\|_p = \|f(x) - P(x) - s_{2^n}(x; f - P)\|_p$$

$$\begin{aligned} &\leq \|f(x) - P(x)\|_p + \|s_{2^n}(x; f - P)\|_p \\ &\leq 2\|f - P\|_p = 2E_{2^n}^{(p)}(f), \end{aligned}$$

by Lemma 2.

LEMMA 4. $E_{2^n}^{(p)}(f) \leq \omega^{(p)}(2^{-n}, f) \leq 2E_{2^n}^{(p)}(f)$.

PROOF. We have, by Lemma 3, for $1 \leq p < \infty$,

$$\begin{aligned} E_{2^n}^{(p)}(f) &\leq \|f - s_{2^n}(x; f)\|_p = \left(\int_G |f(x) - \int_G f(x+u) D_{2^n}(u) du|^p dx \right)^{1/p} \\ &\leq \left\{ \int_G \left(\int_G |f(x) - f(x+u)| D_{2^n}(u) du \right)^p dx \right\}^{1/p} \\ &= \left\{ \int_G \left(\int_{V_{n-1}} |f(x) - f(x+u)| D_{2^n}(u) du \right)^p dx \right\}^{1/p} \\ &\leq \int_{V_{n-1}} D_{2^n}(u) du \left(\int_G |f(x) - f(x+u)|^p dx \right)^{1/p} \\ &\leq \int_{V_{n-1}} D_{2^n}(u) \omega^{(p)}(2^{-n}, f) du = \omega^{(p)}(2^{-n}, f), \end{aligned}$$

which proves the first inequality.

As to the second, take $h \in V_{n-1}$ arbitrarily. Since we have $P(x+h) = P(x)$ for every $P \in \mathfrak{B}_{2^n}$ and for all $x \in G$,

$$\begin{aligned} \|f(x) - f(x+h)\|_p &= \|f(x) - P(x) - f(x+h) + P(x+h)\|_p \\ &\leq \|f(x) - P(x)\|_p + \|f(x+h) - P(x+h)\|_p = 2\|f - P\|_p. \end{aligned}$$

Taking P to be the best approximator,

$$\|f(x) - f(x+h)\|_p \leq 2E_{2^n}^{(p)}(f) \quad \text{for } h \in V_{n-1}.$$

Since $h \in V_{n-1}$ is arbitrary, we have the required inequality. The case $p = \infty$ requires no essential change.

Lemma 4 may be written as

$$\|f - s_{2^n}(x; f)\|_p \leq \omega^{(p)}(2^{-n}, f) \leq 2\|f - s_{2^n}(x; f)\|_p, \text{ for every } n.$$

In his paper [2], Morgenthaler gave the following

DEFINITION. $f(x)$ is said to belong to the class $\text{Lip}^{(p)}\alpha(W)$ if

$$\|f(x) - f(x+h)\|_p = O((\lambda(h))^\alpha) \quad (\alpha > 0).$$

3. Now we are in position to state our theorem :

THEOREM. *The following conditions are equivalent :*

- (1) $f(x) \in \text{Lip}^{(p)}\alpha(W)$
- (2) $\omega^{(p)}(2^{-n}, f) = O(2^{-n\alpha})$
- (3) $E_m^{(p)}(f) = O(m^{-\alpha})$
- (4) $\|f - s_{2^n}(x; f)\|_p = O(2^{-n\alpha})$.

PROOF. Since (2) and (4) are equivalent, we have only to show the equivalence of (1) and (2), (3) and (4). Or it is clear that (1) implies (2) and (3) implies (4), the only ones to be verified are the implications (2) \Rightarrow (1) and (4) \Rightarrow (3).

(2) \Rightarrow (1): For any $0 \neq h \in G$, there is an n such that $h \in V_{n-1} - V_n$; thus by (2)

$$\|f(x+h) - f(x)\|_p \leq A2^{-n\alpha} \leq B(\lambda(h))^\alpha, \text{ giving (1).}$$

(4) \Rightarrow (3): Let $n = m_1$ i.e. $2^n \leq m < 2^{n+1}$, then

$$E_m^{(p)}(f) \leq E_{2^n}^{(p)}(f) \leq \|f - s_{2^n}(x; f)\|_p = O(m^{-\alpha}). \text{ q.e.d.}$$

The corresponding theorem and its proof for the case of the trigonometric system are somewhat more complicated. (See S.B. Steckin [4]).

A standard modification of the above proof gives the following

COROLLARY 1. *Let $\varphi(t)$ be a positive non-decreasing function such that $\varphi(t) \rightarrow 0$, $\varphi(t) = O(\varphi(2t))$ as $t \rightarrow \infty$. Then $\|f(x) - f(x+h)\|_p = O(\varphi(m))$ if and only if $E_m^{(p)}(f) = O(\varphi(m))$, where m is the largest integer not exceeding $1/\lambda(h)$.*

Our theorem also gives

COROLLARY 2. *$f \in \text{Lip}^{(p)}\alpha(W)$ and $g \in \text{Lip}^{(q)}\beta(W)$ imply*

$$f * g \in \text{Lip}^{(r)}(\alpha + \beta)(W),$$

where $\alpha > 0$, $\beta > 0$ and $1/r \geq (1/p) + (1/q) - 1$.

PROOF. It suffices to show that $E_n(f * g) = O(n^{-\alpha-\beta})$. Let (P_n) , (Q_n) be sequences of polynomials with degree $\leq n$ such that $\|f - P_n\|_p = O(n^{-\alpha})$, $\|g - Q_n\|_q = O(n^{-\beta})$. Then

$$\|(f - P_n) * (g - Q_n)\|_r \leq \|f - P_n\|_p \|g - Q_n\|_q = O(n^{-\alpha-\beta})$$

but,

$$(f - P_n) * (g - Q_n) = f * g - P_n * g - f * Q_n + P_n * Q$$

and since $P_n * g + f * Q_n - P_n * Q$ is in \mathfrak{B}_n , the result follows.

The trigonometric analogue requires a modification to differences of higher orders.

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