SOME NOTES ON DIFFERENTIABLE MANIFOLDS WITH ALMOST CONTACT STRUCTURES

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1. Introduction. By an almost contact structure, we mean the (ϕ, ξ, η)-structure defined by Prof. S.Sasaki [4][1], i.e., the structure defined by tensor fields ϕ^i_j, ξ^i and η_j satisfying the following relations,

$$\xi^i\eta_i = 1, \tag{1. 1}$$

$$\phi^i_j\xi^j = 0, \tag{1. 2}$$

$$\phi^i_j\phi^j_k = -\delta^i_k + \xi^i\eta_k. \tag{1. 3}$$

In this paper, we first consider a principal fibre bundle over a manifold with an almost complex structure with a group A^1, where A^1 denotes a 1-dimensional Lie group. And making use of the almost complex structure and an infinitesimal connection in this principal fiber bundle, we shall define an almost contact structure on the bundle space such that the 1-form defined by η_j coincides with the connection form. Next, we shall study the condition for the normality of this almost contact structure, where a normal almost contact structure means the structure such that the tensor defined in [5] vanishes identically. In particular, in the case of a regular contact structure on a compact manifold, we get the result that a necessary and sufficient condition for the manifold to admit a normal contact metric structure associated with the contact structure is that the base manifold of Boothby-Wang's fibering (Cf. [1], Theorems 2 and 3) is a Hodge manifold.

2. Definition of almost contact structures in bundle spaces. Let M' be a $2n$-dimensional differentiable manifold with an almost complex structure defined by a tensor field F of type (1, 1) satisfying the relation

$$F^2_xX = -X \qquad \text{for } x \in M',\ X \in T_x(M'), \tag{2. 1}$$

where $T_x(M')$ denotes the tangent vector space of M' at x.

We consider a principal fiber bundle over M' with group A^1, and denote its bundle space and projection by M and p respectively. Now, we suppose that an infinitesimal connection in $M(M', p, A^1)$ is given and we denote its

1) Numbers in square brackets refer to the bibliography at the end of the paper.

connection form by ω. Since the Lie algebra of A^1 is the real line R with the trivial bracket operation, ω is a real valued Pfaffian form on M with the following properties:

$$\omega(u\bar{s}) = s^{-1}\bar{s} \qquad \text{for } u \in M,\ \bar{s} \in T_s(A^1), \tag{2. 2}$$

$$\omega(\bar{u}s) = \omega(\bar{u}) \qquad \text{for } \bar{u} \in T(M),\ s \in A^1. \tag{2. 3}$$

Let E be a vertical vector field on M defined by the base 1 of the Lie algebra R. So, the relation

$$\omega(E) = 1 \tag{2. 4}$$

holds good.

Next, let $u \in M$ and $x = p(u) \in M'$. If we denote the lifting map defined by this connection by $q_u : T_x(M') \to T_u(M)$, we have

$$q_{ua} = R_a \circ q_u \qquad \text{for } a \in A^1, \tag{2. 5}$$

where R_a denotes the differential of right translation by a. (Throughout this paper, we shall denote the differential of a differentiable map f by the same letter f).

Now we define a linear map ϕ_u of $T_u(M)$ by

$$\phi_u(X) = q_u(F_x \circ p(X)) \qquad \text{for } X \in T_u(M),\ x = p(u).$$

Then from this definition, it follows immediately that ϕ_u's define a differentiable tensor field ϕ of type (1, 1) invariant under the right translation and that the relation

$$\phi(E) = 0 \tag{2. 6}$$

holds good. Moreover, we have

$$\begin{aligned} \phi_u^2(X) &= q_u(F_x \circ p(q_u \circ F_x(p(X)))) \\ &= q_u(F_x^2(p(X))) = -\, q_u(p(X)). \end{aligned}$$

On the other hand, since $q_u(p(X))$ is the horizontal part of X, we have

$$q_u(p(X)) = X - \omega(X)E_u.$$

Therefore, we get

$$\phi^2(X) = -\, X + \omega(X)E. \tag{2. 7}$$

Hence, by virtue of (2. 4), (2. 6) and (2. 7), ϕ, E and ω define an almost contact structure on M. Moreover, since ϕ is invariant under the right translation, the tensor N_j^i for this almost contact structure (Cf. [5]) vanishes identically.

Next, we suppose that an almost Hermitian metric g' is given on M'. If we define a tensor g_u of type (0, 2) on M by

$$g_u(X, Y) = g'(pX, pY) + \omega(X)\omega(Y) \qquad \text{for } X, Y \in T_u(M),$$

we can easily verify that g_u's define a positive definite Riemann metric over M. Moreover, since

$$\begin{aligned} g(\phi X, \phi Y) &= g'(p\phi X, p\phi Y) + \omega(\phi X)\omega(\phi Y) \\ &= g'(FpX, FpY) = g'(pX, pY) \\ &= g(X, Y) - \omega(X)\omega(Y), \end{aligned}$$

and

$$g(E, X) = g'(pE, X) + \omega(E)\omega(X) = \omega(X)$$

for $X, Y \in T_u(M)$, g is associated with the almost contact structure defined above. And with respect to this metric, E is a Killing vector field. Next, if we denote the fundamental 2-form of these two structures on M and M' by ψ and ψ' respectively, we get

$$\psi = p^*(\psi').$$

Summarizing these, we get the following

THEOREM 1. *On the bundle space M of the principal fiber bundle over an almost complex manifold M' with a group A^1, we can define an almost contact structure by a tensor field ϕ, a vector field E and a 1-form ω with the following properties:*

1) *ω is a connection form of an infinitesimal connection on M.*
2) *ϕ is invariant under the transformations generated by the infinitesimal transformation E.*
3) $p\circ\phi = F\circ p$.

Moreover, if M' is an almost Hermitian manifold with metric g', then we can also define an associated Riemann metric g on M such that

4) $g = p^*g' + \omega \otimes \omega$,
5) *E is a Killing vector field.*

3. Conditions for normality. Let N be the tensor of type (1. 2) defined in [5] for the almost contact structure defined in §2, and let $\bar{N}$ be the Nijenhuis tensor of the almost complex structure on M'. Then, by virtue of direct calculations, we see that $\bar{N}$ and N can be expressed in the following form,

$$\text{(3. 1)} \qquad \begin{aligned} N(X, Y) = {}& [X, Y] + \phi[\phi X, Y] + \phi[X, \phi Y] - [\phi X, \phi Y] \\ & - \omega([X, Y])E - d\omega(X, Y)E, \end{aligned}$$

$$\text{(3. 2)} \qquad \bar{N}(X', Y') = [X', Y'] + F[FX', Y'] + F[X', FY'] - [FX', FY'],$$

where X, Y and X', Y' are vector fields on M and M' respectively. Now, let X_0 and Y_0 be elements of $T_u(M)$, then $N_u(X_0, Y_0) = 0$ if and only if $p(N_u(X_0, Y_0)) = 0$ and $\omega(N_u(X_0, Y_0)) = 0$. Let X and Y be right invariant vector fields defined in a neighborhood of u whose values at u are X_0 and Y_0 respectively. Then, $[X, Y]$ and ϕX are right invariant vector fields defined in the same neighborhood as X and Y. Hence, all of these are projectable and the relaions

$$p[X, Y] = [pX, pY] \quad \text{and} \quad p\circ\phi(X) = F\circ p(X)$$

hold. Therefore, making use of (3. 1) and (3. 2), we have

$$\begin{aligned} p(N_u(X_0, Y_0)) &= p(([X, Y] + \phi[\phi X, Y] + \phi[X, \phi Y] - [\phi X, \phi Y] \\ &\qquad\qquad - \omega([X, Y])E - d\omega(X, Y)E)_u) \\ &= ([pX, pY] + F[FpX, pY] + F[pX, FpY] - [FpX, FpY])_{p(u)} \\ &= \bar{N}_{p(u)}(pX_0, pY_0). \end{aligned}$$

Next, since

$$\omega(N_u(X_0, Y_0)) = -\omega([\phi X, \phi Y]_u) - d\omega(X_0, Y_0),$$

and

$$-\omega([\phi X, \phi Y]) = d\omega(\phi X, \phi Y),$$

we see $\omega(N_u(X_0, Y_0)) = 0$ if and only if $d\omega(\phi X_0, \phi Y_0) = d\omega(X_0, Y_0)$. But, since $d\omega$ is a horizontal form invariant under right translations, we can find a unique 2-form Ω' on M' such that

$$d\omega = p^*\Omega'.$$

We call it the curvature form of the connection defined by ω. Then the last condition can be replaced by

$$\Omega'(F(pX_0),\ F(pY_0)) = \Omega'(pX_0, pY_0).$$

Summarizing these, and making use of the fact that p is an onto map, we get the following

THEOREM 2. *The almost contact structure on M defined in* §2 *is normal if and only if the following two conditions are satisfied.*

1) *The almost complex structure on M' is integrable.*

2) *The curvature form of the connection ω is of type* (1, 1) *with respect to the almost complex structure on M'.*

REMARK. The curvature form Ω' defines an element of the Betti part of 2-dimensional integral cohomology group of M'(Cf. [3]).

4. The case of a regular contact structure on a compact manifold.

In this section, we consider the case when a compact manifold admits a regular contact structure defined by a 1-form ω.

In this case, by virtue of the Theorem 2 of Boothby-Wang's paper [1], it is known that "M is a principal fiber bundle over a symplectic space M' whose fundamental 2-form ψ' defines an integral cocycle with group T^1, where T^1 is a 1-dimensional toroidal group. And ω defines an infinitesimal connection in M and its curvature form is ψ'".

Now, we take a Riemann metric g' on M' associated with the form (Cf. [2]), then g' and ψ' define an almost Kählerian structure on M'. And, as is easily seen, the almost contact metric structure on M defined by this almost Kählerian structure and the connection ω is associated with the contact structure defined by ω.

Hence, we get the following

THEOREM 3. *On a compact manifold with a regular contact structure, we can find a Riemann metric associated with the contact structure such that the fundamental vector field is a Killing vector field.*

Next, we study the condition for the existence of a normal contact metric structure associated with this contact structure. We notice that the curvature form ψ' is of type (1, 1) with respect to the almost Kählerian structure. So, if the almost Kählerian structure on M' is integrable, i. e., if M' is a Hodge manifold, then, by virtue of Theorem 2, the almost contact metric structure defined by the Kählerian structure and the connection ω is normal. Therefore, there exists a normal contact metric structure associated with the contact structure defined by ω.

Conversely, we suppose that there exists a normal contact metric structure (ϕ, E, ω, g) associated with the contact structure, then we have

$$\mathfrak{L}(E)\phi = 0, \qquad \mathfrak{L}(E)g = 0,$$

where $\mathfrak{L}(E)$ means a Lie derivative with respect to the vector field E. So, ϕ and g are invariant under the right translations. We define a Riemann metric g' and a tensor field F of type (1, 1) by

$$g'_x(X', Y') = g_u(q_u X', q_u Y'),$$
$$F_x(X') = p\circ\phi_u\circ q_u(X'),$$

where $X', Y' \in T_u(M')$ and $u \in p^{-1}(x)$. Since g and ϕ are invariant under right translations, this definition is independent of the choice of u. Moreover, since

$$\begin{aligned} F_x^2(X') &= p\circ\phi_u\circ q_u(p\circ\phi_u\circ q_u(X')) \\ &= p\circ\phi_u(\phi_u\circ q_u(X') - \omega(\phi_u\circ q_u(X'))E) \\ &= p(-q_u(X') + \omega(q_u(X'))\,E) = -X', \end{aligned}$$

and

$$\begin{aligned} g'_x(F_xX', F_xY') &= g_u(q_u \circ F_xX', q_u \circ F_xY') \\ &= g_u(\phi_u \circ q_uX', \phi_u \circ q_uY') \\ &= g_u(q_uX', q_uY') - \omega(q_uX')\omega(q_uY') \\ &= g'_x(X', Y'), \end{aligned}$$

F and g' define an almost Hermitian structure on M'. And the fact that its fundamental form is ψ' follows from

$$\begin{aligned} g'(X', FY') &= g(qX', qFY') = g(qX', \phi qY') \\ &= d\omega(qX', qY') = (p^*\psi')(qX', qY') \\ &= \psi'(X', Y'). \end{aligned}$$

Therefore, F and g' define an almost Kählerian structure whose fundamental 2-form defines an integral cocycle. And we can easily see that the normal contact metric structure on M is the one defined by the almost Kählerian structure on M' and the connection ω. So, by the normality and Theorem 2, the almost Kählerian structure on M' is integrable, i. e., M' is a Hodge manifold.

So, we have the following

THEOREM 4. *A necessary and sufficient condition for a compact manifold with a regular contact structure to admit an associated normal contact metric structure is that the base manifold of the Boothby-Wang's fibering of M is a Hodge manifold.*

COROLLARY. *If M' is a compact Hodge manifold, then it has over it a canonically associated circle bundle which admits a normal contact metric structure.*

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