APPROXIMATION AND SATURATION OF FUNCTIONS BY ARITHMETIC MEANS OF TRIGONOMETRIC INTERPOLATING POLYNOMIALS

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Let $f(x)$ be a continuous function with period 2π and E_n be the set of equidistant nodal points situated in the interval $0 \leqq x < 2\pi$, that is

$$\xi_0 + 2\pi j/(2n+1) \qquad (j = 0, 1, \cdots, 2n), \qquad (\text{mod. } 2\pi)$$

where ξ_0 is any real number. Then the trigonometric polynomial of order n coinciding with $f(x)$ on E_n is

$$(1) \qquad I_n(x, f) = \frac{1}{\pi}\int_0^{2\pi} f(t) D_n(x-t)\, d\omega_{2n+1}(t),$$

where $D_n(x)$ is the Dirichlet kernel and $\omega_{2n+1}(t)$ is a step function which is associated with E_n. (We shall refer to A. Zygmund [4, Chap. X] these notations and fundamental properties of trigonometric interpolation.) We denote the Fourier expansions of (1) by

$$(2) \qquad I_n(x, f) = \sum_{k=-n}^{n} c_k^{(n)} e^{ikx}$$

$$c_k^{(n)} = \frac{1}{2\pi}\int_0^{2\pi} f(t) e^{-ikt} d\omega_{2n+1}(t).$$

The $\{c_k^{(n)}\}$ are called the k-th Fourier-Lagrange coefficients and for a fixed k, $c_k^{(n)}$ is an approximate Riemann sum for the integral defining Fourier coefficient c_k of $f(x)$, that is

$$c_k = \frac{1}{2\pi}\int_0^{2\pi} f(t) e^{-ikt} dt.$$

Let us denote the partial sums of (1) by

$$I_{n,m}(x, f) = \sum_{k=-m}^{m} c_k^{(n)} e^{ikx} \qquad (m \leqq n),$$

in particular

$$I_n(x, f) = I_{n,n}(x, f).$$

Let $B_{n,\nu}(x,f)$ denote the arithmetic means of $I_{n,m}(x,f)$; thus

$$B_{n,\nu}(x,f) = \frac{1}{\nu+1}\sum_{m=0}^{\nu} I_{n,m}(x,f) \qquad (\nu \leqq n) \tag{3}$$

$$= \sum_{k=-\nu}^{\nu}\left(1-\frac{|k|}{\nu+1}\right)c_k^{(n)}e^{ikx}$$

$$= \frac{1}{\pi}\int_0^{2\pi} f(t)K_\nu(t-x)\,d\omega_{2n+1}(t),$$

where $K_\nu(t)$ is the Fejér kernel, and set

$$B_n(x,f) = B_{n,n}(x,f).$$

In the present note, the author will investigate approximating properties of $B_{n,\nu}(x,f)$. These are analogous to the Fejér means of Fourier series. But the proofs are somewhat delicate.

THEOREM 1. *We have*

$$B_n(x,f) - f(x) = o(n^{-1}) \tag{1^0}$$

uniformly as $n\to\infty$, *if and only if* $f(x)$ *is a constant.*

$$B_{n,\nu}(x,f) - f(x) = O(\nu^{-1}) \tag{2^0}$$

uniformly as $\nu\to\infty$ (*for all* $\nu\leqq n$), *if and only if* $\widetilde{f}(x)$ *satisfies the Lipschitz condition of order* 1.

PROOF. (1^0) From the formula (3),

$$\frac{1}{2\pi}\int_0^{2\pi}\{f(x)-B_n(x,f)\}e^{-ikx}dx = c_k - \left(1-\frac{|k|}{n+1}\right)c_k^{(n)}$$

$$= c_k - c_k^{(n)} + \frac{|k|}{n+1}c_k^{(n)}.$$

If $B_n(x,f)-f(x)=o(n^{-1})$ uniformly, then

$$c_k - c_k^{(n)} + \frac{|k|}{n+1}c_k^{(n)} = o(n^{-1}). \tag{4}$$

When a trigonometric polynomial of order n has approximating degree ε_n for $f(x)$, then the integral of $f(x)$ has the same degree of approximation by its Riemann sums, (Walsh-Sewell [3, Theorem 4]). Since $B_n(x,f)$ is a trigonometric polynomial of order n, and

$$f(x) - B_n(x,f) = o(n^{-1}), \qquad f(x)e^{-ikx} - B_n(x,f)e^{-ikx} = o(n^{-1}) \ (k\leqq n),$$

we have

$$\int_0^{2\pi} f(x)e^{-ikx}dx - c_k^{(n)} = o(n^{-1}), \qquad \text{for fixed } k;$$

that is

$$n(c_k - c_k^{(n)}) = o(1) \qquad \text{as } n \to \infty.$$

Hence from (4)

$$|k|\, c_k^{(n)} \to 0 \qquad \text{as } n \to \infty,$$

and this means that $c_k = 0$, $(k = \pm 1, \pm 2, \dots)$. Thus we have $f(x) = c_0$. The converse is trivial.

(2^0) We suppose that

$$f(x) - B_{n,\nu}(x, f) = O(\nu^{-1})$$

uniformly as $n \geqq \nu \to \infty$. Since the unit ball of L^∞ space is weak* compact, there exist a bounded function $g(x)$ and a subsequence $\{\nu_p\}$ of ν such as

$$(5) \qquad \lim_{n_p \geqq \nu_p \to \infty} \frac{1}{2\pi} \int_0^{2\pi} \nu_p\{f(x) - B_{n_p,\nu_p}(x,f)\}e^{-ikx}dx = \frac{1}{2\pi}\int_0^{2\pi} g(x)e^{-ikx}dx$$

for all integral k. The first term is

$$(6) \qquad \lim_{\nu_p \to \infty} \left\{ \nu_p(c_k - c_k^{(n_p)}) + \frac{\nu_p |k|}{\nu_p + 1} c_k^{(n_p)} \right\} \qquad (\nu_p \leqq n_p).$$

On the other hand if we take $\nu = n$, then the above Walsh-Sewell result yields

$$(7) \qquad n(c_k - c_k^{(n)}) = O(1) \qquad (k \leqq n).$$

When we set $\nu = [n^{1-\delta}]$ $(0 < \delta < 1)$ and select a subsequence $\{n_p\}$ and we set $\nu_p = [n_p^{1-\delta}]$,

then

$$\nu_p(c_k - c_k^{(n_p)}) = n_p^{1-\delta}(c_k - c_k^{(n_p)}) = (n_p)^{-\delta} n_p(c_k - c_k^{(n_p)}) = o(1)$$

from (7). Since $c_k^{(n_p)} \to c_k$, from (5) and (6) we conclude

$$|k| c_k = \frac{1}{2\pi}\int_0^{2\pi} g(x)e^{-ikx}dx, \qquad g(x) \in L^\infty(0, 2\pi),$$

for all integral k. This means that $|k|c_k$ are the Fourier coefficients of $g(x)$ which belongs to the class $L^\infty(0, 2\pi)$. Since $\{c_k\}$ are Fourier coefficients of $f(x)$, it is easy to see that $f'(x) \in L^\infty(0, 2\pi)$. This is equivalent to that $\tilde{f}(x)$ satisfies the Lipschitz condition of order 1.

Conversely if $\tilde{f}'(x) \in L^\infty(0, 2\pi)$, then the Fejér means $\sigma_n(x,f)$ of $f(x)$ are the best approximation (A. Zygmund [4, I, p. 123]), that is

$$f(x) - \sigma_\nu(x,f) = O(\nu^{-1}). \tag{8}$$

$B_{n,\nu}(x,f)$ is a linear method of approximation, and

$$\begin{aligned} B_{n,\nu}(x,f) - f(x) &\\ &= B_{n,\nu}(x, f - \sigma_\nu) - (f - \sigma_\nu) + \{B_{n,\nu}(x, \sigma_\nu) - \sigma_\nu\} \\ &= P_{n,\nu}(x) + Q_{n,\nu}(x), \end{aligned} \tag{9}$$

say. $B_{n,\nu}(x)$ transforms any bounded function to some bounded function, so from (8)

$$P_{n,\nu}(x) = O(\nu^{-1}) \qquad (\nu \leqq n).$$

On the other hand $\sigma_\nu(x)$ is a ν-th order polynomial and $\nu \leqq n$,

$$I_n(x, \sigma_\nu(x)) \equiv \sigma_\nu(x) = \sum_{k=-\nu}^{\nu} \left(1 - \frac{|k|}{\nu + 1}\right) c_k e^{ikx},$$

and

$$B_{n,\nu}(x, \sigma_\nu(x)) = \sum_{k=-\nu}^{\nu} \left(1 - \frac{|k|}{\nu+1}\right)^2 c_k e^{ikx}.$$

Hence

$$\begin{aligned} Q_{n,\nu}(x) &= B_{n,\nu}(x, \sigma_\nu(x)) - \sigma_\nu(x) \\ &= \sum_{k=-\nu}^{\nu} \left\{\left(1 - \frac{|k|}{\nu + 1}\right)^2\right\} c_k e^{ikx} - \sum_{k=-\nu}^{\nu} \left(1 - \frac{|k|}{\nu + 1}\right) c_k e^{ikx} \\ &= -\frac{1}{\nu + 1} \sum_{k=-\nu}^{\nu} \left(1 - \frac{|k|}{\nu + 1}\right) |k| c_k e^{ikx}. \end{aligned}$$

From the assumption $\tilde{f}'(x) \in L^\infty(0, 2\pi)$, the arithmetic means of Fourier series of $\tilde{f}'(x)$ is bounded. Consequently

$$Q_{n,\nu}(x) = O(\nu^{-1}).$$

Collecting the estimates of $P_{n,\nu}(x)$ and $Q_{n,\nu}(x)$, we have the desired result.

THEOREM 2. *If $f(x)$ belongs to the Lipschitz class of order α* $(0 < \alpha < 1)$

then

$$B_{n,\nu}(x,f) - f(x) = O(\nu^{-\alpha}) \qquad (\nu \leqq n)$$

and if $f(x)$ belongs to the Lipschitz class of order 1, *then*

$$B_{n,\nu}(x,f) - f(x) = O(\nu^{-1}\log \nu) \qquad (\nu \leqq n).$$

More generally if $f(x)$ belongs to the class Λ_2^1, then

$$B_{n,\nu}(x,f) - f(x) = O(\nu^{-1} \log \nu).$$

PROOF. $B_{n,\nu}(x,f)$ maps any bounded function to some bounded function. Hence applying author's another result (G. Sunouchi [2, Theorem 1]) to Theorem 1, we get Theorem 2.

Theorem 2 has been proved by Ruban and Krasilinikoff [1] with another method.

LITERATURE

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