SOME PROPERTIES ON MANIFOLDS WITH ALMOST CONTACT STRUCTURES

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Introduction. Recently S.Sasaki [1] has introduced the notion of an almost contact structure over an $(2n+1)$-dimensional differentiable manifold M (differentiability means class C^∞ always in the following) by three tensors ϕ, ξ, η of types (1, 1), (0, 1) and (1, 0) on M satisfying some relations. The importance of this structure is in the fact that if M has a contact structure defined by a 1-form η, then η induces an almost contact (metric) structure taking a suitable metric in M, and in this contact case, many interesting results have been obtained. But for an almost contact structure which is not necessarily a contact structure, it seems to me that few results are known. In this paper, we consider mainly an almost contact structure.

In their paper [2], S.Sasaki and Y.Hatakeyama defined four tensors $N^i{}_{jk}$, N_{jk}, $N^i{}_j$ and N_j for an almost contact structure by the method of constructing an almost complex structure in $M \times R$ (R denotes the additive group of real number field) and computing its Nijenhuis tensor. Regarding this product space as a trivial bundle space over M, we consider to generalize this method to a principal fibre bundle P with a base space M and a structural group A which is a 1-dimensional real abelian Lie group.

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1. Nijenhuis tensors of almost contact structures. Let M be an almost contact manifold and its defining tensor fields of types (1, 1), (1, 0) and (0,1) be ϕ, ξ and η respectively. We take a real 1-dimensional abelian Lie group A, and denote its Lie algebra over R by $\mathfrak{A}$. Suppose that there exists a differentiable principal A-bundle $P(M, p, A)$ over M with a projection $p: P \to M$. Then we can take in this principal bundle a differentiable infinitesimal connection with a connection form ω which is an $\mathfrak{A}$-valued 1-form on P satisfying the following two relations:

(1. 1) $\qquad \omega(\bar{u}s) = \omega(\bar{u}), \qquad u \in P,\ s \in A,\ \bar{u} \in T_u(P),$

(1. 2) $\qquad \omega(u\bar{s}) = s^{-1}\bar{s}, \qquad u \in P,\ s \in A,\ \bar{s} \in T_s(A),$

where $T_u(P)$ (resp. $T_s(A)$) denotes the tangent space to P (resp. A) at $u \in P$ (resp.

$s \in A$). Since $\mathfrak{A}$ is a 1-dimensional vector space over R with a generator α, we can identify $\mathfrak{A}$ with R by the mapping $r \in R \leftrightarrow r\alpha \in \mathfrak{A}$. Then every $\mathfrak{A}$-valued mapping can be considered as an R-valued mapping. In this way we consider ω to be an R-valued 1-form.

Let $F(P)$ (resp. $F(M)$) denotes the set of all real valued differentiable functions on P(resp. on M), and let $\mathfrak{X}(P)$ (resp. $\mathfrak{X}(M)$) be an $F(P)$-module (resp. $F(M)$-module) of the totality of differentiable vector fields on P(resp. on M). Then the latter is at the same time a vector space over R. Let $\mathfrak{X}^r(P)$ be a vector subspace of $\mathfrak{X}(P)$ over R consisting of the right invariant differentiable vector field with respect to the group operation of A. As any element of $\mathfrak{X}(P)$ is generated by vertical and horizontal vector fields, $\mathfrak{X}(P)$ is generated by $\mathfrak{X}^r(P)$ over $F(P)$. Moreover it is easily seen that every vector field $X \in \mathfrak{X}(P)$ belongs to $\mathfrak{X}^r(P)$ if and only if it is projectable and $\omega(X)$ is in the dual image of $F(M)$ by projection p, i. e., $pX \in \mathfrak{X}(M)$, and $\omega(X) \in p^*F(M)$.

Now using the almost contact structure (ϕ, ξ, η) on M and the connection form ω, we define an almost complex structure on P as follows: If we take a right invariant vector field X on P, then as $\omega(X) \in p^*F(M)$, there exists a unique function $\bar{\omega}(X) \in F(M)$ such that

$$p^*\bar{\omega}(X) = \omega(X)$$

on P. Then we define a vector field $JX \in \mathfrak{X}(P)$ by the two equations:

$$\omega(JX) = - p^*\eta(pX), \tag{1. 3}$$

$$p(JX) = \phi(pX) + \bar{\omega}(X)\xi. \tag{1. 4}$$

If we take a vector field $X \in \mathfrak{X}^r(P)$ and a function $f \in F(P)$ such that $fX \in \mathfrak{X}^r(P)$, then f must satisfy $f(us) = f(u)$ for $u \in P$, $s \in A$. Therefore, there exists a function $\bar{f} \in F(M)$ such that $f = p^*\bar{f}$, and we have

$$p(J(fX)) = \phi(\bar{f}pX) + \bar{\omega}(fX)\xi = \bar{f}(pJX),$$
$$\omega(J(fX)) = - p^*\eta(\bar{f}pX) = f\omega(JX).$$

And so the relation

$$JfX = fJX$$

holds good. Next we extend the above definition to an arbitrary vector field $X \in \mathfrak{X}(P)$. For this X, as there exist functions $f_\lambda \in F(P)$ and vector fields $X_\lambda \in \mathfrak{X}^r(P)$ such that $X = \sum_\lambda f_\lambda X_\lambda$ on P, we define $JX \in \mathfrak{X}(P)$ by

$$JX = \sum_\lambda f_\lambda JX_\lambda.$$

By virtue of the above consideration, this definition is consistent. J is a tensor field on P of type (1, 1). It is noticed that the tensor field J is completely dependent on the connection ω on P.

THEOREM 1. *J is an almost complex structure on P which commutes with the operation of A.*

PROOF. For every vector field $X \in \mathfrak{X}^r(P)$, we can first see that $JX \in \mathfrak{X}^r(P)$ according to $p(JX) \in \mathfrak{X}(M)$, $\omega(JX) \in p^*F(M)$. Therefore we have $R_sJX = JR_sX = JX$ for $X \in \mathfrak{X}^r(P)$, $s \in A$. If $f \in F(P)$, then

$$R_sJ(fX) = (R_s{}^*f)(R_sJX) = J(R_s^*f(R_sX)) = JR_s(fX),$$

which shows that the tensor field J commutes with group operation. For any $X \in \mathfrak{X}^r(P)$, as JX also belongs to $\mathfrak{X}^r(P)$, we have

$$\begin{aligned}\omega(J^2X) &= -p^*\eta(pJX) = -p^*\eta(\phi pX + \bar{\omega}(X)\xi)\\ &= -p^*\bar{\omega}(X)p^*\eta(\xi) = -\omega(X),\\ p(J^2X) &= \phi(pJX) + \bar{\omega}(JX)\xi = \phi(\phi pX + \bar{\omega}(X)\xi) + (-\eta(pX)\xi)\\ &= -pX + \eta(pX)\xi - \eta(pX)\xi = -pX.\end{aligned}$$

Therefore J satisfies for all $X \in \mathfrak{X}^r(P)$ the equation

$$J^2X = -X.$$

Since J is a tensor field on P and any vector field is generated by right invariant vector fields, this equation must hold on $\mathfrak{X}(P)$. Q.E.D.

In the following we calculate the Nijenhuis tensor of this almost complex structure J on P. It is defined by the tensor N of type (1, 2)

$$N(X, Y) = [X, Y] + J[JX, Y] + J[X, JY] - [JX, JY].$$

We have only to calculate it for the right invariant vector fields. Formerly S.Sasaki and Y.Hatakeyama [2] has defined tensors $N^i{}_{jk}$, N_{jk}, $N^i{}_j$ and N_j associated to the almost contact structure (ϕ, ξ, η) on M by the following way. Let $P_0 = M \times R$ be a trivial R-bundle over M with a natural integrable connection ω_0, then we can define an almost complex structure J_0 on P_0 by the same way as above. In fact, for a right invariant vector field $X + \Lambda$, $X \in \mathfrak{X}(M)$, $\Lambda \in \mathfrak{X}(R)$, such that $R_s\Lambda_t = \Lambda_{ts}(t, s \in R)$, we have

$$J_0(X + \Lambda) = (\phi X + \omega_0(\Lambda)\xi) + (-\eta(X)Z_\alpha),$$

where α denotes a generator of the Lie algebra of R such that $\omega_0(Z_\alpha) = 1$, Z_α being a fundamental vector field corresponding to α. We can calculate straightforwardly the Nijenhuis tensor field $N_0(X + \Lambda, Y + \Gamma)$ of this almost complex structure J_0, and if we define the tensor fields N_1, N_2, N_3 and N_4 of types respectively (1, 2), (0, 2), (1, 1) and (0, 1) on M by

$$N_0(X, Y) = N_1(X, Y) + N_2(X, Y)Z_\alpha,$$
$$N_0(X, Z_\alpha) = N_3(X) + N_4(X)Z_\alpha,$$

then we have the following equations.

$$\begin{aligned} N_1(X, Y) &= [X, Y] + \phi[\phi X, Y] + \phi[X, \phi Y] - [\phi X, \phi Y] \\ &\quad + (Y\cdot\eta(X) - X\cdot\eta(Y))\xi, \\ N_2(X, Y) &= (\pounds(\phi X)\eta)Y - (\pounds(\phi Y)\eta)X, \\ N_3(X) &= (\pounds(\xi)\phi)Y, \\ N_4(X) &= -(\pounds(\xi)\eta)X. \end{aligned}$$

These are the same results as those of [2].

Now returning to the case of general principal A-bundle with a connection ω, we also calculate the Nijenhuis tensor of the almost complex structure on P defined by (1. 3) and (1. 4). We notice that for any function $f = p^*\bar{f} \in F(P)$ and for any projectable vector field $X \in \mathfrak{X}(P)$, the relation

$$\pounds(X)p^*\bar{f} = p^*(\pounds(pX)\bar{f})$$

holds good. As $d\omega$ is a horizontal 2-form on P, there exists a 2-form χ on M such that $d\omega = p^*\chi$. If X, Y are right invariant vector fields on P, so are JX, JY, $[X, Y]$ and so on. Therefore we can compute $pN(X, Y)$, $\omega(N(X, Y))$ and we have

LEMMA 1. *For all right invariant vector fields X,Y on P, we have*

$$\begin{aligned} pN(X, Y) &= N_1(pX, pY) + \bar{\omega}(Y)N_3(pX) - \bar{\omega}(X)N_3(pY) \\ &\quad - \{\chi(pJX, pY) + \chi(pX, pJY)\}\xi, \\ \omega(N(X, Y)) &= p^*N_2(pX, pY) + \omega(Y)p^*N_4(pX) - \omega(X)p^*N_4(pY) \\ &\quad - \{d\omega(X, Y) - d\omega(JX, JY)\}. \end{aligned}$$

From this Lemma, we have the following result:

THEOREM 2. *If M^{2n+1} is a normal almost contact manifold, then every almost complex structure of a principal A-bundle $P(M^{2n+1}, p, A)$ associated to an infinitesimal connection ω is integrable if and only if the connection satisfies the relation*

$$d\omega(JX, Y) + d\omega(X, JY) = 0 \tag{1. 5}$$

for all vector fields X and Y on P.

PROOF. It is well known that in an almost contact structure, the vanishing of the tensor N_1 means the vanishing of the other tensors N_2, N_3 and N_4. In this case Lemma 1 shows that relations

$$pN(X, Y) = -(\chi(pJX, pY) + \chi(pX, pJY))\xi,$$
$$\omega N(X, Y) = -(d\omega(X, Y) - d\omega(JX, JY))$$

hold good for $X, Y \in \mathfrak{X}^r(P)$. Therefore it is evident that the vanishing of the tensor N is equvalent to (1. 5). Q.E.D.

For any vector fields $X', Y' \in \mathfrak{X}(M)$, if we take their lift $\widetilde{X}, \widetilde{Y}$ which are right invariant horizontal vector fields on P, then we obtain, by virtue of Lemma 1, the following relations:

$$pN(\widetilde{X}, \widetilde{Y}) = N_1(X', Y') - \{\chi(\phi X', Y') + \chi(X', \phi Y')\}\xi,$$
$$\omega N(X, Y) = p^* N_2(X', Y') - \{d\omega(X, Y) - d\omega(JX, JY)\}.$$

On the other hand, if M is a contact manifold with a contact form η, then an almost contact structure can be induced from η choosing a suitable metric on M. In this case the tensors N_2 and N_4 vanish. Therefore, if we suppose that the almost complex structure on P defined by a contact structure on M and an infinitesimal connection on P is integrable, then M must be normal, and moreover ω satisfies (1. 5). In fact, by virtue of the vanishing of N, we have

$$N_1(X', Y') = \{\chi(\phi X', Y') + \chi(X', \phi Y')\}\xi.$$

Since $N_2 = 0$ is equivalent to $\eta N_1 = 0$, we get $N_1 = 0$. We rewrite this result as a theorem.

THEOREM 3. *Suppose that an almost complex structure on P is defined by a contact structure on M^{2n+1} and an infinitesimal connection ω on P, then the almost complex structure J is integrable if and only if M^{2n+1} is a normal contact manifold and ω satisfies* (1. 5).

2. Infinitesimal (ϕ, ξ, η)-transformations. In this section we shall study some relations between the vector fields on P and vector fields on M. We suppose that P has an almost complex structure J defined by an almost contact structure (ϕ, ξ, η) on M and an infinitesimal connection ω on P. A vector field X on P which verifies the condition

$$\mathfrak{L}(X)J = 0$$

is called an almost analytic vector field. We denote the set of all almost analytic vector fields on P by $\mathfrak{g}$. In the next place, we call a vector field X' on M an infinitesimal (ϕ, ξ, η)-transformation when it satisfies the conditions

$$\mathfrak{L}(X')\phi = 0, \qquad \mathfrak{L}(X')\eta = 0,$$

and the set of all of them is denoted by $\mathfrak{h}$. Since A is an abelian group, a fundamental vector field Z_α on P corresponding to a generator α of $\mathfrak{A}$ such that $\omega(Z_\alpha) = 1$ is right invariant. By virtue of Theorem 1, as the tensor J is invariant by the operation of A on P, we have $\mathfrak{L}(Z_\alpha)J = 0$, that is, Z_α is an almost analytic vector field. At first we shall seek for the condition under which JZ_α is almost analytic. Since JZ_α is right invariant, we have by virtue of Lemma 1 for any vector field $X \in \mathfrak{X}^r(P)$,

$$pN(X, Z_\alpha) = N_3(pX) - \chi(pX, \xi)\xi,$$
$$\omega N(X, Z_\alpha) = p^*\{N_4(pX) + \chi(\phi pX, \xi)\}.$$

On the other hand we have for any vector field $X \in \mathfrak{X}(P)$

$$\begin{aligned} N(X, Z_\alpha) &= (\mathfrak{L}(Z_\alpha)J)JX + (\mathfrak{L}(JZ_\alpha)J)X \\ &= (\mathfrak{L}(JZ_\alpha)J)X, \end{aligned}$$

and so the conditions $N(X, Z_\alpha) = 0$ for all $X \in \mathfrak{X}(P)$ and JZ_α belongs to $\mathfrak{g}$ are equivalent with each other. Therefore if we assume that the curvature form $d\omega = p^*\chi$ on P satisfies the condition

$$\chi(X', \xi) = 0 \tag{2. 1}$$

for all $X' \in \mathfrak{X}(M)$, then JZ_α belongs to $\mathfrak{g}$ if and only if the tensor N_3 of M in consideration vanishes.

THEOREM 4. *When the condition* (2. 1) *is verified, then* JZ_α *is almost analytic if and only if the tensor* N_3 *vanishes.*

It is noteworthy that the relation (1. 5) implies (2. 1). In particular, when P is a product bundle $M \times R$ and ω is an integrable connection ($d\omega = 0$), then JZ_α on P is almost analytic if and only if $N_3 = 0$. Moreover in contact case, if the vector field JZ_α is almost analytic on P, then we have $N_3 = 0$ and the connection ω satisfies (2. 1). In fact, as we know, in the almost contact structure associated with a contact structure the tensor N_4 vanishes. So we have

$$\chi(\phi X', \xi) = -N_4(X') = 0$$

for all $X' \in \mathfrak{X}(M)$, and this means that the condition (2. 1) is valid.

Taking into consideration of the fact that JZ_α is a lift of ξ with respect to the connection ω, we proceed to study the condition in order that the lift of a vector field on M is almost analytic on P. For this purpose, we shall prove the following lemma.

LEMMA 2. *For any right invariant vector fields* X *and* Y *on* P *on which an almost complex structure* J *is given by* (1. 3) *and* (1. 4), *the relations*

$$\begin{aligned} p((\mathfrak{L}(X)J)Y) = (\mathfrak{L}(X')\phi)Y' &+ \bar{\omega}(Y)[X', \xi] \\ &+ (\mathfrak{L}(Y')\bar{\omega}(X))\xi + \chi(X', Y')\xi \end{aligned}$$

and

$$\omega((\pounds(X)J)Y) = -\ p^*(\pounds(X')\eta)Y' - \pounds(JY)\omega(X) - d\omega(X, JY)$$

hold good, where X' and Y' denote the projections of X and Y.

PROOF. Since JX, JY, $[X,Y]$ and so on are in $\mathfrak{X}^r(P)$ if $X, Y \in \mathfrak{X}^r(P)$, we have

$$\begin{aligned} p(\pounds(X)J\cdot Y) &= p\{[X, JY] - J[X,Y]\} \\ &= (\pounds(X')\phi)Y' + \bar{\omega}(Y)[X',\xi] + (\pounds(Y')\cdot\bar{\omega}(X))\xi + \chi(X',Y')\xi. \end{aligned}$$

In the same way, we see that

$$\begin{aligned} \omega((\pounds(X)J)Y) &= \omega\{[X, JY] - J[X,Y]\} \\ &= -\ d\omega(X, JY) - \pounds(JY)\cdot\omega(X) - p^*(\pounds(X')\cdot\eta(Y')) \\ &\quad + p^*(\eta([X',Y']) \\ &= -\ p^*(\pounds(X')\eta\cdot Y') - \pounds(JY)\cdot\omega(X) - d\omega(X, JY). \end{aligned}$$ Q.E.D.

Now we suppose that M is an almost contact manifold and P has an integrable connection ω. An example for such a bundle space P is for instance a product bundle $M \times R$ with a natural trivial connection (see §1). In this case, if M is normal almost contact, the almost complex structure J defined on P associated with ω is integrable by virtue of Theorem 2, and we have the following theorem :

THEOREM 5. *Let $P(M^{2n+1}, p, A)$ be a principal bundle which has an integrable connection ω and a base space M^{2n+1} with an almost contact manifold. If $\omega(X)$ is constant on P for a right invariant vector field X, then X belongs to $\mathfrak{G}$ if and only if pX belongs to $\mathfrak{h}$. In other words, if X is an almost analytic right invariant vector field, then $\omega(X) =$ constant if and only if pX is an infinitesimal (ϕ, ξ, η)-transformation.*

PROOF. We first suppose that $\omega(X) =$ const. on P for a vector field $X \in \mathfrak{X}^r(P)$. Then, by virtue of Lemma 2, for any $Y \in \mathfrak{X}^r(P)$ we have the relations

$$\begin{aligned} p((\pounds(X)J)Y) &= (\pounds(X')\phi)Y' + \bar{\omega}(Y)[X',\xi], \\ \omega((\pounds(X)J)Y) &= -\ p^*(\pounds(X')\eta)Y'. \end{aligned}$$

If $\pounds(X)J = 0$, then replacing the vector field Y by the fundamental vector field Z_α in the first equation, we have $[X',\xi] = 0$ and therefore $X' \in \mathfrak{h}$. Conversely, if X' belongs to $\mathfrak{h}$, then it can be verified easily that $[X',\xi] = 0$. And we have at once $(\pounds(X)J)Y = 0$ for all $Y \in \mathfrak{X}^r(P)$ and therefore X is an almost analytic vector field. Next we suppose a vector field $X \in \mathfrak{X}^r(P)$ to be almost analytic. Then according to Lemma 2, we have

$$(\pounds(X')\phi)Y' + \bar{\omega}(Y)[X',\xi] + (\pounds(Y')\cdot\bar{\omega}(X))\xi = 0,$$

$$- p^*(\pounds(X')\eta)Y' - (\pounds(JY))\cdot\omega(X) = 0.$$

In the same way as above, we can see easily that $\omega(X) = \text{const.}$ on P is equivalent to $X' \in \mathfrak{h}$. Q.E.D.

COROLLARY 1. *On the same principal bundle as Theorem* 5, *every vector field X' on M^{2n+1} belongs to $\mathfrak{h}$ if and only if its lift $\widetilde{X}$ with respect to the connection ω belongs to $\mathfrak{g}$.*

PROOF. Since the lift $\widetilde{X}$ is right invariant and satisfies $\omega(\widetilde{X}) = 0$ on P, by virtue of Theorem 5 we have $\pounds(\widetilde{X})J = 0$ if and only if $X' = p\widetilde{X} \in \mathfrak{h}$. Q.E.D.

COROLLARY 2. *In a normal almost contact manifold M^{2n+1}, for every infinitesimal (ϕ, ξ, η)-transformation X', $\phi X'$ belongs to $\mathfrak{h}$ if and only if $\eta(X') = constant$ on M^{2n+1}.*

PROOF. We take a principal A-bundle P with an integrable connection ω, and give it a complex structure J by (1. 3) and (1. 4). For any $X' \in \mathfrak{h}$, if we take a lift $\widetilde{X}$ in P with respect to ω, then from the above Corollary 1, $\widetilde{X}$ is an almost analytic vector field on P. Since J is a complex structure we have $\pounds(J\widetilde{X})J = 0$. By virtue of Theorem 5, $\omega(J\widetilde{X})$ is constant on P if and only if $p(J\widetilde{X})$ belongs to $\mathfrak{h}$, that is to say, $-\eta(X')$ is constant on M if and only if $\phi X'$ belongs to $\mathfrak{h}$. Q.E.D.

It is known that in a contact manifold, the Lie algebra of all infinitesimal contact transformations of M is infinite dimensional. While in an almost contact structure associated with the contact structure, an infinitesimal transformation of M which leaves invariant thetensors ϕ and η is necessarily a Killing vector field. Hence the set of all such infinitesimal transformations is a finite dimensional Lie algebra. In a manifold with an almost contact structure, we can get the following theorem under an additional assumption of compactness on M.

THEOREM 6. *Let M^{2n+1} be a compact differentiable manifold with a normal almost contact structure. Then the Lie algebra $\mathfrak{h}$ of all infinitesimal (ϕ, ξ, η)-transformations on M^{2n+1} is finite dimensional.*

PROOF. Take a principal circle bundle $P(M, p, S^1)$ over M with an integrable connection ω, and we define an almost complex structure J on P associated with this connection, by (1. 3) and (1. 4). Then by virtue of Theorem 2, P is a complex manifold with complex structure J. Now we denote by $\mathfrak{g}_0$ a set of all right invariant almost analytic vector fields X on P such that $\omega(X) = \text{constant}$.

It can be easily verified that $\mathfrak{g}_0$ is a subalgebra of $\mathfrak{g}$. We can identify the Lie algebra $\mathfrak{A}$ of S^1 with a 1-dimensional subalgebra of $\mathfrak{g}_0$ generated by a fundamental vector field Z_α. Then there exists a natural exact sequence

$$0 \to \mathfrak{A} \to \mathfrak{g}_0 \to \mathfrak{h} \to 0.$$

To any $X' \in \mathfrak{h}$, let its lift be $\widetilde{X} \in \mathfrak{g}_0$. Then this mapping $X' \to \widetilde{X}$ is clearly a splitting of the above exact sequence, and therefore we have

$$\mathfrak{g}_0 \cong \mathfrak{A} \oplus \mathfrak{h},$$

which means that $\mathfrak{h}$ may be considered as a subalgebra of $\mathfrak{g}_0$. From the assumption of compactness of M, P is also a compact complex manifold. Therefore $\mathfrak{g}$ is a finite dimensional Lie algebra. Hence its subalgebra $\mathfrak{g}_0$, and also $\mathfrak{h}$ is finite dimensional. Q.E.D.

3. Almost contact metric structures. Suppose M be a manifold with an almost contact metric structure (ϕ, ξ, η, g), and as in the previous sections consider a principal A-bundle $P(M, p, A)$ with base space M. We give to P an almost complex structure making use of an infinitesimal connection ω by the equations (1. 3) and (1. 4). Moreover, since M has a Riemannian metric g, we can give a Riemannian metric to the manifold P by the following relation

$$G(X, Y) = p^*g(pX, pY) + \omega(X)\omega(Y) \tag{3. 1}$$

for all right invariant vector fields X, Y. If fX, hY is in $\mathfrak{X}^r(P)$ for functions f, h on P and vector fields $X, Y \in \mathfrak{X}^r(P)$, then we can easily verify that

$$G(fX, hY) = fhG(X, Y),$$

therefore we can extend this definition to the vector fields on $\mathfrak{X}(P)$. The tensor G of type (0, 2) is clearly symmetric. Moreover for any $X \in \mathfrak{X}^r(P)$ we have

$$G(X, X) = p^*g(pX, pX) + \omega(X)\omega(X) \geqq 0,$$

and the vanishing of $G(X,X)$ means the vanishing of $g(pX, pX)$ and $\omega(X)$ and therefore X is at the same time vertical and horizontal, which implies $X = 0$. Therefore G is positive definite. As this metric G is invariant by the operation of the group A on the right, that is $R_s^*G = G$, the fundamental vector field Z_α is a unit Killing vector field with respect to the metric G. We shall prove the following

THEOREM 7. *The Riemannian metric G and the almost complex structure J give an almost Hermitian structure on P.*

PROOF. For any two vector fields $X, Y \in \mathfrak{X}^r(P)$, we know that JX, $JY \in \mathfrak{X}^r(P)$, and from the definition (3. 1), we have

$$G(JX, JY) = p^*g(pJX, pJY) + \omega(JX)\omega(JY)$$

$$= p^*\{g(\phi pX, \phi pY) + \bar{\omega}(X)\bar{\omega}(Y)g(\xi, \xi) + \eta(pX)\eta(pY)\}$$
$$= p^*g(pX, pY) + \omega(X)\omega(Y) = G(X, Y).$$

Therefore the metric G and the almost complex structure J give an almost Hermitian structure on P and its fundamental 2-form is given by

$$\Omega(X, Y) = G(X, JY). \qquad \text{Q.E.D.}$$

Next we shall study when this almost Hermitian structure reduces to a Kählerian structure. For this porpose, we give some lemmas.

LEMMA 3. *For any vector fields* $X, Y, Z \in \mathfrak{X}^r(P)$, *we have*

$$(3.2) \qquad 2G(\nabla_X Y, Z) = 2p^*g(\bar{\nabla}_{X'}Y', Z') + \omega(X)d\omega(Y, Z) - \omega(Y)d\omega(Z, X) + \omega(Z)d\omega(Y, X) + 2\omega(Z)(\mathfrak{L}(X)\cdot\omega(Y)),$$

where ∇ *(resp.* $\bar{\nabla}$*) denotes the covariant differentiation with respect to the Riemannian connection on* P *(resp. on* M*).*

PROOF. From the formula

$$2G(\nabla_X Y, Z) = \mathfrak{L}(X)G(Y, Z) + \mathfrak{L}(Y)G(Z, X) - \mathfrak{L}(Z)G(X, Y) - G(X, [Y, Z]) + G(Y, [Z, X]) + G(Z, [X, Y])$$

for any Riemannian metric G and for any vector fields X, Y, Z, we can verify Lemma 3 without difficulty.

LEMMA 4. *For any vector fields* $X, Y, Z \in \mathfrak{X}^r(P)$, *we have*

$$(3.3) \qquad 2p^*g((\nabla_X Y)' - \bar{\nabla}_{X'}Y', Z') = \omega(X)d\omega(Y, Z) + \omega(Y)d\omega(X, Z),$$
$$(\nabla_X\omega)Y + (\nabla_Y\omega)X = 0, \qquad \text{i. e., } \omega \text{ is } G\text{-Killing.}$$

PROOF. From the definition of metric G, we have

$$G(\nabla_X Y, Z) = p^*g((\nabla_X Y)', Z') + \omega(\nabla_X Y)\omega(Z)$$

and by virtue of (3. 2) we have also

$$2p^*g((\nabla_X Y)' - \bar{\nabla}_{X'}Y', Z') = \omega(X)d\omega(Y, Z) - \omega(Y)d\omega(Z, X) + \omega(Z)\{(\nabla_X\omega)Y + (\nabla_Y\omega)X\}.$$

If we replace Z in this equation by vertical vector field Z_α, then we have

$$(\nabla_X\omega)Y + (\nabla_Y\omega)X = 0, \qquad \text{for all } X, Y \in \mathfrak{X}^r(P).$$

This proves Lemma 4.

LEMMA 5. For any vector fields $X, Y, Z \in \mathfrak{X}^r(P)$, we have

$$(3.4) \qquad 2\eta((\nabla_X Y)' - \bar{\nabla}_{X'}Y') = \omega(X)\chi(Y', \xi) - \bar{\omega}(Y)\chi(\xi, X'),$$

$$2g(\phi((\nabla_X Y)' - \bar{\nabla}_{X'} Y'), Z') = -\bar{\omega}(X)\chi(Y', \phi Z') + \bar{\omega}(Y)\chi(\phi Z', X').$$

PROOF. The projections of JZ_α and $JZ-\omega(Z)JZ_\alpha$ are ξ and $\phi Z'$. Therefore if we replace Z by JZ_α or $JZ - \omega(Z)JZ_\alpha$ in (3. 3), then we can obtain (3 .4) from the anti-symmetry of ϕ with respect to metric g.

LEMMA 6. *For any vector fields $X, Y, Z \in \mathfrak{X}^r(P)$, we have*

$$\begin{aligned}(3.\ 5)\qquad 2G((\nabla_X J)Y, Z) = p^*[&2g((\bar{\nabla}_{X'}\phi)Y', Z') + \eta(Y')\chi(Z', X') - \eta(Z')\chi(Y', X') \\ &+ \bar{\omega}(Y)\{\chi(X', \phi Z') + 2((\bar{\nabla}_{X'}\eta)Z')\} - \bar{\omega}(Z)\{\chi(X', \phi Y') + 2((\bar{\nabla}_{X'}\eta)Y')\}] \\ &+ \omega(X)\{d\omega(JY, Z) + d\omega(Y, JZ)\}.\end{aligned}$$

PROOF. We may deduce it by straightforward calculation using (3. 4) and the fact that ω is a G-Killing form on P.

If we denote the fundamental 2-form of the almost Hermitian structure on P by Ω, then $P(J, G)$ is a Kählerian manifold if Ω is a closed form and J is integrable. As is well known, these conditions are equivalent to $\nabla J = 0$. Investigating ∇J, we can deduce the following theorem.

THEOREM 8. *If the almost Hermitian manifold $P(J, G)$ is Kählerian, then we have for all $X, Y \in \mathfrak{X}^r(P)$,*

$$(3.\ 6)\qquad d\omega(JX, Y) + d\omega(X, JY) = 0,$$

$$(3.\ 7)\qquad 2g((\nabla_{X'}\phi)Y', Z') = -\eta(Y')\chi(Z', X') + \eta(Z')\chi(Y', X'),$$

$$(3.\ 8)\qquad 2(\nabla_{X'}\eta)Y' = \chi(\phi X', Y').$$

Therefore η is a closed form and M^{2n+1} has a normal almost contact metric structure. Conversely, if the structure tensors (ϕ, ξ, η, g) and connection ω satisfy the conditions (3. 6), (3. 7) *and* (3. 8), *then $P(J, G)$ is a Kählerian manifold.*

PROOF. Suppose that $P(J, G)$ is a Kählerian manifold. Then as the almost complex structure J is covariant constant, we see that the right hand side of (3. 5) vanishes for any vector fields $X, Y, Z \in \mathfrak{X}^r(P)$. Therefore, if we replace X with the vertical vector field Z_α, then we have

$$d\omega(JY, Z) + d\omega(Y, JZ) = 0$$

for $Y, Z \in \mathfrak{X}^r(P)$. We can deduce from this relation the following

$$\chi(\phi X', Y') + \chi(X', \phi Y') = 0,$$
$$\chi(X', \xi) = 0$$

for all $X'. Y' \in \mathfrak{X}(M)$. While if we replace Y with the vertical vector field Z_α, then we get (3. 8)

$$\chi(X', \phi Z') + 2(\bar{\nabla}_{X'}\eta)Z' = 0$$

for $X', Z' \in \mathfrak{X}(M)$. Substituting these equations into (3. 5), we have (3. 7). Then we have

$$\begin{aligned} 2d\eta(X', Y') &= 2(\bar{\nabla}_{X'}\eta)Y' - 2(\bar{\nabla}_{Y'}\eta)X' \\ &= \chi(\phi X', Y') - \chi(\phi Y', X') = 0. \end{aligned}$$

Therefore η is a closed form on M, Next, as the almost complex structure J is integrable, the Nijenhuis tensor N of J vanishes. Therefore, from Lemma 1 and (3. 6) we can obtain $N_1 = 0$ at once. The converse statement in the theorem is evident. Q.E.D.

As is proved in [2], a manifold with an almost contact structure admits a symmetric (ϕ, ξ, η)-connection if and only if the almost contact structure is normal and η is closed. Therefore, we can see by virtue of Theorem 8 that the following is true: if the almost Hermitian manifold $P(J, G)$ is a Kählerian manifold, then M admits a symmetric (ϕ, ξ, η)-connection. In general, the metrical connection on M is not always (ϕ, ξ, η)-connection. But we can obtain the following corollary from Theorem 8.

COROLLARY. *Suppose that the almost Hermitian manifold $P(J, G)$ is Kählerian manifold and that ξ is a Killing vector field on M^{2n+1}. Then ω is an integrable connection and the metrical connection of the base space M^{2n+1} is a (ϕ, ξ, η)-connection. The converse is also true.*

By virtue of Theorem 8, we see that if P is a Kählerian manifold, then $d\eta = 0$, and so M can not be a contact manifold. Therefore, on a contact manifold M we can not admit a Kählerian A-bundle defined by (1. 3), (1. 4) and (3. 1) using an infinitesimal connection ω.

Next we study whether there exists a differentiable function ρ on P such that a metric $\widetilde{G}$ defined by

$$\widetilde{G} = \rho G$$

and the almost complex structure J give an almost Kählerian structure on P. For this purpose, we calculate the fundamental 2-form Ω on P, then we have

$$\Omega = p^*\psi + p^*\eta \wedge \omega,$$

where ψ is a 2-form on M such that

$$\psi(X', Y') = g(X', \phi Y').$$

If we calculate $d\widetilde{\Omega} = d\rho \wedge \Omega + \rho d\Omega$, then we obtain

(3. 9) $i(JZ_\alpha)i(Z_\alpha)\,d\widetilde{\Omega} = (Z_\alpha\rho)\omega + (JZ_\alpha\rho)p^*\eta - \rho p^*(\mathfrak{L}(\xi)\eta) - d\rho.$

On the other hand, if we take an adapted coframe $(\eta_1,\cdots,\eta_{2n},\eta)$ in a neighborhood U of the almost contact structure on M, then we can take $(p^*\eta_1,\cdots,p^*\eta_{2n},p^*\eta,\omega)$ as a coframe in $p^{-1}(U)$. Let its dual frame be $(Z_1,\cdots,Z_{2n}, JZ_\alpha, Z_\alpha)$. Then for any $\rho\in F(P)$, we have

$$d\rho = (Z_\alpha\rho)\omega + (JZ_\alpha\rho)p^*\eta + \sum_{k=1}^{2n}(Z_k\rho)(p^*\eta_k),$$

$$p^*(\mathfrak{L}(\xi)\eta) = \sum_{k=1}^{2n} A_k(p^*\eta_k) \qquad \text{for some } A_k\in p^*F(M),$$

because $\mathfrak{L}(\xi)\eta$ has no η-component. Then we have the following

THEOREM 9. *Suppose that the almost Hermitian structure* (J,G) *on* P *is defined by an integrable connection* ω *and an almost contact metric structure associated with a contact structure* η *on* M^{2n+1}. *Then the almost Hermitian structure* $(J,\widetilde{G})$, $\widetilde{G}=\rho G$, *defines an almost Kählerian structure on* P *if and only if the function* ρ *satisfies the equation*

$$d\rho = -\rho\omega.$$

PROOF. Since M is a contact manifold, we have

$$\mathfrak{L}(\xi)\eta = 0,$$
$$\Omega = p^*d\eta + p^*\eta\wedge\omega.$$

Let $P(J,\widetilde{G})$ is an almost Kählerian manifold, then $d\widetilde{\Omega}=0$, and therefore we have

$$d\rho = (JZ_\alpha\rho)p^*\eta + (Z_\alpha\rho)\omega$$

by virtue of (3. 9). We can easily see that

$$\begin{aligned}0 = d\widetilde{\Omega} &= d\rho\wedge\Omega + \rho d\Omega\\ &= (Z_\alpha\rho+\rho)\omega\wedge p^*d\eta + (JZ_\alpha\rho)p^*(\eta\wedge d\eta).\end{aligned}$$

As $\omega\wedge p^*d\eta$ and $p^*(\eta\wedge d\eta)$ is mutually independent forms on P, we obtain

$$Z_\alpha\rho + \rho = 0, \qquad (JZ_\alpha\rho) = 0.$$

Therefore, the necessity of the theorem is proved. Sufficiency is evident. Q.E.D.

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