TOPOLOGY OF POSITIVELY PINCHED KAEHLER MANIFOLDS

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1. Introduction. The purpose of the present paper is to show that if the curvature of a complete Kaehler manifold M of complex dimension m does not deviate much from that of the complex projective space $P_m(C)$, then $\pi_i(M) = \pi_i(P_m(C))$ for all i. Results in the same direction have been obtained by Rauch [15], Klingenberg [11] and do Carmo [6].

To state our result more explicitly, we introduce some notations and give a few definitions. Let J be the tentor field defining the complex structure of M. Let q be a real 2-dimensional subspace of the tangent space $T_x(M)$ at a point x of M and let X and Y be an orthonormal basis for q. We define the angle $\alpha(q)$, $0 \leqq \alpha(q) \leqq \pi/2$, between the two planes q and $J(q)$ by

$$\cos \alpha(q) = |(X, JY)|,$$

where the inner product (X, JY) is defined by the Kaehler metric. It is a simple matter to verify that $\alpha(q)$ depends only on q. We set

$$\overline{K}(q) = (1 + 3\cdot\cos^2\alpha(q))/4.$$

For a Kaehler manifold M we have three kinds of pinchings. Let $K(q)$ denote the sectional curvature of M. Then, the *Riemannian pinching* of M is greater than $\delta, \delta > 0$, if there is a positive number L such that

$$\delta L < K(q) \leqq L \qquad \text{for all } q.$$

The *Kaehlerian pinching* of M is greater than δ if there is a positive number L such that

$$\delta L\cdot \overline{K}(q) < K(q) \leqq L\cdot \overline{K}(q) \qquad \text{for all } q.$$

Finally, the *holomorphic pinching* of M is greater than δ if there is a positive number L such that

$$\delta L < K(q) \leqq L \qquad \text{for all } q \text{ such that } J(q) = q.$$

REMARKS. 1) If $J(q) = q$, then $\overline{K}(q) = 1$.

2) If the Kaehlerian pinching of M is greater than δ, then the holomorphic pinching and the Riemannian pinching of M are, respectively, greater than δ and $\delta/4$.

3) The Kaehlerian pinching of a complex projective space with Fubini-

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Study metric is 1 (see, for instance, [17]). Consequently, its holomorphic pinching and Riemannian pinching are, respectively, 1 and 1/4.

Now, our result may be stated as follows:

THEOREM. *Let M be a complete Kaehler manifold of complex dimension m with Kaehlerian pinching $> 4/7$. Then, $\pi_i(M) = \pi_i(P_m(C))$ for all i.*

This improves slightly Klingenberg's constant 16/25 obtained in [11]. Whereas his method is based on Morse theory, our result is based on Sphere Theorem of Berger [2] and Klingenberg [10] (in particular, for odd dimensional Riemannian manifolds) which may be stated as follows:[1)]

Every simply connected, complete Riemannian manifold with Riemannian pinching $> 1/4$ is homeomorphic with a sphere.

Although 1/4 is the best possible constant for even dimensional Riemannian manifolds, it is an open question whether Sphere Theorem for odd dimensional Riemannian manifolds holds for a smaller constant. In §2 we shall state our main result in such a way that any sharpening of Sphere Theorem for odd dimensional Riemannian manifolds would result in the reduction of 4/7 to a smaller constant.

In §6 we shall give miscellaneous results obtained by the same method.

I conclude this introduction by expressing my thanks to Klingenberg and do Carmo for showing me the manuscripts of their papers [11] and [6] from which I learned the notion of Kaehlerian pinching.

2. An outline of the proof. We know that a sphere S^{2m+1} of dimension $2m+1$ is a principal circle bundle over $P_m(C)$. The main idea is to generalize this situation, that is, to construct a principal circle bundle P over M such that the universal covering space of P is homeomorphic with S^{2m+1}. Then the exact homotopy sequences of the fibrings $S^1 \rightarrow S^{2m+1} \rightarrow P_m(C)$ and $S^1 \rightarrow P \rightarrow M$ give an isomorphism $\pi_i(M) \approx \pi_i(P_m(C))$ for $i \geqq 2$. On the other hand, M is simply connected by a theorem of Synge [14] or by a theorem of the author [12] so that $\pi_i(M) = \pi_i(P_m(C))$ for all i.

We shall first show that the theorem stated in the introduction is an immediate consequence of the following

THEOREM 1. *Let M be a complete Kaehler manifold with Kaehlerian pinching $> \delta$. Then there exist a principal circle bundle P over M and a Riemannian metric on P with Riemannian pinching $> \delta/(4-3\delta)$.*

1) Tsukamoto simplified part of the proof of Sphere Theorem [16].

If we set $\delta = 4/7$, then $\delta/(4-3\delta) = 1/4$ so that the universal covering space of P is homeomorphic with a sphere and, by the preceding argument, $\pi_i(M) = \pi_i(P_m(C))$ for all i.

We shall outline here the proof of Theorem 1. In §3 we consider, in general, a principal circle bundle P over a Riemannian manifold M of real dimension n with metric $ds^2 = \sum_{i=1}^{n} (\theta_i)^2$. Let γ be a 1-form on P defining a connection in P. Let a and b be real numbers and consider the Riemannian metric $d\sigma^2 = \pi^*(ds^2) + (ab\gamma)^2$ on P, where π is the projection of P onto M. We use two constants a and b instead of just one for purely computational reason. We express the curvature of $d\sigma^2$ in terms of those of ds^2 and γ. In §4 we shall show that if $b\cdot d\gamma = \pi^*\left(\sum J_{ij}\theta^i \wedge \theta^j\right)$ where J_{ij} are the components of J with respect to $\theta^1, \cdots, \theta^n$ and if M is of Kaehlerian pinching $> \delta$, then P is of Riemannian pinching $> \delta/(4-3\delta)$. In §5 we find a circle bundle P and a connection form γ such that $b\cdot d\gamma$ with a suitable b is sufficiently close to $\pi^*\left(\sum J_{ij}\theta^i \wedge \theta^j\right)$ in a certain sense, thus completing the proof of Theorem 1.

3. Riemannian structure on a circle bundle.

Throughout §3, let P be a principal circle bundle over an n-dimensional manifold M with projection π, ds^2 a Riemannian metric on M and γ a 1-form on P defining a connection in the bundle P. Functions on M such as components of tensor fields on M are considered sometimes as functions on P in a natural way without any change of notations. We shall also agree on that indices i, j, k and l run from 1 to n and indices α, β, λ and μ run from 0, 1, to n.

Let a and b be arbitrary real numbers fixed throughout §3. Let $d\sigma^2 = \pi^*(ds^2) + (ab\gamma)^2$. Then $d\sigma^2$ is a Riemannian metric on P. We shall now study the structure equations of the Riemannian connections defined by ds^2 and $d\sigma^2$ and also the connection given by γ. In studying the Riemannian connections of ds^2 and $d\sigma^2$ we shall not consider frame bundles but shall use exclusively forms defined on the base manifolds M and P.

Let U be a small open set in M in which ds^2 is given by

$$ds^2 = \sum_j (\theta^j)^2,$$

where $\theta^1, \cdots, \theta^n$ are 1-forms defined on U. Let $(\omega^i{}_j)$ be a skew-symmetric matrix of 1-forms on U which defines the Riemannian connection of M so that we have the following structure equations:

$$d\theta^i = -\sum_j \omega^i{}_j \wedge \theta^j,$$

$$d\omega^i{}_j = -\sum_k \omega^i{}_k \wedge \omega^k{}_j + \Omega^i{}_j$$

with

$$\Omega^i{}_j = 1/2 \sum_{k,l} K_{ijkl}\theta^k \wedge \theta^l,$$

where K_{ijkl} are the components of the curvature tensor with respect to $\theta^1, \cdots, \theta^n$.

Next, we shall study the connection defined by γ. Since the structure group S^1 of P is abelian, the structure equation is given by

$$d\gamma = \Gamma,$$

where Γ is the curvature form of γ and can be written as follows:

$$\Gamma = \pi^*\Big(\sum_{i,j} A_{ij}\theta^i \wedge \theta^j\Big), \quad A_{ij} = -A_{ji}.$$

Finally, we shall study the Riemannian connection defined by $d\sigma^2$. Set

$$\varphi^0 = ab\gamma,$$
$$\varphi^i = \pi^*(\theta^i),$$

so that $d\sigma^2 = \sum_\alpha (\varphi^\alpha)^2$.

PROPOSITION 1. *Set*

$$\psi^0{}_0 = 0,$$
$$\psi^i{}_0 = -\psi^0{}_i = -\sum_j abA_{ij}\varphi^j,$$
$$\psi^i{}_j = \pi^*(\omega^i{}_j) - abA_{ij}\varphi^0.$$

Then (ψ^α_β) *defines the Riemannian connection on* P *with respect to* $d\sigma^2$.

PROOF. Evidently, (ψ^α_β) is skew-symmetric. To prove that (ψ^α_β) defines a linear connection of the manifold P, let V be another small open subset of M on which $ds^2 = \sum_j (\bar\theta^j)^2$. Then

$$\bar\theta^i = \sum_j s^i{}_j\theta^j \qquad \text{on } U \cap V,$$

where $(s^i{}_j)$ takes values in $O(n)$. Let $(\bar\omega^i{}_j)$ and $(\bar\Omega^i{}_j)$ be the connection form and the curvature form of the Riemannian connection given by ds^2 with respect to the basis $\bar\theta^1, \cdots, \bar\theta^n$; they are defined on V. Set

$$d\gamma = \Gamma = \pi^*\Big(\sum_{i,j} \bar A_{ij}\bar\theta^i \wedge \bar\theta^j\Big)$$

and

$$\bar{\varphi}^0 = ab\gamma,$$
$$\bar{\varphi}^i = \pi^*(\bar{\theta}^i).$$

Using $\bar{\varphi}^\alpha$ and $\bar{A}_{ij}$ we define $(\bar{\psi}^\alpha_\beta)$ in the same way as (ψ^α_β).

Since both $(\omega^i{}_j)$ and $(\bar{\omega}^i{}_j)$ define the same Riemannian connection, they are related to each other as follows:

$$\bar{\omega}^i{}_j = \sum_{k,l} s^i{}_k \omega^k{}_l s^j{}_l - \sum_k ds^i{}_k s^j{}_k,$$

or, in short,

$$\bar{\omega} = s\omega s^{-1} - ds\cdot s^{-1}, \text{ where } s = (s^i{}_j),\ \omega = (\omega^i{}_j) \text{ and } \bar{\omega} = (\bar{\omega}^i{}_j).$$

On the other hand, we have

$$\bar{A}_{ij} = \sum_{k,l} s^i{}_k A_{kl} s^i{}_l,$$

$$\bar{\varphi}^\alpha = \sum_\beta t^\alpha_\beta \varphi^\beta, \qquad \text{where } t^i{}_j = s^i{}_j,\ t^0{}_j = t^i{}_0 = 0,\ t^0{}_0 = 1.$$

A straightforward computation shows

$$\bar{\psi}^\alpha_\beta = \sum_{\lambda,\mu} t^\alpha_\lambda \psi^\lambda_\mu t^\beta_\mu - \sum_\lambda dt^\alpha_\lambda t^\beta_\lambda,$$

which means that (ψ^α_β) defines a linear connection of the manifold P.

To see that it actually defines the Riemannian connection, it suffices to prove that the connection has no torsion. By a simple calculation, we obtain

$$d\varphi^0 + \sum_\mu \psi^0{}_\mu \wedge \varphi^\mu = ab \sum_{k,l} A_{kl}\varphi^k \wedge \varphi^l + ab \sum_{k,l} A_{kl}\varphi^l \wedge \varphi^k = 0,$$

$$\begin{aligned} d\varphi^i + \sum_\mu \psi^i{}_\mu \wedge \varphi^\mu &= \pi^*(d\theta^i) + \sum_j (\pi^*(\omega^i{}_j) - abA_{ij}\varphi^0) \wedge \varphi^j \\ &\quad - ab \sum_j A_{ij}\varphi^j \wedge \varphi^0 \\ &= \pi^*\Big(d\theta^i + \sum_j \omega^i{}_j \wedge \theta^j\Big) = 0. \end{aligned}$$ QED.

PROPOSITION 2. *If* (Ψ^α_β) *is the curvature form of the connection defined by* (ψ^α_β), *then*

$$\Psi^0{}_0 = 0,$$

$$\Psi^i{}_0 = -\Psi^0{}_i = -a^2b^2 \sum_{k,l} A_{ik}A_{kl}\varphi^l \wedge \varphi^0 - ab \sum_{k,l} A_{ik;l}\varphi^l \wedge \varphi^k,$$

$$\Psi^i{}_j = \pi^*(\Omega^i{}_j) - \sum_{k,l} a^2b^2(A_{ij}A_{kl} + A_{ik}A_{jl})\varphi^k \wedge \varphi^l$$

$$- ab \sum_k A_{ij;k} \varphi^k \wedge \varphi^0,$$

where

$$\sum_k A_{ij;k} \theta^k = dA_{ij} - \sum_k A_{ik} \omega^k{}_j + \sum_k A_{kj} \omega^i{}_k.$$

PROOF. The proof is a straightforward calculation using Proposition 1 and the structure equation

$$\Psi^\alpha_\beta = d\psi^\alpha_\beta + \sum_\lambda \psi^\alpha_\lambda \wedge \psi^\lambda_\beta. \qquad \text{QED.}$$

REMARK. The covariant derivative of the tensor field A_{ij} with respect to the Riemannian connection of M is given precisely by $A_{ij;k}$.

The components $R_{\alpha\beta\lambda\mu}$ of the curvature tensor of the Riemannian manifold P are defined by

$$\Psi^\alpha_\beta = 1/2 \sum_{\lambda,\mu} R_{\alpha\beta\lambda\mu} \varphi^\lambda \wedge \varphi^\mu.$$

PROPOSITION 3. *The curvature* $R_{\alpha\beta\lambda\mu}$ *is expressed by* K_{ijkl} *and* A_{ij} *as follows*:

1) $R_{ijkl} = K_{ijkl} - a^2b^2(2A_{ij}A_{kl} + A_{ik}A_{jl} - A_{il}A_{jk}),$

2) $R_{i0k0} = a^2b^2 \sum_l A_{il}A_{kl},$

3) $R_{i0kl} = ab(A_{ik;l} - A_{il;k}) = - abA_{kl;i}.$

Formulas 1), 2) *and* 3) *determine all components* $R_{\alpha\beta\lambda\mu}$.

PROOF. From Proposition 2, we obtain

$$1/2 \sum_{\lambda,\mu} R_{ij\lambda\mu} \varphi^\lambda \wedge \varphi^\mu = \sum_{k,l} [1/2K_{ijkl} - a^2b^2(A_{ij}A_{kl} + A_{ik}A_{jl})]\varphi^k \wedge \varphi^l$$

$$+ ab \sum_k A_{ij;k} \varphi^0 \wedge \varphi^k.$$

Skew-symmetrizing the coefficients of $\varphi^\lambda \wedge \varphi^\mu$ in the right hand side and equating with $(1/2)R_{ij\lambda\mu}$, we obtain 1) and the first equality of 3). Formula 2) follows similarly from Proposition 2. Finally, the equality $R_{i0kl} = - abA_{kl;i}$ may be also derived from Proposition 2, but the equality $A_{ik;l} - A_{il;k} = - A_{kl;i}$ is equivalent to the fact that the form $\sum_{i,j} A_{ij}\theta^i \wedge \theta^j$ is closed. QED.

4. Algebraic propositions. As in the preceding section, we assume that $1 \leqq i,j,k,l \leqq n$ and $0 \leqq \alpha,\beta,\lambda,\mu \leqq n$. In this section, K_{ijkl} will be a set of real numbers subject to the same algebraic conditions as the Riemannian curvature tensor, i.e.,

$$K_{ijkl} = -K_{jikl} = -K_{ijlk} = K_{klij},$$
$$K_{ijkl} + K_{iklj} + K_{iljk} = 0.$$

From now on we assume that n is even. Let $J = (J_{ij})$ be a skew-symmetric matrix such that $JJ = -I$ or $\sum_j J_{ij}J_{jk} = -\delta_{ik}$. We set

$$S_{ijkl} = K_{ijkl} - a^2(2J_{ij}J_{kl} + J_{ik}J_{jl} - J_{il}J_{jk}),$$
$$S_{i0k0} = -S_{i00k} = -S_{0ik0} = S_{0i0k} = a^2\delta_{ik},$$
$$S_{\alpha\beta\lambda\mu} = 0 \qquad \text{otherwise.}$$

REMARK. If we replace bA_{ij} and $A_{ij;k}$ in Proposition 3 by J_{ij} and 0 respectively, then $R_{\alpha\beta\lambda\mu} = S_{\alpha\beta\lambda\mu}$.

It is easy to see that the set of numbers $S_{\alpha\beta\lambda\mu}$ satisfy the same algebraic conditions as the curvature tensor $R_{\alpha\beta\lambda\mu}$.

Let $\boldsymbol{R}^{n+1}$ be the vector space of $(n+1)$-tuples of real numbers. If $X = (X^0, X^1, \cdots, X^n)$ and $Y = (Y^0, Y^1, \cdots, Y^n)$ are elements in $\boldsymbol{R}^{n+1}$, then their inner product (X,Y) is defined by $(X,Y) = \sum_\alpha X^\alpha Y^\alpha$. For each 2-dimensional subspace p of $\boldsymbol{R}^{n+1}$, we define $S(p)$ as follows. Let X and Y form an orthonormal basis for p. Then

$$S(p) = \sum_{\alpha,\beta,\lambda,\mu} S_{\alpha\beta\lambda\mu} X^\alpha Y^\beta X^\lambda Y^\mu.$$

Then $S(p)$ is independent of X and Y and we have

$$S(p) = \sum_{i,j,k,l} S_{ijkl} X^i Y^j X^k Y^l + \sum_{i,k} S_{i0k0} X^i Y^0 X^k Y^0$$
$$+ \sum_{i,k} S_{i00l} X^i Y^0 X^0 Y^l + \sum_{i,k} S_{0jk0} X^0 Y^j X^k Y^0 + \sum_{j,l} S_{0j0l} X^0 Y^j X^0 Y^l.$$

Let ξ and η be the elements of $\boldsymbol{R}^n$ given by

$$\xi = (X^1, \cdots, X^n), \qquad \eta = (Y^1, \cdots, Y^n).$$

Then

$$J\xi = \left(\sum_j J_{1j}X^j, \cdots, \sum_j J_{nj}X^j\right), \quad J\eta = \left(\sum_j J_{1j}Y^j, \cdots, \sum_j J_{nj}Y^j\right).$$

The inner product in $\boldsymbol{R}^n$ is defined also in the usual way. Then

$$\sum S_{ijkl} X^i Y^j X^k Y^l = \sum K_{ijki} X^i Y^j X^k Y^l - 3a^2(\xi, J\eta)^2,$$

$$\sum S_{i0k0} X^i Y^0 X^k Y^0 = a^2(\xi,\xi) Y^0 Y^0,$$

$$\sum S_{i00l} X^i Y^0 X^0 Y^l = -a^2(\xi,\eta) X^0 Y^0,$$

$$\sum S_{0jk0} X^0 Y^j X^k Y^0 = -a^2(\xi,\eta) X^0 Y^0,$$

$$\sum S_{0j0l}X^0Y^jX^0Y^l = a^2(\eta,\eta)X^0X^0.$$

By adding these five equalities, we obtain

$$S(p) = \sum K_{ijkl}X^iY^jX^kY^l - 3a^2(\xi,J\eta)^2 + a^2[(\xi,\xi)Y^0Y^0 - 2(\xi,\eta)X^0Y^0 + (\eta,\eta)X^0X^0].$$

Since $(\xi,\xi) = 1 - X^0X^0$, $(\xi,\eta) = - X^0Y^0$ and $(\eta,\eta) = 1 - Y^0Y^0$, we have

PROPOSITION 4.

$$S(p) = \sum K_{ijkl}X^iY^jX^kY^l - 3a^2(\xi,J\eta)^2 + a^2(X^0X^0 + Y^0Y^0).$$

Let q be a 2-dimensional subspace of $\boldsymbol{R}^n$ and let $U = (U^1,\dots,U^n)$ and $V = (V^1,\dots,V^n)$ form an orthonormal basis for q. Define $K(q)$ and $\alpha(q)$, $0 \leqq \alpha(q) \leqq \pi/2$, by

$$K(q) = \sum_{i,j,k,l} K_{ijkl}U^iV^jU^kV^l,$$

$$\cos\,\alpha(q) = |(U,JV)|.$$

Then both $K(q)$ and $\alpha(q)$ depend only on q, not on U and V.

Assume that ξ and η are linearly independent and let q be the 2-dimensional subspace of $\boldsymbol{R}^n$ spanned by them. Then the vectors U and V defined as follows form an orthonormal basis for q.

$$U = \xi/(\xi,\xi)^{1/2},$$
$$V = [(\xi,\xi)\eta - (\xi,\eta)\xi]/[(\xi,\xi)((\xi,\xi)(\eta,\eta) - (\xi,\eta)^2)]^{1/2}.$$

Consequently, we have

$$(\xi,J\eta)^2 = [(\xi,\xi)(\eta,\eta) - (\xi,\eta)^2]\cos^2\alpha(q),$$

$$\sum K_{ijkl}X^iY^jX^kY^l = [(\xi,\xi)(\eta,\eta) - (\xi,\eta)^2]K(q).$$

On the other hand, we have

$$(\xi,\xi)(\eta,\eta) - (\xi,\eta)^2 = (1 - X^0X^0)(1 - Y^0Y^0) - X^0X^0Y^0Y^0 = 1 - X^0X^0 - Y^0Y^0.$$

The above three equalities and Proposition 4 imply 1) of the following proposition.

PROPOSITION 5. 1) *If ξ and η are linearly independent so that they span a subspace q, then*

$$S(p) = (1 - X^0X^0 - Y^0Y^0)[K(q) - 3a^2\cos^2\alpha(q)] + a^2(X^0X^0 + Y^0Y^0);$$

2) *If ξ and η are linearly dependent, then $S(p) = a^2$.*

PROOF. If ξ and η are dependent, then the first two terms in the right hand side of the formula in Proposition 4 vanish. On the other hand, $1-X^0X^0 - Y^0Y^0 = (\xi,\xi)(\eta,\eta) - (\xi,\eta)^2 = 0$. Thus, the last term in the formula of Proposition 4 is equal to a^2. QED.

As in §1, we set

$$\overline{K}(q) = (1 + 3\cos^2\alpha(q))/4.$$

PROPOSITION 6. *In Proposition 4, let a be any positive number not greater than 1/2. If ξ and η span a 2-dimensional subspace q of $\boldsymbol{R}^n$ and if*

$$4a^2\overline{K}(q) \leqq K(q) \leqq \overline{K}(q),$$

then

$$a^2 \leqq S(p) \leqq 1 - 3a^2.$$

PROOF. Since $1 - X^0X^0 - Y^0Y^0 = (\xi,\xi)(\eta,\eta) - (\xi,\eta)^2$ (as we have seen before Proposition 5), we have, by Schwarz's inequality,

$$1 - X^0X^0 - Y^0Y^0 \geqq 0.$$

Since $K(q) \geqq 4a^2\overline{K}(q) = a^2(1 + 3\cos^2\alpha(q))$, we have

$$K(q) - 3a^2\cos^2\alpha(q) \geqq a^2 > 0.$$

On the other hand, since $K(q) \leqq \overline{K}(q)$ and $4a^2 \leqq 1$, we have

$$K(q) - 3a^2\cos^2\alpha(q) \leqq [1 + 3(1 - 4a^2)\cos^2\alpha(q)]/4 \leqq [1 + 3(1 - 4a^2)]/4 = 1 - 3a^2.$$

We shall first find an upper bound for $S(p)$.

$$\begin{aligned} S(p) &\leqq (1 - X^0X^0 - Y^0Y^0)(1 - 3a^2) + a^2(X^0X^0 + Y^0Y^0) \\ &= 1 - 3a^2 + (4a^2 - 1)(X^0X^0 + Y^0Y^0) \leqq 1 - 3a^2. \end{aligned}$$

We shall next find a lower bound for $S(p)$.

$$S(p) \geqq (1 - X^0X^0 - Y^0Y^0)a^2 + a^2(X^0X^0 + Y^0Y^0) = a^2. \qquad \text{QED.}$$

Let A_{ij} and $A_{ij;k}$ be real numbers subject to the same algebraic conditions as tensor fields A_{ij} and $A_{ij;k}$ in §3. Explicitly,

$$\begin{aligned} A_{ij} &= - A_{ji}, \\ A_{ij;k} &= - A_{ji;k}, \\ A_{ij;k} &+ A_{ki;j} + A_{jk;i} = 0. \end{aligned}$$

Then, define $R_{\alpha\beta\lambda\mu}$ by the formulas in Proposition 3 so that they satisfy the same algebraic conditions as the curvature tensor. For each 2-dimensional subspace p of $\boldsymbol{R}^{n+1}$ with an orthonormal basis $X = (X^0, X^1, \cdots, X^n)$ and $Y = (Y^0,Y^1,\cdots,Y^n)$, we set

$$R(p) = \sum_{\alpha,\beta,\lambda,\mu,} R_{\alpha\beta\lambda\mu} X^\alpha Y^\beta X^\lambda Y^\mu.$$

PROPOSITION 7. *Let a be any fixed positive number. Given a positive number ε, there is a positive number ρ such that*

$$|R(p) - S(p)| < \varepsilon$$

if $\sum_{i,j} |bA_{ij} - J_{ij}|^2 < \rho$ *and* $\sum_{i,j,k} |bA_{ij;k}|^2 < \rho$.

PROOF. As we remarked earlier, if we set $bA_{ij} = J_{ij}$ and $A_{ij;k} = 0$, then $R(p) = S(p)$. Since $R(p)$ depends continuously on A_{ij} and $A_{ij;k}$, our conclusion follows. QED.

5. Construction of a circle bundle. In this section, we shall complete the proof of Theorem 1. Let M be a complete Kaehler manifold with Kaehlerian pinching $> 4a^2$, where a is a positive number not greater than $1/2$. By normalizing metric, we may assume that the sectional curvature $K(q)$ satisfies the following inequality:

$$4a^2\bar{K}(q) < K(q) \leqq \bar{K}(q).$$

By a theorem of Synge [14] or by a theorem of Myers [13], M is compact. Let $ds^2 = \sum_j (\theta^j)^2$ be the Kaehler metric of M and J_{ij} the components of the complex structure tensor J with respect to $\theta^1,\cdots,\theta^n$. Using notations of §3 and §4, we state

PROPOSITION 8. *Given any positive number ρ, there exist a harmonic 2-form $\sum_{i,j} A_{ij}\, \theta^i \wedge \theta^j$ on M representing an element of $H^2(M;Z)$ and a real number b such that*

$$\sum_{i,j} |J_{ij} - bA_{ij}|^2 < \rho \text{ and } \sum_{i,j,k} |bA_{ij;k}|^2 < \rho,$$

where $A_{ij;k}$ denote the components of the covariant derivative of A_{ij}.

PROOF. From the theory of elliptic partial differential equations (see, for instance, [5], [8]) we infer that there exists a positive constant C such that, for every harmonic form $\sum_{i,j} B_{ij}\theta^i \wedge \theta^j$ on M, we have

$$\sum_{i,j,k} |B_{ij;k}|^2 \leqq C \cdot \text{maximum of} \sum_{i,j} |B_{ij}|^2.$$

Given $\rho > 0$, Let $\rho_1 = \min\{\rho/C, \rho\}$. Since $H^2(M;Z)$ form a basis in $H^2(M;\boldsymbol{R})$, the set of $\{b\alpha; b \in \boldsymbol{R}$ and $\alpha \in H^2(M;Z)\}$ is dense in $H^2(M;\boldsymbol{R})$. Hence, there are a real number b and a harmonic form $\sum_{i,j} A_{ij}\theta^i \wedge \theta^j$ representing an element of $H^2(M; Z)$ such that $\sum_{i,j} |bA_{ij} - J_{ij}|^2 < \rho_1$. Set $B_{ij} = bA_{ij} - J_{ij}$. Then $B_{ij;k} = bA_{ij;k}$. Hence,

$$\sum_{i,j,k} |bA_{ij;k}|^2 = \sum_{i,j,k} |B_{ij;k}|^2 < C\rho_1 \leqq \rho. \qquad \text{QED.}$$

For each $x \in M$ and each plane p in $\boldsymbol{R}^{n+1}$, we define $S(p)$ using the sets of numbers $K_{ijkl}(x)$, $J_{ij}(x)$ and a as in §4. Then, the assumption $4a^2\overline{K}(q) < K(q) \leqq \overline{K}(q)$ for all q implies by 2) of Proposition 5 and by Proposition 6 the following inequalities:

$$a^2 < S(p) < 1 - 3a^2.$$

Note that, since we have a strict inequality $4a^2\overline{K}(q) < K(q)$, we have also the strict inequalities $a^2 < S(p) < 1 - 3a^2$. Since M is compact, there is a positive number ε such that

$$a^2 + \varepsilon < S(p) < 1 - 3a^2 - \varepsilon \qquad \text{for all } x \in M \text{ and all } p.$$

Corresponding to this positive number ε, we take a positive number ρ given by Proposition 7. Then choose a number b and a harmonic 2-form $\sum_{i,j} A_{ij}\theta^i \wedge \theta^j$ as in Proposition 8. Assuming for the moment the existence of a principal circle bundle P over M and a connection form γ on P such that $d\gamma = \pi^*\left(\sum_{i,j} A_{ij}\theta^i \wedge \theta^j\right)$, we shall finish the proof of Theorem 1. By means of $\varphi^0, \varphi^1, \dots, \varphi^n$ we can identify $\boldsymbol{R}^{n+1}$ with each tangent space of P. Thus, we denote by p a plane in $\boldsymbol{R}^{n+1}$ and also the corresponding plane in each tangent space of the manifold P, so that $R(p)$ in Proposition 7 can be now considered as the sectional curvature of the Riemannian manifold P. By our choice of ε and by Proposition 7, we have

$$a^2 < S(p) - \varepsilon < R(p) < S(p) + \varepsilon < 1 - 3a^2.$$

Thus, if M is of Kaehlerian pinching $> 4a^2$, then P is of Riemannian pinching $> a^2/(1 - 3a^2)$. If we replace $4a^2$ by δ, then we have Theorem 1.

Now, the only thing which has to be proved is the following proposition.

PROPOSITION 9. *Given a harmonic 2-form* $\sum_{i,j} A_{ij}\theta^i \wedge \theta^j$ *representing an element of* $H^2(M; Z)$, *there are a principal circle bundle* P *and a connection form* γ *on* P *such that* $d\gamma = \pi^*\left(\sum_{i,j} A_{ij}\theta^i \wedge \theta^j\right)$.

PROOF. The exact sequence $0 \to Z \to \boldsymbol{R} \to S^1 \to 0$ induces an exact sequence of the cohomology groups of M with coefficients in the corresponding sheaves of germs of mappings. In particular, we have

$$H^1(M; \underline{S}^1) \approx H^2(M; Z),$$

where $\underline{S}^1$ is the sheaf of germs of differentiable mappings into S^1. The group $H^1(M; \underline{S}^1)$ can be considered as the set of all principal circle bundles over M. The isomorphism $H^1(M; \underline{S}^1) \approx H^2(M; Z)$ is given explicitly as follows. Let P be an element of $H^1(M; \underline{S}^1)$, i.e., a principal circle bundle over M. Let γ be a connection form on P. Then $d\gamma = \pi^*(\alpha)$, where α is a closd 2-form on M. The cohomology class of α is the element of $H^2(M; Z)$ corresponding to P.

Therefore, given a harmonic 2-form $\sum A_{ij}\theta^i \wedge \theta^j$ representing an element of $H^2(M; Z)$, let P be the corresponding principal circle bundle over M and γ' be any connection form on P, so that the closed 2-form $\sum B_{ij}\theta^i \wedge \theta^j$ defined by $d\gamma' = \pi^*\left(\sum B_{ij}\theta^i \wedge \theta^j\right)$ is cohomologous to the form $\sum A_{ij}\theta^i \wedge \theta^j$. Let β be a 1-form on M such that

$$\sum A_{ij}\theta^i \wedge \theta^j - \sum B_{ij}\theta^i \wedge \theta^j = d\beta.$$

Set $\gamma = \gamma' + \pi^*(\beta)$. It is easy to verify that γ is a connection form on P and that $d\gamma = \pi^*\left(\sum A_{ij}\theta^i \wedge \theta^j\right)$. QED.

6. Miscellaneous results. We shall show that the same method can be applied to Kaehler manifolds with positive holomorphic pinching. The following proposition is due to Berger [3][2]:

PROPOSITION 10. *Let M be a Kaehler manifold such that*

$$\delta \leqq K(q) \leqq 1 \qquad \text{for all } q \text{ with } J(q) = q.$$

Then

$$(-5 + 7\delta + 6\cos^2\alpha(q))/8 \leqq K(q) \leqq (7 - 5\delta + 6\cos^2\alpha(q))/8$$

for all q.

PROOF. For any two linearly independent vectors X and Y, we shall denote by $k(X,Y)$ the sectional curvature by the plane spanned by X and Y so that

$$k(X,Y) = K(X,Y,X,Y)/[(X,X)(Y,Y) - (X,Y)^2],$$

where K on the right hand side denotes the Riemannian curvature tensor.

2) Because of some errors in Berger's paper, we give here a complete proof.

Let q be any plane in the tangent space $T_x(M)$ at a point x of M and let X and Y form an orthonormal basis for q so that

$$K(q) = k(X,Y) = K(X,Y,X,Y).$$

Making use of Bianchi's identity and the following formulas:

$$k(X,Y) = k(JX,JY),$$
$$K(JX,Y,JX,Y) = k(JX,Y)\sin^2\alpha(q),$$

we obtain, for any real numbers a and b,

$$(a^2+b^2)^2 k(aX+bY,J(aX+bY)) = a^4k(X,JX) + b^4k(Y,JY) + 2a^2b^2E + ua^3b + vab^3,$$

where

$$E = k(X,Y) + 3k(JX,Y)\sin^2\alpha(q).$$

Replacing b by $-b$, we obtain a similar equality. By adding the two equalities thus obtained, we have

$$(a^2+b^2)^2[k(aX+bY,J(aX+bY)) + k(aX-bY,J(aX-bY))] = 2a^4k(X,JX) + 2b^4k(Y,JY) + 4a^2b^2E.$$

From our assumption, we obtain the following inequalities:

$$\delta(a^2+b^2)^2 \leqq a^4k(X,JX) + b^4k(Y,JY) + 2a^2b^2E \leqq (a^2+b^2)^2.$$

By setting $a = b = 1$, we have

$$4\delta - k(X,JX) - k(Y,JY) \leqq 2E \leqq 4 - k(X,JX) - k(Y,JY).$$

Hence,

$$2\delta - 1 \leqq E \leqq 2 - \delta.$$

Proceeding in the same way with

$$(a^2 + 2ab\cos\alpha(q) + b^2)^2 k(aX+bJY,J(aX+bJY)) = a^4k(X,JX) + b^4k(Y,JY) + 2a^2b^2F + u'a^3b + v'ab^3$$

where

$$F = 3k(X,Y) + k(JX,Y)\cos^2\alpha(q),$$

we obtain the following inequalities:

$$\delta[(a^2+b^2)^2 + 4a^2b^2\cos^2\alpha(q)] \leqq a^4k(X,JX) + b^4k(Y,JY) + 2a^2b^2F \leqq [(a^2+b^2)^2 + 4a^2b^2\cos^2\alpha(q)].$$

By setting $a = b = 1$, we have

$$2\delta - 1 + 2\delta\cos^2\alpha(q) \leqq F \leqq 2 - \delta + 2\cos^2\alpha(q).$$

Finally, we have

$$(7\delta - 5 + 6\delta\cos^2\alpha(q))/8 \leqq (3F - E)/8 \leqq (7 - 5\delta + 6\cos^2\alpha(q))/8.$$

Since $3F - F = k(X,Y)$, this completes the proof. QED.

PROPOSITION 11. *With the same notations as in Proposition* 5, *if*

$$(7\delta - 5 + 6\delta\cos^2\alpha(q))/8 \leqq K(q) \leqq (7 - 5\delta + 6\cos^2\alpha(q))/8$$

and if

$$7\delta - 5 \leqq 8a^2 \leqq 2\delta.$$

then

$$(7\delta - 5)/8 \leqq S(p) \leqq (13 - 5\delta - 24a^2)/8.$$

PROOF. By Proposition 5, we have

$$\begin{aligned} S(p) &\leqq (1 - X^0X^0 - Y^0Y^0)[7 - 5\delta + 6(1 - 4a^2)\cos^2\alpha(q)]/8 + a^2(X^0X^0 + Y^0Y^0) \\ &\leqq (1 - X^0X^0 - Y^0Y^0)[7 - 5\delta + 6 - 24a^2)/8 + a^2(X^0X^0 + Y^0Y^0) \\ &= (13 - 5\delta - 24a^2)/8 + (32a^2 - 13 + 5\delta)(X^0X^0 + Y^0Y^0)/8 \\ &\leqq (13 - 5\delta - 24a^2)/8. \end{aligned}$$

Also, by Proposition 5, we have

$$\begin{aligned} S(p) &\geqq (1 - X^0X^0 - Y^0Y^0)[7\delta - 5 + 6(\delta - 4a^2)\cos^2\alpha(q)]/8 + a^2(X^0X^0 + Y^0Y^0) \\ &\geqq (1 - X^0X^0 - Y^0Y^0)(7\delta - 5)/8 + a^2(X^0X^0 + Y^0Y^0) \\ &= (7\delta - 5)/8 + (8a^2 - 7\delta + 5)(X^0X^0 + Y^0Y^0)/8 \\ &\geqq (7\delta - 5)/8. \end{aligned}$$

QED.

In particular, if we set

$$\delta = 4a^2,$$

then

$$(7\delta - 5)/8 \leqq S(p) \leqq (13 - 11\delta)/8.$$

Thus, the method we used in the proof of Theorem 1 gives

THEOREM 2. *Let M be a complete Kaehler manifold with holomorphic pinching* $> \delta$. *Then, there are a principal circle bundle P over M and a Riemannian metric on P with Riemannian pinching* $> (7\delta - 5)/(13 - 11\delta)$.

If $\delta = 11/13$, then $(7\delta - 5)/(13 - 11\delta) = 1/4$. Hence,

COROLLARY. *Let M be a complete Kaehler manifold with holomorphic pinching* $> 11/13$. *Then*

$$\pi_i(M) = \pi_i(P_m(C)) \qquad \textit{for all } i. \ (m = \dim_C M).$$

REMARK. According to Berger [3], if a Kaehler manifold M of complex dimension >1 is of Riemannian pinching $>\delta$, then M is of holomorphic pinching $>\delta(8\delta+1)/(1-\delta)$. Hence, if M is of Riemannian pinching $\geqq 0.23$, then M is of holomorphic pinching $>11/13$. Thus, if M is of Riemannian pinching $\geqq 0.23$, then $\pi_i(M)=\pi_i(P_m(C))$ for all i.

In [1], Berger proved the following theorem:

Let P be a $(2m+1)$-dimensional compact Riemannian manifold with Riemannian pinching $>2(m-1)/8m-5$. Then, $H^2(P;\boldsymbol{R})=0$.

His result, combined with Theorem 1, gives

THEOREM 3. *Let M be a complete Kaehler manifold of complex dimension m. If M is of Kaehlerian pinching $>8(m-1)/14m-11$, then*

$$\dim H^2(M;\boldsymbol{R})=1.$$

PROOF. By Theorem 1, we can construct a principal circle bundle P over M with a Riemannian metric with Riemannian pinching $>2(m-1)/8m-5$. By the result of Berger, $H^2(\widetilde{P};\boldsymbol{R})=0$ where $\widetilde{P}$ is the universal covering manifold of P. Hence, $\pi_2(P)=\pi_2(\widetilde{P})=H_2(\widetilde{P};Z)$ is finite. By the exact homotopy sequence of the fibring $S^1\rightarrow P\rightarrow M$,

$$\pi_2(M)\approx Z+\text{a finite group}.$$

By Hurewicz isomorphism, $H_2(M;\boldsymbol{R})=\boldsymbol{R}$. QED.

THEOREM 4. *Let M be a complete Kaehler manifold with holomorphic pinching $>(22m-17)/(26m-19)$, where m is the complex dimension of M. Then*

$$\dim H^2(M;\boldsymbol{R})=1.$$

PROOF. The proof is quite similar to that of Theorem 3. The only change is the use of Theorem 2 in place of Theorem 1. QED.

REMARK. For $m=2$, this result is weaker than that of Berger [3] who shows that if M is a complete Kaehler manifold of complex dimension 2 with holomorphic pinching $>1/2$, then $\dim H^2(M;\boldsymbol{R})=1$.

Let M be a complete Kaehler manifold of complex dimension m with Riemannian pinching $>\delta$, where δ is the positive number defined by $(22m-17)/(26m-19)=\delta(8\delta+1)/(1-\delta)$. *Then*, $\dim H^2(M;\boldsymbol{R})=1$. The proof is by the reasoning given in the remark following Theorem 2. Again, for $m=2$, this result is weaker than those of Berger [3] and Andreotti-Frankel [9]. Berger assumes only $\delta>0$. Andreotti and Frankel proves that if $\delta>0$, then

M is homeomorphic with $P_2(C)$.

The proof of Theorem 1 gives also the following result:

THEOREM 5. *If M is an Einstein-Kaehler manifold with positive scalar curvature, then we can construct a principal circle bundle P over M and an Einstein metric with positive scalar curvature on P.*

PROOF. We recall that an Einstein metric is a Riemannian metric such that the Ricci tensor is a constant multiple of the metric tensor. If M is an Einstein-Kaehler manifold, there exists a constant c such that the 2-form $c\sum J_{ij}\theta^i \wedge \theta^j$ represents the first Chern class which is an element of $H^2(M;Z)$. Let P be the principal circle bundle corresponding to the cohomology class represented by $c\sum J_{ij}\theta^i \wedge \theta^j$ and let γ be a connection form such that $d\gamma = \pi^*\left(c\sum J_{ij}\theta^i \wedge \theta^j\right)$. In Proposition 3, we have then

$$A_{ij} = cJ_{ij}.$$

If we set $b = 1/c$ in Proposition 3, then we have

$$R_{ij} = \sum_{\lambda} R_{i\lambda j\lambda} = K_{ij} - 2a^2\delta_{ij},$$
$$R_{i0} = 0,$$
$$R_{00} = \sum_{\lambda} S_{0\lambda 0\lambda} = na^2.$$

Since $K_{ij} = h\delta_{ij}$ for some positive constant h, set $a = (h/(n+2))^{1/2}$. Then, $R_{\alpha\beta} = n/(n+2)\delta_{\alpha\beta}$. QED.

The construction of the bundle P and the metric $d\sigma^2$ in Theorem 5 is natural in the sense that *P is a space of constant positive curvature if and only if M is a space of constant positive holomorphic (sectional) curvature.* In fact, suppose that M is a space of constant positive holomorphic curvature. By normalizing the metric, we may assume that the holomorphic curvature of M is equal to 1. Then $K(q) = \overline{K}(q)$ for all q (cf. Proposition 10) and $K_{ij} = h\delta_{ij}$, where $h = (n+2)/4$. Hence, $a = 1/2$ in the proof of Theorem 5. By Proposition 6, the metric on P is of sectional curvature $1/4$. Conversely, assume that the sectional curvature of P is constant. Then, by Proposition 6, $R(p) = a^2$ for all p. By the same proposition, we have $K(q) = 4a^2\overline{K}(q)$ for all q, thus proving our assertion.

We shall prove a Kaehlerian analogue of the following result of Berger [4]:

Let P be a compact manifold with a 1-parameter family of Riemannian *metrics* $d\sigma^2(t)$, $-\varepsilon < t < \varepsilon$, *such that*

1) *For each* t, $d\sigma^2(t)$ *is an Einstein metric*;

2) *The metric* $d\sigma^2(0)$ *is of constant positive curvature*:

3) *The family* $d\sigma^2(t)$ *is real analytic in* t.

Then, for each t, $d\sigma^2(t)$ *is of constant positive curvature.*

Our result may be stated as follows:

THEOREM 6. *Let* M *be a compact manifold with a* 1-*parameter family of complex structures* $J(t)$ *and a* 1-*parameter family of Riemannian metrics* $ds^2(t)$, $-\varepsilon < t < \varepsilon$, *such that*

1) *For each* t, $ds^2(t)$ *is an Einstein-Kaehler metric with respect to the complex structure* $J(t)$;

2) *The metric* $ds^2(0)$ *is of constant positive holomorphic curvature with respect to the complex structure*;

3) *The families* $J(t)$ *and* $ds^2(t)$ *are real analytic in* t.

Then, for each t, $ds^2(t)$ *is of constant positive holomorphic curvature with respect to the complex structure* $J(t)$.

PROOF. For each t, construct a principal circle bundle $P(t)$ and a Riemannian metric $d\sigma^2(t)$ on $P(t)$ as in Theorem 5. Because of the observation we made after Theorem 5 and the above result of Berger, it suffices to prove that $P(t)$ is independent of t and that $d\sigma^2(t)$ is real analytic in t. Since $P(t)$ corresponds to the first Chern class of the complex structure $J(t)$ under the isomorphism $H^1(M;\underline{S}^1)\approx H^2(M;Z)$ (cf. the proof of Theorem 5) and since the first Chern class of $J(t)$ depending continuously on t and lying in the discrete subgroup $H^2(M;Z)$ of $H^2(M;\boldsymbol{R})$ must be independent of $t, P(t)$ is independent of t. Let $P = P(t)$. For each t, let $\alpha(t)$ be the harmonic 2-forms on M representing the first Chern class with respect to the Kaehler structure defined by $J(t)$ and $ds^2(t)$. Since $\alpha(t)$ can be expressed in terms of the Ricci tensor of $ds^2(t)$ and the complex structure $J(t)$ (cf.[7]), the family $\alpha(t)$ is real analytic in t. As in the proof of Theorem 5, let γ be a connection form on P such that $d\gamma = \pi^*(\alpha(0))$. Note that such a connection form γ is not unique. Since, for each t, $\alpha(t)$ is cohomologous to $\alpha(0)$, there exists a 1-form $\beta(t)$ such that $\alpha(t) = \alpha(0) + d\beta(t)$. We shall show that it is possible to construct a family $\beta(t)$ real analytic in t. Let C^k be the space of real k-forms on M. Let δ be the adjoint of d and $\Delta = d\delta + \delta d$ the Laplacian defined by the metric $ds^2(0)$. From the theory of harmonic integrals, we infer that $\delta d\delta C^2 = \delta dC^1 = \delta C^2$ and hence that the Laplacian Δ maps δC^2 isomorphically onto itself. (Our assertions follow from the decomposition theorem: $C^k = dC^{k-1} + \delta C^{k+1} + H^k$, where H^k denotes the space of harmonic k-forms). Let

$$\beta(t) = \Delta^{-1}[\delta(\alpha(t) - \alpha(0))],$$

where Δ^{-1} is considered as the inverse of the isomorphism $\Delta : \delta C^2 \to \delta C^2$. The family $\beta(t)$ thus constructed is real analytic in t. Set

$$\gamma(t) = \gamma + \pi^*(\beta(t)).$$

Then, for each t, $\gamma(t)$ is a connection form on P such that $d\gamma(t) = \pi^*(\alpha(t))$. For each t, we define constants $a(t)$ and $b(t)$ as in the proof of Theorem 5 and set

$$d\sigma^2(t) = \pi^*(ds^2(t)) + (a(t)b(t)\gamma(t))^2.$$

Since $a(t)$ and $b(t)$ are obviously real analytic in t, $d\sigma^2(t)$ is also real analytic in t. This completes the proof of Theorem 6. QED.

7. Concluding remarks. In this section we shall explain a few related problems.

1) Every compact Hermitian symmetric space without flat factor is of positive holomorphic pinching. Is every compact Kaehler manifold with positive holomorphic pinching with a Hermitian symmetric space without flat factor ?

2) In particular, is the following statement true ? If M is a compact Kaehler manifold with positive holomorphic pinching, then $H^{p,q}(M; C) = 0$ for $p \neq q$.

3) Let M be a compact Kaehler manifold with holomorphic pinching $> 1/2$. Is M homeomorphic with $P_m(C)$? If K_1, K_2 and K are the Riemannian curvature tensors of Kaehler manifolds M_1, M_2 and $M_1 \times M_2$ respectively and if X_1, X_2 and $X = c_1X_1 + c_2X_2$ are tangent vectors of M_1, M_2 and $M_1 \times M_2$ respectively, then $K(X,JX,X,JX) = c_1{}^4K_1(X_1,JX_1,X_1, JX_1) + c_2{}^4K_2(X_2,JX_2,X_2,JX_2)$. It follows that if $M_1 = M_2 = P_m(C)$ with Fubini-Study metric, then $M_1 \times M_2$ is of holomorphic pinching exactly $1/2$.

4) Are there compact Kaehler manifolds with positive Kaehlerian pinching which are not homeomorphic with $P_m(C)$?

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