ON FIBERINGS OF SOME NON-COMPACT CONTACT MANIFOLDS

SHÛKICHI TANNO

(Received June 18, 1963)

Introduction. Every contact manifold M ($\dim M \neq 1$) with η as a contact form has a contact metric structure (ϕ, ξ, η, g) [6] such that

$$d\eta(X, Y) = w(X, Y) \overset{\text{def}}{\equiv} g(X, \phi Y),$$

for any differentiable vector fields X, Y on M. W.M.Boothby and H.C.Wang [5] proved that, if η is a regular contact form on a compact manifold M, then M is a principal fiber bundle over a symplectic manifold B with structural group and fiber the circle group S^1. Further, η defines a connection in the bundle M in such a way that $d\eta = \pi^*\Omega$ is the equation of structure of the connection, where π is the natural projection $M \to B$ and Ω is the fundamental 2-form on B. In this case, as is verified recently [9], M is a K-contact manifold. That is, we can find a suitable Riemannian metric g associated with η such that ξ is a Killing vector field.

This note is devoted to give fiberings of some non-compact K-contact manifolds. We remark that, it is only in the case where M is non-compact that there may exist an infinitesimal transformation which leaves ϕ invariant, but not η invariant. If such an infinitesimal transformation exists on a complete K-contact manifold, then M is a principal fiber bundle over an almost Kählerian manifold with structural group and fiber the real line R. And η is an infinitesimal connection in M. Further, if M is simply-connected, then M is reduced to the product bundle.

1. Lie algebra of infinitesimal transformations. By L_ϕ we denote the set of all infinitesimal transformations which leave ϕ invariant and by $\mathfrak{A}$ that of all infinitesimal automorphisms of the contact metric structure. Let $Z' \in L_\phi$ and $Z' \bar{\in} \mathfrak{A}$, then for some constant σ, $\mathfrak{L}(Z')\eta = \sigma\eta$. As $\sigma \neq 0$, we define $Z = -\sigma^{-1}Z'$, so that $\mathfrak{L}(Z)\eta = -\eta$ is satisfied. This Z plays an important rôle throughout this note. We put $L = (\alpha Z : \alpha \in R)$, then we have the following

THEOREM 1-1. *In a differentiable manifold with a contact metric structure, the relation*

$$L_\phi = \mathfrak{A} \oplus L \qquad (\textit{direct sum})$$

is valid. And the followings are evident,

$$[\mathfrak{A}, \mathfrak{A}] \subset \mathfrak{A}, \qquad [L, L] = 0,$$
$$[\mathfrak{A}, L] \subset \mathfrak{A}.$$

PROOF. For any $X \in L_\phi$, there corresponds a real number α satisfying $\mathfrak{L}(X)\eta = \alpha\eta$. Then X is decomposed as follows:

$$X = (X + \alpha Z) + (-\alpha Z).$$

Clearly, $\mathfrak{L}(X + \alpha Z)\phi = 0$ and $\mathfrak{L}(X + \alpha Z)\eta = 0$ follow. This being said, it is known that $X + \alpha Z$ leaves g and ξ invariant, and so $X + \alpha Z \in \mathfrak{A}$. As $\mathfrak{A}$ is a subalgebra of the Lie algebra composed of all Killing vector fields on M, $\mathfrak{A}$, and hence L_ϕ, is finite dimensional. If M is compact, of course, $L_\phi = \mathfrak{A}$.

2. An almost contact metric structure associated with Z. We assume that there exists a vector field Z on M which satisfies $\mathfrak{L}(Z)\eta = -\eta$. Using this Z we define $\bar{\eta}$ as follows:

$$\bar{\eta} = \eta + i(Z)w, \tag{2. 1}$$

where $i(Z)$ is the interior product operator by Z. Then, η fulfils the following two relations:

$$\bar{\eta}(\xi) = 1, \tag{2. 2}$$

$$d\bar{\eta} = 0. \tag{2. 3}$$

A linear operator $\bar{\phi}$ of the family $\mathfrak{X}(M)$ of all vector fields on M to itself is defined as follows:

$$\bar{\phi}X = \phi X - \bar{\eta}(\phi X)\xi, \qquad X \in \mathfrak{X}(M). \tag{2. 4}$$

Then the next formulas hold good:

$$\bar{\eta}\bar{\phi} = 0, \qquad \bar{\phi}\xi = 0, \tag{2. 5}$$

$$\bar{\phi}\bar{\phi}X = -X + \bar{\eta}(X)\xi, \qquad X \in \mathfrak{X}(M). \tag{2. 6}$$

And the definition of the new metric is:

$$\begin{aligned}\bar{g}(X, Y) = g(X, Y) &+ w(Z, X)\cdot\eta(Y) + w(Z, Y)\cdot\eta(X) \\ &+ w(Z, X)\cdot w(Z, Y), \qquad X, Y \in \mathfrak{X}(M).\end{aligned} \tag{2. 7}$$

If we notice that $w(Z, X) = \bar{\eta}(X) - \eta(X)$, (2. 7) may be written in the simplified form as

$$\bar{g}(X, Y) = g(X, Y) + \bar{\eta}(X)\cdot\bar{\eta}(Y) - \eta(X)\cdot\eta(Y). \tag{2. 8}$$

LEMMA 2-1. *The tetrad* $(\bar{\phi}, \xi, \bar{\eta}, \bar{g})$ *defines an almost contact metric structure on* M.

PROOF. By virtue of (2. 2) and (2. 8), we see that

$$\bar{\eta}(X) = \bar{g}(\xi, X), \qquad X \in \mathfrak{X}(M). \tag{2. 9}$$

Further, the next relation is valid,

$$\bar{g}(\bar{\phi}X, \bar{\phi}Y) = \bar{g}(X, Y) - \bar{\eta}(X)\cdot\bar{\eta}(Y), \qquad X, Y \in \mathfrak{X}(M). \tag{2.10}$$

In fact, from the definition of $\bar{\phi}$ it follows that $\eta\bar{\phi} = -\bar{\eta}\phi$, and so we have

$$\eta(\bar{\phi}X)\cdot\eta(\bar{\phi}Y) = \bar{\eta}(\phi X)\cdot\bar{\eta}(\phi Y).$$

And from (2. 5), the relation $\bar{\eta}(\bar{\phi}X)\cdot\bar{\eta}(\bar{\phi}Y) = 0$ follows. Again from (2. 4) we can deduce the following

$$g(\bar{\phi}X, \bar{\phi}Y) = g(\phi X, \phi Y) + \bar{\eta}(\phi X)\cdot\bar{\eta}(\phi Y).$$

Putting $\bar{\phi}X$ and $\bar{\phi}Y$ instead of X and Y into (2. 8) and utilizing preceding three equalities, we obtain

$$\bar{g}(\bar{\phi}X, \bar{\phi}Y) = g(\phi X, \phi Y). \tag{2.11}$$

As $g(\phi X, \phi Y) = g(X, Y) - \eta(X)\cdot\eta(Y)$, (2. 8) and (2.11) give the verification to (2.10). To see that $\bar{g}$ is positive definite, we transform (2. 8) identifying X and Y,

$$\bar{g}(X, X) = g(\phi X, \phi X) + \bar{\eta}(X)^2.$$

Hence, g is non-negative. Moreover, if $\bar{g}(X, X) = 0$, then $\bar{\eta}(X) = 0$, and $\phi X = 0$. The latter $\phi X = 0$ implies $\bar{\phi}X = 0$, and so $X = 0$. Though it is redundant, ξ and the vector of the form $\bar{\phi}X, X \in \mathfrak{X}(M)$, are orthogonal with respect to $\bar{g}$. (2. 2), (2. 5), (2. 6), (2. 9) and (2.10) are the required conditions for $\bar{\phi}, \xi, \bar{\eta}$ and $\bar{g}$ to define an almost contact metric structure on M.

LEMMA 2-2. *We have the following relation*:

$$\bar{g}(X, \bar{\phi}Y) = w(X, Y), \qquad X, Y \in \mathfrak{X}(M). \tag{2.12}$$

PROOF. By virtue of (2. 4), (2. 5) and (2. 8), we have

$$\begin{aligned}\bar{g}(X, \bar{\phi}Y) &= g(X, \bar{\phi}Y) - \eta(X)\cdot\eta(\bar{\phi}Y)\\ &= g(X, \phi Y) - \bar{\eta}(\phi Y)\cdot g(X, \xi) - \eta(X)\cdot\eta(\bar{\phi}Y)\\ &= w(X, Y).\end{aligned}$$

3. Maximal integral manifolds of the distribution $\bar{\eta} = 0$. The distribution on M defined by $\bar{\eta} = 0$ is completely integrable on account of closedness of the 1-form $\bar{\eta}$. Therefore, for any point p of M, there is a maximal integral manifold $W = W(p)$ through p. Let $\iota : W \to M$ be the injection map. Then, since $\bar{\phi}$ maps $\iota(\mathfrak{X}(W))$ onto itself in some sense, we can define $\bar{\phi}'$ as follows:

$$\bar{\phi}'X' = \iota^{-1}\bar{\phi}(\iota X'), \qquad X' \in \mathfrak{X}(W). \tag{3. 1}$$

Furthermore, we can define $\bar{g}' = \iota^*\bar{g}$.

PROPOSITION 3-1. *If a contact manifold M admits a vector field Z satisfying $\mathfrak{L}(Z)\eta = -\eta$. Then every (maximal) integral manifold W of the distribution $\eta + i(Z)d\eta = 0$ has an almost Kählerian structure.*

PROOF. For any vector field $X' \in \mathfrak{X}(W)$, $\bar{\eta}(\iota X') = 0$ holds good. Thus, $\bar{\phi}'\bar{\phi}'X' = -X'$ follows from (3. 1). And we get

$$\bar{g}'(\bar{\phi}'X', \bar{\phi}'Y') = \bar{g}(\iota\bar{\phi}', \iota\bar{\phi}'Y') = \bar{g}(\bar{\phi}\iota X', \bar{\phi}\iota Y')$$
$$= \bar{g}(\iota X', \iota Y') = \bar{g}'(X', Y').$$

Thereby, the pair $(\bar{\phi}', \bar{g}')$ defines an almost Hermitian structure on W. Denoting by Ω the fundamental 2-form and using Lemma 2-2, we have

$$\Omega(X', Y') = \bar{g}'(X', \bar{\phi}'Y') = \bar{g}(\iota X', \iota\bar{\phi}'Y')$$
$$= \bar{g}(\iota X', \bar{\phi}\iota Y') = w(\iota X', \iota Y')$$
$$= \iota^* d\eta(X', Y').$$

Hence, we see that $\Omega = d\iota^*\eta$. Q. E. D.

The Nijenhuis tensor N for the almost contact structure is given by

$$N(X, Y) = [X, Y] + \phi[\phi X, Y] + \phi[X, \phi Y] - [\phi X, \phi Y] + (Y\cdot\eta(X) - X\cdot\eta(Y))\xi, \tag{3. 2}$$

for $X, Y \in \mathfrak{X}(M)$. And the Nijenhuis tensor N' for the almost complex structure $\bar{\phi}'$ on W is expressed as follows:

$$N'(X', Y') = [X', Y'] + \bar{\phi}'[\bar{\phi}'X', Y'] + \bar{\phi}'[X', \bar{\phi}'Y'] - [\bar{\phi}'X', \bar{\phi}'Y'], \tag{3. 3}$$

for $X', Y' \in \mathfrak{X}(W)$. If we put $N^1 = \mathfrak{L}(\xi)\phi$, then several relations hold good between N^1 and N if η is a contact form [7], and from those we extract the followings for the later use,

$$\phi\cdot N(X, Y) + N(X, \phi Y) - \eta(Y)\cdot N^1(X) = 0, \tag{3. 4}$$

$$N(X, \xi) + \phi\cdot N^1(X) = 0, \tag{3. 5}$$

$$\phi\cdot N^1(X) + N^1(\phi X) = 0, \tag{3. 6}$$

$$N^1(\xi) = 0, \quad \eta\cdot N^1(X) = 0, \tag{3. 7}$$

$$\eta\cdot N(X, Y) = 0. \tag{3. 8}$$

LEMMA 3-2. *For any vector fields X, Y on a contact manifold M, the next identities are valid,*

$$\bar{\phi}[\bar{\phi}X, Y] = \phi[\phi X, Y] - \bar{\eta}(\phi[\bar{\phi}X, Y])\xi - \bar{\eta}(\phi X)[\xi, \phi Y] + \bar{\eta}(\phi X)N^1(Y), \tag{3. 9}$$

$$[\bar{\phi}X, \bar{\phi}Y] = [\phi X, \phi Y] - \bar{\eta}(\phi Y)[\phi X, \xi] - \bar{\eta}(\phi X)[\xi, \phi Y] \tag{3.10}$$

$$- \{\bar{\phi}(X)\cdot\bar{\eta}(\phi Y) - (\bar{\phi}Y)\cdot\bar{\eta}(\phi X)\}\xi.$$

PROOF. (3. 9) follows from direct calculation in which we use the relation

$$\phi[\xi, Y] = [\xi, \phi Y] - N^1(Y).$$

(3.10) is obtained similarly and so we shall omit the details.

LEMMA 3-3. *For any point p of M and $X', Y' \in \mathfrak{X}(W(p))$, we put $X = \iota X'$, $Y = \iota Y'$, then*

$$\iota N_p'(X', Y') = N_p(X, Y) + \bar{\eta}_p(\phi X)\cdot N_p^1(Y) - \bar{\eta}_p(\phi Y)\cdot N_p^1(X) - K_p(X, Y)\xi_p, \tag{3.11}$$

where we have put

$$K_p(X, Y) = \bar{\eta}_p(\phi[X, \bar{\phi}Y] + \phi[\bar{\phi}X, Y]) - (\bar{\phi}Y)_p\eta(\bar{\phi}X) + (\bar{\phi}X)_p\eta(\bar{\phi}Y) + Y_p\eta(X) - X_p\eta(Y).$$

PROOF. As the value, for instance, of $N(X, Y)$ at p depends upon only the magnitudes of tangent vectors X_p, Y_p at p and is independent of their extensions, we can assume that X (or Y) is defined in each sufficiently small neighborhood U in M by $X_{\exp t\xi\cdot q} = \exp(t\xi)X_q$, $q \in$ {slice in U passing through p}, $|t| < \varepsilon$, ε being a positive number depending on the point p. Then, we see that the term $\iota N_p'(X', Y')$ is equal to

$$[X, Y]_p + \bar{\phi}_p[\bar{\phi}X, Y]_p + \bar{\phi}_p[X, \bar{\phi}Y]_p - [\bar{\phi}X, \bar{\phi}Y]_p,$$

since, for example, $\iota\bar{\phi}'[\bar{\phi}'X', Y']$ is easily seen to be equal to $\bar{\phi}[\bar{\phi}\iota X', \iota Y']$. Consequently, by Lemma 3-2, we have (3.11).

THEOREM 3-4. *In a normal contact manifold M admitting Z as above, every (maximal) integral manifold W of the distribution $\bar{\eta} = 0$ is Kählerian. Conversely, in a K-contact manifold M, if for any point p of M, $W(p)$ is Kählerian, then M is normal.*

PROOF. We have $N = 0$ in a normal contact manifold M by definition. And, as is known [7], $N^1 = 0$ follows from $N = 0$. So, by Lemma 3-3, we get

$$\iota N_p'(X', Y') = - K_p(\iota X', \iota Y')\xi_p,$$

for any p of W and X', $Y' \in \mathfrak{X}(W)$. As $\bar{\eta}\cdot\iota N'(X', Y') = 0$, $K_p(\iota X', \iota Y')$ must vanish everywhere in W. Thus $N'(X', Y') = 0$ follows.

In the next place, suppose that M is a K-contact manifold, that is $N^1 = 0$. This is equivalent to the fact that ξ is a Killing vector field with respect to the metric g. Then (3.11) of Lemma 3-3 implies

$$\iota N_p'(X', Y') = N_p(\iota X', \iota Y') - K_p(\iota X', \iota Y')\xi_p.$$

Therefore, for any point p of M if $W(p)$ is Kählerian, we obtain

$$N_p(\iota X', \iota Y') = K_p(\iota X', \iota Y')\xi_p.$$

By (3. 8) we get $K_p(\iota X', \iota Y') = 0$. And hence $N_p(\iota X', \iota Y') = 0$. Further, if we put (3. 5) into consideration, our assertion may be seen to be true.

LEMMA 3-5. *Let M be a K-contact manifold. Then ξ is a Killing vector field also with respect to the metric $\bar{g}$, and consequently ξ and $\bar{\eta}$ are parallel fields in the Riemannian geometry of $\bar{g}$.*

PROOF. By assumption, we know that $\mathfrak{L}(\xi)g = 0$ and $\mathfrak{L}(\xi)\eta = 0$. And as $\mathfrak{L}(\xi)\bar{\eta} = 0$ is readily seen, $\mathfrak{L}(\xi)\bar{g} = 0$ follows from (2. 8). Since ξ and $\bar{\eta}$ are related by (2. 9), and $\bar{\eta}$ is a closed form, they are both parallel fields with respect to the Riemannian connection defined by $\bar{g}$.

THEOREM 3-6. *In a simply-connected K-contact manifold M with the property that ξ generates a 1-parameter global group of isometries (in particular, with completeness), we suppose that there exists a vector field Z satisfying $\mathfrak{L}(Z)\eta = -\eta$. Then M is a product bundle of an almost Kählerian manifold W and the additive group of real numbers R. And the given contact form η is a connection form in the bundle M. Moreover, M is normal if and only if W is Kählerian.*

PROOF. We take an arbitrary point p of M and by W we denote the maximal integral manifold through p. Let $\exp(t\xi)$, $t \in R$, be a 1-parameter group of isometries with respect to g. Of course, for any $t \in R$, $\exp(t\xi)$ is also isometric with respect to $\bar{g}$ by Lemma 3-5. By $g(\xi, \xi) = \bar{g}(\xi, \xi) = 1$, the canonical parameter is also the arc-length with respect to both g and $\bar{g}$. We define the map $\mu : R \times W \to M$ by

$$\mu(t, q) = \exp(t\xi)q, \qquad t \in R,\ q \in W.$$

Then μ is well-defined into map. Now, let x be an arbitrary point of M, then x has a neighborhood $U(x)$ with local coordinates (y^k), $k = 1, 2, \cdots, 2n+1$ ($\dim M = 2n+1$), such that each slice (y^{2n+1} = constant) is an integral manifold of the distribution $\bar{\eta} = 0$ [1]. For the slice passing through x, say $\lambda(x)$, if $\exp(t\xi)\cdot\lambda(x)$ meets $U(x)$, the intersection is contained in some slice, because $\bar{\eta}$ defining the distribution is invariant by $\exp(t\xi)$ for any $t \in R$. By this reason, we can assume that $U(x) = \{\exp(t\xi)\cdot\lambda(x),\ -\varepsilon(x) < t < \varepsilon(x)\}$, where $\varepsilon(x)$ is determined by the condition that, for any $d \in \lambda(x)$, $\{\exp(t\xi)d,\ |t| < \varepsilon(x)\} \cap \lambda(x) = d$. Under these preparations, we can show that μ is onto. Namely, suppose u be any point of M, and join u and p by a curve l. Then, by the standard argument, the curve is covered by finite neighborhoods $U(x_\alpha)$, $\alpha = 0, 1, \cdots, f$, $x_\alpha \in l$ ($x_0 = u$, $x_f = p$) in such a way that $U(x_\beta) \cap U(x_{\beta+1})$ contains a point $u_{\beta+1} \in l$ for each $\beta = 0, 1, 2, \cdots, f-1$. Then, for the slice $\lambda(u)$ passing through u, there exists a number t_1, such that $\exp(t_1\xi)u_1 \in \lambda(u)$ and the slice passing through u_1 is identical with $\exp(-t_1\xi)\cdot\lambda(u)$ in $U(u)$. By two similar processes for (x_1, u_1) and (x_1, u_2), we get t_2 having a property that the slice passing through u_1 in $U(x_1)$ is the image of the slice passing through u_2 in $U(x_1)$ by $\exp(t_2\xi)$. Hence, $\exp(-t_1-t_2)\xi\cdot\lambda(u)$ is contained in the maximal integral manifold through u_2. After finite steps, we have numbers $t_1, \cdots, t_{f+1}$, for which $\bar{u} \equiv \exp\left(-\sum_{\gamma=1}^{f+1} t_\gamma\right)\xi\cdot u \in W$ holds. Therefore, μ maps $R \times W$ onto M. However, it is not difficult to see that there corresponds to l uniquely the curve $\bar{l}$ in W joining p and $\bar{u}$. So we know that $\bar{u}$ is irrespective of the choice of the curve which joins p and u. In fact, let l_1 be another curve connecting p and u, then we have a homotopy l_r, $0 \leqq r \leqq 1$, $l_0 = l$, since M is simply-connected. $\bar{u}_r$ corresponding to l_r is a curve in W and is also contained in the set $R(u) = \{\exp(t\xi)u,\ t \in R\}$, and $R(u)$ meets W at most countably many times. Consequently, $\bar{u}_r = \bar{u}$. Hence, μ is one-one and $R \times W$ and M are homeomorphic. The action R_s of $s \in R$ is defined on each fiber $R(\bar{u})$, $\bar{u} \in W$, by $R_s u = \exp(t+s)\xi\cdot\bar{u}$, for $u = \exp(t\xi)\bar{u}$, that is to say, $R_s = \exp(s\xi)$. By this, M is a principal fiber bundle over an almost Kählerian manifold W with fiber and structural group R. And it may be proved that η is a connection form in M [5]. If we denote by π the natural projection $M \to W$, then $d\eta = \pi^*\Omega$ (Proposition 3-1). Now, if W is Kählerian, the (maximal) integral manifold $W(x)$ through any point x of M is Kählerian. By Theorem 3-4, M is normal when and only when W is Kählerian.

REMARK 1. By virtue of Lemma 3-5, we see that the integral submanifold W is totally geodesic with respect to the metric $\bar{g}$.

REMARK 2. If we assume the completeness of the metric $\bar{g}$, then our decomposition follows easily from the de Rham's Theorem.

As an example, we know that the contact metric structure given to Euclidean space E^{2n+1} $(n \geqq 1)$ admits the vector $Z : \mathfrak{L}(Z)\eta = -\eta$.

4. The case where M is not simply-connected. To establish the fibering, we have to show first of all that the contact form η is regular. For this purpose, we give some lemmas.

LEMMA 4-1. *For any x of M, Z is related to ξ by*

$$(\exp(t\xi)Z)_x = Z_x + t\xi_x. \tag{4. 1}$$

PROOF. Define $\Theta(t) = \exp(t\xi)Z - Z - t\xi$. Then $\Theta(0) = 0$. And easily we see that

$$\left(\frac{d\Theta(t)}{dt}\right)_s = \exp(s\xi)\cdot\lim_{t\to 0}\frac{\exp(t\xi)Z - Z}{t} - \xi$$
$$= \exp(s\xi)\cdot\mathfrak{L}(Z)\xi - \xi = 0,$$

since, $\mathfrak{L}(Z)\xi = \xi$. Thus $\Theta(t)$ is identically equal to 0.

LEMMA 4-2. *The contact form η is regular. Namely, the distribution defined by ξ is regular.*

PROOF. Assume that there is a point p such that any coordinate system at p is not regular with respect to the distribution ξ. And let $W(p, r)$ be the open connected submanifold of the integral manifold $W(p)$ through p composed of the points whose distances from p are less than (sufficiently small) r with respect to g. Then we take $U = \{\exp(t\xi)q : q \in W(p, r),\ |t| < b\}$ as a coordinate neighborhood of p, where b is a sufficiently small (§3) fixed positive number. By the hypothesis, there exist two points x and y in $W(p, r)$ such that x and y are in the same leaf $R(x)$. That is, we get a number s for which $y = \exp(s\xi)x$ holds good. As $\exp(s\xi)^* g_y = g_x$, with the help of Lemma 4-1, we have

$$g_x(Z_x, \xi_x) = g_y(\exp(s\xi)Z_x, \xi_y)$$
$$= g_y(Z_y, \xi_y) + g_y(s\xi_y, \xi_y).$$

And hence

$$|\eta_x(Z) - \eta_y(Z)| = |s| > b. \tag{4. 2}$$

On the other hand, r may be taken so that $|\eta_x(Z) - \eta_y(Z)|$ is smaller than b, in contradiction to the inequality (4. 2). Q. E. D.

Similarly we have

LEMMA 4-3. *Each leaf $R(p)$, $p \in M$, cannot be a circle, but homeomorphic to the real line R.*

THEOREM 4-4. *If in a K-contact manifold M, ξ generates a 1-parameter group of isometries (in particular, M is complete). And if there exists a vector field Z such that $\mathfrak{L}(Z)\eta = -\eta$, in particular if $L_\phi \neq \mathfrak{A}$, then M is a principal fiber bundle over an almost Kählerian manifold M/ξ with R as the structural group and fiber. Denoting by Ω the fundamental 2-form on M/ξ and by π the projection, the relation $d\eta = \pi^*\Omega$ holds good. Of course, η defines a connection in M. Further M is normal if and only if M/ξ is Kählerian.*

PROOF. Since the contact form is regular, $B = M/\xi$ has a differentiable structure [4]. And it can be shown that B is a Hausdorff space. The proof of the fact that M is a principal bundle over B with R as the structural group and fiber and that η is a connection form in M is almost similar to [5]. And so we shall omit it. Let $\widetilde{X}$ and $\widetilde{Y}$ be two vector fields on B. If the vector fields $\widetilde{X}^*$ and $\widetilde{Y}^*$ on M satisfy $\eta(\widetilde{X}^*) = \eta(\widetilde{X}^*) = 0$ and $\pi_p \widetilde{X}^* = \widetilde{X}_{\pi p}$, $\pi_p \widetilde{Y}^* = \widetilde{Y}_{\pi p}$, for any πp, $p \in M$, we call $\widetilde{X}^*, \widetilde{Y}^*$ the lifts of $\widetilde{X}, \widetilde{Y}$ with respect to η. The Riemannian metric $\widetilde{g}$ and almost complex structure $\widetilde{\phi}$ in B are defined by

$$\widetilde{g}_{\widetilde{p}}(\widetilde{X}, \widetilde{Y}) = g_p(\widetilde{X}^*, \widetilde{Y}^*), \tag{4. 3}$$

$$\widetilde{\phi}_{\widetilde{p}} \widetilde{X}_{\widetilde{p}} = \pi_p \phi_p \widetilde{X}^*_p, \tag{4. 4}$$

where $\widetilde{p} \in B$ and p is an arbitrary point such that $\pi p = \widetilde{p}$. We see that the right hand sides of (4. 3) and (4. 4) are independent of the choice of $p \in \widetilde{p}$, since η, g and ϕ are invariant under the transformation $R_t = \exp(t\xi)$. Then the fundamental 2-form Ω on B satisfies

$$\begin{aligned} \pi^*\Omega(\widetilde{X}^*, \widetilde{Y}^*) &= \widetilde{g}(\widetilde{X}, \widetilde{\phi}\widetilde{Y}) = g(\widetilde{X}^*, (\widetilde{\phi}\widetilde{Y})^*) \\ &= g(\widetilde{X}^*, \phi\widetilde{Y}^*) = d\eta(\widetilde{X}^*, \widetilde{Y}^*). \end{aligned}$$

Hence we can deduce $d\eta = \pi^*\Omega$, because any tangent vector V to M is expressed as a sum of $\eta(V)\xi$ and some lift. We refer to [9] for the verification that M is normal when and only when B is Kählerian. We remark that the base space B may be understood also by another approach if M is complete with respect to $\widetilde{g}$ in §3 [3].

References

[1] C. CHEVALLEY, Theory of Lie Groups, Princeton 1946.

[2] G. DE RHAM, Sur la réductibilité d'un espace de Riemann, Comm. Math. Helv., 26(1952), 328–344.

[3] A. G. WALKER, The fibring of Riemannian manifolds, Proc. London Math. Soc., 3(1953), 1–19.

[4] R. S. PALAIS, A global formulation of the Lie theory of transformation groups, Memoirs of Amer. Math. Soc. 22. 1957.

[5] W. M. BOOTHBY AND H. C. WANG, On contact manifolds, Ann. of Math. 68(1958), 721–734.

[6] S. SASAKI, On differentiable manifolds with certain structures which are closely related to almost contact structure I, Tôhoku Math. Journ., 12(1960), 459–476.

[7] S. SASAKI AND Y. HATAKEYAMA, On differentiable manifolds with certain structures which are closely related to almost contact structure II, Tôhoku Math. Journ., 13(1961), 281–294.

[8] S. TANNO, Note on infinitesimal transformations over contact manifolds, Tôhoku Math. Journ., 14(1962), 416–430.

[9] Y. HATAKEYAMA, Some notes on differentiable manifolds with almost contact structures Tôhoku Math. Journ., 15(1963), 176–181.

TÔHOKU UNIVERSITY.