A NOTE ON SATURATION AND BEST APPROXIMATION

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Let $f(x)$ be an integrable function with period 2π and let its Fourier series be

$$\frac{a_0}{2} + \sum_{\nu=1}^{\infty} (a_\nu \cos \nu x + b_\nu \sin \nu x) \equiv \sum_{\nu=0}^{\infty} A_\nu(x).$$

We write $W^\lambda = W^{(p)\lambda}$ $(1 \leqq p \leqq \infty)$ for the set of $f(x)$ for which

$$f^{[\lambda]}(x) \sim \sum_{\nu=1}^{\infty} \nu^\lambda A_\nu(x)$$

represents respectively

the Fourier series of a function of $L^\infty(0, 2\pi)$ $(p = \infty)$
the Fourier series of a function of $L^p(0, 2\pi)$ $(1 < p < \infty)$
the Fourier-Stieltjes series of a bounded measure on $(0, 2\pi)$ $(p = 1)$.

We prove the following

THEOREM. *Let $\lambda > 0$ and let $T = (T_n)$ be a linear approximation process with*

(1) $\|T_n(f)(x)\| \leqq M_1 \|f\|$

(2) $\|f(x) - T_n(f)(x)\| \leqq M_2 n^{-\lambda} \|f^{[\lambda]}\|$ for $f \in W^\lambda$.

Then, $E_n(f) = E_n^{(p)}(f)$ being the best approximation,

$E_n(f) = O(n^{-\alpha}\psi(n))$ *implies* $\|f(x) - T_n(f)(x)\| = O(n^{-\alpha}\psi(n))$,

where $0 < \alpha < \lambda$ and $\psi(x)$ is a positive continuous function and that $\psi(x)/x$ is non-increasing for large x and

$$\int_1^n \frac{\psi(x)}{x} dx = O(\psi(n)).$$

If $\psi(x) = x^\beta$ $(0 < \beta \leqq 1)$ the theorem is equivalent to that of Sunouchi [1], who used the moving average: our proof is based on a slightly generalized form of Bernstein's inequality.

LEMMA 1. *Let $P_n(x)$ be a trigonometric polynomial of degree n. Then*

$$\|P_n^{[\alpha]}(x)\| \leqq A_\alpha n^\alpha \|P_n(x)\|,$$

where $\alpha > 0$ and A_α is a constant depending on α only.

This is essentially known: we give a proof only for the sake of completeness. Cf. [4, chap. 3. Lemma (13. 16)].

PROOF. *Let $D_n(t)$ be the Dirichlet kernel of order n. Then*

$$\begin{aligned} P_n^{[\alpha]}(x) &= \frac{1}{\pi}\int_0^{2\pi} P_n(x+t)D_n^{[\alpha]}(t)\,dt \\ &= \frac{1}{\pi}\int_0^{2\pi} P_n(x+t)\left\{\sum_{\nu=1}^{n}\nu^\alpha\cos\nu t + \sum_{\nu=1}^{n-1}\nu^\alpha\cos(2n-\nu)t\right\}dt \\ &= \frac{1}{\pi}\int_0^{2\pi} P_n(x+t)\left\{\sum_{\nu=1}^{n-1}2\nu^\alpha\cos nt\,\cos(n-\nu)t + n^\alpha\cos nt\right\}dt \\ &= \frac{2n^\alpha}{\pi}\int_0^{2\pi} P_n(x+t)\cos nt\sum_{\nu=0}^{n-1}\left(1-\frac{\nu}{n}\right)^\alpha\cos\nu t\,dt \\ &\qquad -\frac{n^\alpha}{\pi}\int_0^{2\pi} P_n(x+t)\cos nt\,dt. \end{aligned}$$

Thus (applying the generalized Minkowski inequality if necessary)

$$\begin{aligned} \|P_n^{[\alpha]}(x)\| \leqq \frac{2n^\alpha}{\pi}\int_0^{2\pi} \|P_n(x+t)\|\left|\sum_{\nu=0}^{n}\left(1-\frac{\nu}{n}\right)^\alpha\cos\nu t\right|dt \\ +\frac{n^\alpha}{\pi}\int_0^{2\pi}\|P_n(x+t)\|dt. \end{aligned}$$

The result now follows upon observing that the norm of a function is translation invariant and $\int_0^{2\pi}\left|\sum_{\nu=0}^{n-1}\left(1-\frac{\nu}{n}\right)^\alpha\cos\nu t\right|dt \leqq M_\alpha$ $(\alpha > 0)$.

LEMMA 2. *Let α be a positive number and $P_n(x)$ $(n = 1, 2, \cdots)$ be a sequence of trigonometric polynomials of degree n such that*

$$\|f(x) - P_n(x)\| \leqq \varphi(n)/n^{\alpha-1} \qquad (n = 1, 2, \cdots),$$

where $\varphi(x)$ is a positive continuous function, non-increasing for large x. Then

$$\|P_n^{[\alpha]}(x)\| \leqq A + Bn\varphi(n) + C\int_1^n \varphi(x)\,dx,$$

A, B, and C being independent of n.

PROOF. (cf. [3, pp. 26—27]) Fix Such a natural number a that $\varphi(x)$ is nonincreasing for $x \geqq 2[a-1]$. (We write $2[j]$ instead of 2^j, for the sake of typographic convenience). We have, for $j \geqq a$,

$$\begin{aligned}\|P_{2[j]}-P_{2[j+1]}\| &\leqq \|P_{2[j]}-f\|+\|P_{2[j+1]}-f\| \\ &\leqq \varphi(2[j])2[-j(\alpha-1)]+\varphi(2[-(j+1)])2[-(j+1)(\alpha-1)] \\ &\leqq (1+2[1-\alpha]\varphi(2[j])2[-j(a-1)].\end{aligned}$$

Since $P_{2[j]}-P_{2[j+1]}$ is a trigonometric polynomial of degree $2[j+1]$, Lemma 1 gives

$$\begin{aligned}\|P_{2[j]}^{[\alpha]}-P_{2[j+1]}^{[\alpha]}\| &\leqq A_\alpha 2[(j+1)\alpha]\, 2[-j(\alpha-1)\varphi(2[j]) \\ &\leqq A_\alpha 2[j]\varphi(2[j]).\end{aligned}$$

Summing over $a \leqq j \leqq m-1$,

$$\begin{aligned}\|P_{2[m]}^{[\alpha]}-P_{2[a]}^{[\alpha]}\| &\leqq A_\alpha \sum_{j=a}^{m-1} 2[j]\varphi(2[j]) \leqq A_\alpha \sum_{j=a}^{m-1}(2[j]-2[j-1])\varphi(2[j]) \\ &\leqq A_\alpha \sum_{j=a}^{m-1}\int_{2[j-1]}^{2[j]}\varphi(x)dx = A_\alpha\int_{2[a-1]}^{2[m-1]}\varphi(x)\,dx.\end{aligned}$$

Given $n \geqq 2[a-1]$, let m be so chosen that $2[m] \leqq n < 2[m+1]$. Then

$$\|P_n-P_{2[m]}\| \leqq \varphi(n)\cdot n^{-\alpha+1}+\varphi(2[m])2[-m(\alpha-1)]$$

implies (by Lemma 1)

$$\begin{aligned}\|P_n^{[\alpha]}-P_{2[m]}^{[\alpha]}\| &\leqq A_\alpha n^\alpha\{\varphi(n)n^{-\alpha+1}+\varphi(2[m])2[-m(\alpha-1)]\} \\ &\leqq A_\alpha\{n\varphi(n)+2[m]\varphi(2[m])\} \\ &\leqq A_\alpha n\varphi(n)+\int_{2[m-1]}^{2[m]}\varphi(x)\,dx.\end{aligned}$$

Collecting these estimates, we obtain

$$\begin{aligned}\|P_n^{[\alpha]}\| &\leqq \|P_n^{[\alpha]}-P_{2[m]}^{[\alpha]}\|+\|P_{2[m]}^{[\alpha]}-P_{2[a]}^{[\alpha]}\|+\|P_{2[a]}^{[\alpha]}\| \\ &\leqq A_{a,\alpha}+B_\alpha n\varphi(n)+C_\alpha\int_1^{2[m]}\varphi(x)\,dx,\end{aligned}$$

q. e. d.

PROOF OF THE THEOREM. Let $P_n(x)$ $(n=1, 2,\dots)$ be trigometric polynomials for which

$$\|f(x)-P_n(x)\| \leqq M_3 n^{-\alpha}\psi(n) = M_3 n^{-(\alpha-1)}\psi(n)/n.$$

Lemma 2, with $\varphi(x)=M_3\,\psi(x)/x$, gives

$$\|P_n^{[\alpha]}(x)\| \leqq A + B\psi(n) + C\int_1^n \frac{\psi(x)}{x}\,dx \leqq M_4\psi(n)$$

and, by Lemma 1,

$$\|P_n^{[\lambda]}(x)\| = \|(P_n^{[\alpha]})^{[\lambda-\alpha]}(x)\| \leqq M_5\ n^{\lambda-\alpha}\psi(n).$$

The hypothesis (2) of our theorem now gives

$$\|P_n(x) - T_n(P_n)(x)\| \leqq M_2 n^{-\lambda} M_5 n^{\lambda-\alpha}\psi(n) = M_6 n^{-\alpha}\psi(n).$$

The proof is completed upon observing

$$\|f(x) - T_n(f)(x)\| \leqq \|f - P_n\| + \|P_n - T_n(P_n)\| + \|T_n(f - P_n)\|$$
$$\leqq (1 + M_1)\|f - P_n\| + \|P_n - T_n(P_n)\|$$

by (1).

REMARK 1. The hypothesis (2) is certainly satisfied if the process T is saturated with order $n^{-\lambda}$ and the class W^{λ}. (cf. for example [2]).

REMARK 2. If $T_n(f)(x)$ is a polynomial of degree n, the inverse implication in the conclusion of our theorem is trivially true, and the conclusion may be stated as follows:

$$E_n(f) = O(n^{-\alpha}\psi(n)) \Leftrightarrow \|f(x) - T_n(f)(x)\| = O(n^{-\alpha}\psi(n)).$$

If we take $\psi(x) = x^{\beta}(\log x)^{\gamma}$ $(0 < \beta < 1,\ -\infty < \gamma < \infty)$ our theorem leads to the following

COROLLARY. $E_n(f) = O(n^{-\alpha}(\log n)^{\gamma})$
$\Rightarrow \|f(x) - T_n(f)(x)\| = O(n^{-\alpha}(\log n)^{\gamma}) \qquad (0 < \alpha < \lambda).$

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