ON CONTACT STRUCTURES OF REAL AND COMPLEX MANIFOLDS

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Introduction. Recently, the theory of contact manifolds has been developed by many authors. As well-known, a contact form ω on a $(2n+1)$-dimensional differentiable manifold M is by definition a global 1-form such that $\omega \wedge d\omega^n \neq 0$ on whole M. Through a theorem of E. Cartan, the condition $\omega \wedge d\omega^n \neq 0$ means that there exist local coordinates $(x_1, \cdots, x_n, y_1, \cdots, y_n, z)$ of M, where the contact form ω may be written as

$$\omega = dz - \sum_{i=1}^{n} y_i dx_i.$$

In the theory of contact manifolds treated by the method of differential forms, it seems that the above local expression of the contact form has played an important rôle.

In the present paper, we shall show first that the fundamental relations concerning the contact structures can be derived easily without use of the theorem of E. Cartan, in more general forms. In the following sections, we shall arrange the theory of contact manifolds in our view-point.

Moreover, in §8 we shall get some results on the existence of dynamic contact structures over complex analytic manifolds, and in §9 and §10 we shall investigate into the infinitesimal transformations of cosymplectic manifolds.

1. Vector fields and differential forms. In the beginning, let us arrange some notions, for the later use, related to the vector fields and the differential forms on a C^∞-manifold M.

Let U be an open set of the C^∞-manifold M, and let $\mathfrak{A}(U)$, $\mathfrak{B}(U)$ and $\mathfrak{A}^p(U)$ denote respectively the ring of all real valued C^∞-functions on U, the $\mathfrak{A}(U)$-module of all C^∞ vector fields on U, and the $\mathfrak{A}(U)$-module of all p-forms on U. Then, $\mathfrak{A}(U)$, $\mathfrak{B}(U)$, $\mathfrak{A}^p(U)$ are also regarded as R-modules, where R denotes the real number field, and with respect to the natural restrictions

$$\mathfrak{A}(U) \to \mathfrak{A}(V),\ \mathfrak{B}(U) \to \mathfrak{B}(V),\ \mathfrak{A}^p(U) \to \mathfrak{A}^p(V),$$

for open sets $U \supset V$, each of them constructs a presheaf of R-modules. In the followings, we shall write as

$$\mathfrak{A} = \mathfrak{A}(U),\ \mathfrak{B} = \mathfrak{B}(U),\ \mathfrak{A}^p = \mathfrak{A}^p(U),$$

abbreviating the open set U.

A *vector field* $X \in \mathfrak{B}$ is a map

$$X : \mathfrak{A} \to \mathfrak{A},\quad f \to Xf,$$

satisfying the following axioms.

(i) R-linear.

(ii) Derivation: $X(fg) = (Xf)g + f(Xg),\quad f, g \in \mathfrak{A}.$

(iii) Sectional : $(Xf)(x_0) = 0$, if $f \equiv 0$ on a neighborhood of a point $x_0 \in M$.

It is easily seen that $Xc = 0$ for a constant $c \in R \subset \mathfrak{A}$. Moreover, the product

$$\mathfrak{B} \times \mathfrak{B} \to \mathfrak{B},\quad (X, Y) \to [X, Y]$$

can be defined by

$$[X, Y]f = X(Yf) - Y(Xf),\ f \in \mathfrak{A},$$

and the R-module $\mathfrak{B}$ becomes a Lie-algebra.

A *p-form* $\varphi \in \mathfrak{A}^p$ is a map

$$\varphi : \mathfrak{B} \times \cdots \times \mathfrak{B} \to \mathfrak{A},\quad (X_1, \cdots, X_p) \to \varphi(X_1, \cdots, X_p),$$

satisfying the following conditions.

(i) $\mathfrak{A}$-multilinear.

(ii) Alternate.

(iii) Commutative with the restrictions.

In particular, we set $\mathfrak{A}^0 = \mathfrak{A}$, and we can see $\mathfrak{A}^p = 0$ for $p > \dim M$.

The *exterior product* of forms

$$\mathfrak{A}^p \times \mathfrak{A}^q \to \mathfrak{A}^{p+q},\quad (\varphi, \psi) \to \varphi \wedge \psi,$$

can be defined uniquely so that the following conditions may be satisfied.

(i) $\mathfrak{A}$-bilinear.

(ii) Associative.

(iii) $1 \wedge \varphi = \varphi$, for $\varphi \in \mathfrak{A}^p$.

(iv) $(\varphi_1 \wedge \cdots \wedge \varphi_p)(X_1, \cdots, X_p) = \det(\varphi_i(X_j))$, for $\varphi_i \in \mathfrak{A}^1, X_j \in \mathfrak{B}$.

With respect to this multiplication, the direct sum

$$\mathfrak{A}^* = \sum_p \mathfrak{A}^p$$

becomes a graded $\mathfrak{A}$-algebra, and it is obvious that the exterior product is anti-commutative:

$$\varphi \wedge \psi = (-1)^{pq}\psi \wedge \varphi, \quad \text{for } \varphi \in \mathfrak{A}^p, \psi \in \mathfrak{A}^q.$$

Therefore, if p is odd, it holds that

$$\varphi^2 = \varphi \wedge \varphi = 0, \text{ for } \varphi \in \mathfrak{A}^p.$$

The *exterior derivation* of forms

$$d: \mathfrak{A}^p \to \mathfrak{A}^{p+1}, \ \varphi \to d\varphi$$

can be defined uniquely so that the map d may satisfy the following axioms.

(i) R-linear.

(ii) Anti-derivation: $d(\varphi \wedge \psi) = d\varphi \wedge \psi + (-1)^p \varphi \wedge d\psi$, $\varphi \in \mathfrak{A}^p$, $\psi \in \mathfrak{A}^q$.

(iii) Order 2: $d \circ d = 0$.

(iv) $df(X) = Xf$, for $f \in \mathfrak{A}^0$, $X \in \mathfrak{B}$.

Then, the R-algebra $\mathfrak{A}^*$ with the derivation d becomes a cochain complex. It is notable that any p-form $\varphi \in \mathfrak{A}^p$ can be written locally as a finite sum

$$\varphi = \sum_k g_k df_{k1} \wedge \cdots \wedge df_{kp}, \quad g_k, f_{kj} \in \mathfrak{A},$$

which is evident if we observe an expression of the form φ in local coordinates of M, and then the exterior derivative of the form φ is given by

$$d\varphi = \sum_k dg_k \wedge df_{k1} \wedge \cdots \wedge df_{kp}.$$

The *inner product* for a vector field $X \in \mathfrak{B}$,

$$i(X): \mathfrak{A}^p \to \mathfrak{A}^{p-1}, \quad \varphi \to i(X)\varphi,$$

can be defined by the relation

$$i(X)\varphi(Y_1, \cdots, Y_{p-1}) = \varphi(X, Y_1, \cdots, Y_{p-1}), \ Y_j \in \mathfrak{B}.$$

Then, the map $i(X)$ is characterized by the following axioms.

(i) $\mathfrak{A}$-linear.

(ii) Anti-derivation.

(iii) Order 2: $i(X) \circ i(X) = 0$.

(iv) $i(X)1 = 0$.

(v) $i(X)\varphi = \varphi(X)$, for $\varphi \in \mathfrak{A}^1$.

Moreover, we have the map

$$i: \mathfrak{B} \to \mathrm{Hom}_{\mathfrak{A}} (\mathfrak{A}^p, \mathfrak{A}^{p-1}), \quad X \to i(X),$$

having the properties

(i) $\mathfrak{A}$-linear,

(ii) $i(X)\circ i(Y) = -i(Y)\circ i(X), \quad X, Y \in \mathfrak{B}$.

The *Lie derivative* of forms with respect to a vector field $X \in \mathfrak{B}$,

$$\mathfrak{L}(X): \mathfrak{A}^p \to \mathfrak{A}^p, \quad \varphi \to \mathfrak{L}(X)\varphi,$$

can be defined by the formula

$$\mathfrak{L}(X) = i(X)\circ d + d\circ i(X).$$

Then the map $\mathfrak{L}(X)$ is characterized by the following axioms.

(i) R-linear.

(ii) Derivation: $\mathfrak{L}(X)(\varphi \wedge \psi) = \mathfrak{L}(X)\varphi \wedge \psi + \varphi \wedge \mathfrak{L}(X)\psi, \quad \varphi, \psi \in \mathfrak{A}$.

(iii) $\mathfrak{L}(X)f = Xf$, for $f \in \mathfrak{A}$.

(iv) $\mathfrak{L}(X)df = d(Xf)$, for $f \in \mathfrak{A}$.

In regard to the product of the Lie-algebra $\mathfrak{B}$, we have the formulas

$$i([X, Y]) = \mathfrak{L}(X)\circ i(Y) - i(Y)\circ\mathfrak{L}(X),$$
$$\mathfrak{L}([X, Y]) = \mathfrak{L}(X)\circ\mathfrak{L}(Y) - \mathfrak{L}(Y)\circ\mathfrak{L}(X), \quad X, Y \in \mathfrak{B},$$

which are proved easily if we observe that their right hand sides satisfy the axioms of the inner product $i([X, Y])$ and the Lie derivative $\mathfrak{L}([X, Y])$ respectively.

2. The canonical field and the Lagrange brackets of an almost contact manifold. A C^∞-manifold M of odd dimension $2n+1$ is said to be an *almost contact manifold,* if a global 1-form $\omega \in \mathfrak{A}^1(M)$ and a global 2-form $\pi \in \mathfrak{A}^2(M)$ are given so that they satisfy the condition $\omega \wedge \pi^n \neq 0$ at every point of M, where the forms ω, π are called the *almost contact forms* on M.

If the almost contact forms ω, π are given on M, then there exist uniquely a global vector field $E \in \mathfrak{B}(M)$ and maps

$$l: \mathfrak{A}^1 \to \mathfrak{B}, \quad \varphi \to l(\varphi),$$
$$L: \mathfrak{A} \to \mathfrak{B}, \quad f \to L(f),$$

which satisfy respectively the followng formulas.

(1) $(Eg)\omega \wedge \pi^n = dg \wedge \pi^n, \ g \in \mathfrak{A}$.

(2) $l(\varphi)g\omega \wedge \pi^n = n\varphi \wedge dg \wedge \omega \wedge \pi^{n-1}, \ g \in \mathfrak{A}$.

(3) $L(f)g\omega \wedge \pi^n = ndf \wedge dg \wedge \omega \wedge \pi^{n-1}, \ g \in \mathfrak{A}$.

In fact, the functions $Eg, l(\varphi)g, L(f)g \in \mathfrak{A}$ are uniquely determined, since the $(2n+1)$-form $\omega \wedge \pi^n$ gives a base of the $\mathfrak{A}$-module $\mathfrak{A}^{2n+1}(M)$, and it is obvious from the formulas that the maps $E, l(\varphi), L(f)$: $\mathfrak{A} \to \mathfrak{A}$ satisfy the axioms of vector field in the preceding section.

The vector field E is called the *canonical field* of the almost contact structure and both the maps l, L are called the *Lagrange brackets* of the almost contact structure. From the definitions, we can easily obtain the following

PROPOSITION 1. *Let l, L be the Lagrange brackets of the almost contact structure.*

1° $L(f) = l(df)$, $f \in \mathfrak{A}$.

2° *The map* $l : \mathfrak{A}^1 \to \mathfrak{B}$ *is* $\mathfrak{A}$*-linear, that is,*
$l(f_1\varphi_1 + f_2\varphi_2) = f_1 l(\varphi_1) + f_2 l(\varphi_2)$, $f_1, f_2 \in \mathfrak{A}$, $\varphi_1, \varphi_2 \in \mathfrak{A}^1$.

3° *The map* $L : \mathfrak{A} \to \mathfrak{B}$ *is* R*-linear and a derivation, that is,*
$L(c_1 f_1 + c_2 f_2) = c_1 L(f_1) + c_2 L(f_2)$,
$L(f_1 f_2) = f_1 L(f_2) + f_2 L(f_1)$, $c_1, c_2 \in R$, $f_1, f_2 \in \mathfrak{A}$,

4° $L(f)g = - L(g)f$, $f, g \in \mathfrak{A}$,

Let us notice that each 1-form $\theta \in \mathfrak{A}^1$ can be written locally as a finite sum

$$\theta = \sum_k h_k dg_k, \quad h_k, g_k \in \mathfrak{A},$$

and then the inner product $i(X)\theta$ for any vector field $X \in \mathfrak{B}$ is given by

$$i(X)\theta = \theta(X) = \sum_k h_k X g_k.$$

Then, we can write respectively the formulas (1), (2), (3) in more general forms:

(1)′ $\theta(E)\omega \wedge \pi^n = \theta \wedge \pi^n$, $\theta \in \mathfrak{A}^1$,

(2)′ $\theta(l(\varphi))\omega \wedge \pi^n = n\varphi \wedge \theta \wedge \omega \wedge \pi^{n-1}$, $\theta, \varphi \in \mathfrak{A}^1$,

(3)′ $\theta(L(f))\omega \wedge \pi^n = ndf \wedge \theta \wedge \omega \wedge \pi^{n-1}$, $\theta \in \mathfrak{A}^1, f \in \mathfrak{A}$.

THEOREM 1. *Let ω, π be the almost contact forms on a $(2n + 1)$-dimensional manifold M. Then, the canonical field E and the Lagrange brackets l, L are characterized by the following properties.*

1° $X = E \in \mathfrak{B}(M)$, *if and only if*
(i) $i(X)\omega = 1$,
(ii) $i(X)\pi = 0$.

2° $X = l(\varphi) \in \mathfrak{B}$ *for* $\varphi \in \mathfrak{A}^1$, *if and only if*
(i) $i(X)\omega = 0$,
(ii) $i(X)\pi = \varphi(E)\omega - \varphi$.

3° $X = L(f)$ *for* $f \in \mathfrak{A}$, *if and only if*
(i) $i(X)\omega = 0$,
(ii) $i(X)\pi = (Ef)\omega - df$.

PROOF. Let us notice that any $(2n+2)$-form on a $(2n+1)$-dimensional manifold is identically zero, and that the inner product $i(Y)$ for a vector field $Y \in \mathfrak{V}$ is an anti-derivation on the forms.

1° Putting $\theta = \omega$ into the formula (1)′, we have

$$\omega(E)\omega \wedge \pi^n = \omega \wedge \pi^n,$$

which implies $i(E)\omega = \omega(E) = 1$. Moreover, putting $\theta = i(Y)\pi$ and $Y \in \mathfrak{V}$ into (1)′, we have

$$\pi(Y, E)\omega \wedge \pi^n = i(Y)\pi \wedge \pi^n = \frac{1}{n+1} i(Y)\pi^{n+1} = 0,$$

since the inner product $i(Y)$ is an anti-derivation and the $(2n+2)$-form π^{n+1} is identically zero. It follows that

$$(i(E)\pi)(Y) = \pi(E, Y) = -\pi(Y, E) = 0$$

for any vector field Y. This proves $i(E)\pi = 0$.

Conversely, assume that a vector field X has the properties (i), (ii). Then, considering the $(2n+2)$-form $dg \wedge \omega \wedge \pi^n = 0$, we have

$$\begin{aligned} &i(X)(dg \wedge \omega \wedge \pi^n) = 0 \\ &= (Xg)\omega \wedge \pi^n - dg \wedge \omega(X)\pi^n + dg \wedge \omega \wedge ni(X)\pi \wedge \pi^{n-1} \\ &= (Xg)\omega \wedge \pi^n - dg \wedge \pi^n. \end{aligned}$$

This proves

$$(Xg)\omega \wedge \pi^n - dg \wedge \pi^n = 0$$

which is nothing but the formula (1). Hence $X = E$.

2° Assume that $X = l(\varphi)$. Putting $\theta = \omega$ into the formula (2)′, we have

$$\omega(X)\omega \wedge \pi^n = n\varphi \wedge \omega \wedge \omega \wedge \pi^{n-1} = 0$$

which implies $i(X)\omega = \omega(X) = 0$. Moreover, putting $\theta = i(Y)\pi$, $Y \in \mathfrak{V}$, into (2)′, and considering the $(2n+2)$-form $\varphi \wedge \omega \wedge \pi^n = 0$, we have

$$\begin{aligned} &\pi(Y, X)\omega \wedge \pi^n = n\varphi \wedge \pi\, i(Y)\pi \wedge \omega \wedge \pi^{n-1}, \\ &i(Y)(\varphi \wedge \omega \wedge \pi^n) = 0 \\ &= \varphi(Y)\omega \wedge \pi^n - \varphi \wedge \omega(Y)\pi^n + n\varphi \wedge \omega \wedge i(Y)\pi \wedge \pi^{n-1}. \end{aligned}$$

Therefore, applying the formula (1)′, we have

$$\begin{aligned} \pi(Y, X)\omega \wedge \pi^n &= \varphi(Y)\omega \wedge \pi^n - \varphi \wedge \omega(Y)\pi^n \\ &= \{\varphi(Y) - \omega(Y)\varphi(E)\}\omega \wedge \pi^n. \end{aligned}$$

It follows that

$$(i(X)\pi)(Y) = -\pi(Y, X) = \varphi(E)\omega(Y) - \varphi(Y)$$

for any vector field Y. This proves

$$i(X)\pi = \varphi(E)\omega - \varphi.$$

Conversely, assume that a vector field X has the properties (i), (ii). Then, considering the $(2n+2)$-form $dg \wedge \omega \wedge \pi^n = 0$, we have

$$\begin{aligned} & i(X)(dg \wedge \omega \wedge \pi^n) = 0 \\ &= (Xg)\omega \wedge \pi^n - dg \wedge \omega(X)\pi^n + dg \wedge \omega \wedge ni(X)\pi \wedge \pi^{n-1} \\ &= (Xg)\omega \wedge \pi^n + ndg \wedge \omega \wedge (\varphi(E)\omega - \varphi) \wedge \pi^{n-1} \\ &= (Xg)\omega \wedge \pi^n - ndg \wedge \omega \wedge \varphi \wedge \pi^{n-1}. \end{aligned}$$

This proves

$$(Xg)\omega \wedge \pi^n - n\varphi \wedge dg \wedge \omega \wedge \pi^{n-1} = 0,$$

which is nothing but the formula (2). Hence, $X = l(\varphi)$.

3° This is a special case of 2° where $\varphi = df$.

Now, we define a vector field $K(f) \in \mathfrak{V}$ for a function $f \in \mathfrak{A}$, by the relation

$$K(f) = fE + L(f).$$

Then, we have the map

$$K : \mathfrak{A} \to \mathfrak{V},\ f \to K(f),$$

and, from the Proposition 1 and the Theorem 1, we can obtain easily the followings.

PROPOSITION 2. *The map $K : \mathfrak{A} \to \mathfrak{V}$ staisfies the conditions*:

(i) *R-linear,*

(ii) $K(fg) = fK(g) + gK(f) - fgE,\ \ f, g \in \mathfrak{A}.$

THEOREM 2. *Let ω, π be the almost contact forms on M. Then the vector field $X = K(f) \in \mathfrak{V}$ for a function $f \in \mathfrak{A}$ is characterized by the properties*:

(i) $i(X)\omega = f,$

(ii) $i(X)\pi = (Ef)\omega - df.$

COROLLARY. *The R-linear map $K : \mathfrak{A} \to \mathfrak{V}$ is an injection, and the left inverse map of K is given by the 1-form ω: $\mathfrak{V} \to \mathfrak{A}$.*

PROOF. $\omega(K(f)) = i(K(f))\,\omega = f$ for $f \in \mathfrak{A}$. This proves that the map $\omega \circ K : \mathfrak{A} \to \mathfrak{A}$ is identical.

At the end of the present section, let us remark on some relations between the vector fields and the 1-forms of the almost contact manifold M. Let ω, π be the almost contact forms on M.

A vector field $X \in \mathfrak{B}$ is said to be *horizontal*, if $\omega(X) = 0$, and a p-form $\varphi \in \mathfrak{A}^p$ is said to be *basic*, if $i(E)\varphi = 0$. Then, we have exact sequences of $\mathfrak{A}$-modules

$$0 \to \mathfrak{W} \to \mathfrak{B} \xrightarrow{\omega} \mathfrak{A} \to 0,$$

$$0 \to \mathfrak{B}^1 \to \mathfrak{A}^1 \xrightarrow{i(E)} \mathfrak{A} \to 0,$$

where $\mathfrak{W}$, $\mathfrak{B}^1$ denote respectively the $\mathfrak{A}$-module of all horizontal fields, and the $\mathfrak{A}$-module of all basic 1-forms. The $\mathfrak{A}$-linear maps $r(E): \mathfrak{A} \to \mathfrak{B}$, $r(\omega): \mathfrak{A} \to \mathfrak{A}^1$ defined by $r(E)f = fE$, $r(\omega)f = f\omega$ for $f \in \mathfrak{A}$ give respectively splittings of the above exact sequences, namely both the $\mathfrak{A}$-linear maps $\omega \circ r(E)$ and $i(E) \circ r(\omega)$ are identical on $\mathfrak{A}$.

Now, consider the $\mathfrak{A}$-linear map $\pi: \mathfrak{B} \to \mathfrak{A}^1$ defined by $\pi(X) = i(X)\pi$ for $X \in \mathfrak{B}$. Then we have the following

PROPOSITION 3. *The $\mathfrak{A}$-linear map π gives a bijection between the $\mathfrak{A}$-modules $\mathfrak{W}$ and $\mathfrak{B}^1$, and its inverse map is given by the Lagrange bracket $-l$ restricted on $\mathfrak{B}^1$.*

Accordingly, there exists a natural isomorphism between the exact sequences of $\mathfrak{A}$-modules such as

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{W} \arrow[r] \arrow[d, shift right, "-l"'] & \mathfrak{B} \arrow[r, "\omega"] \arrow[d, shift left, "\alpha"] & \mathfrak{A} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathfrak{B}^1 \arrow[r] \arrow[u, shift right, "\pi"'] & \mathfrak{A}^1 \arrow[r, "i(E)"] \arrow[u, shift left, "\alpha^{-1}"] & \mathfrak{A} \arrow[r] & 0,
\end{tikzcd}$$

where the maps α, α^{-1} are given by

$$\alpha = \pi + r(\omega)\circ\omega, \quad \alpha^{-1} = -l + r(E)\circ i(E).$$

PROOF. A 1-form $\pi(X) = i(X)\pi \in \mathfrak{A}^1$ for any vector field X is basic, since $i(E)\circ i(X)\pi = \pi(X, E) = 0$. A vector field $l(\varphi) \in \mathfrak{B}$ for any 1-form $\varphi \in \mathfrak{A}^1$ is horizontal, since $\omega(l(\varphi)) = 0$. For a basic 1-form $\varphi \in \mathfrak{B}^1$, it holds that

$$i(l(\varphi))\pi = \varphi(E)\omega - \varphi = -\varphi,$$

since $\varphi(E) = 0$. This proves that the map $\pi\circ(-l)$ is identical on $\mathfrak{B}^1$.

Moreover, if $i(X)\pi = 0$ for a horizontal field $X \in \mathfrak{W}$, then applying the Theorem 1, we have $X = l(0) = 0$, because the conditions $i(X)\omega = 0$, $i(X)\pi = 0$ are satisfied. This proves that the map $\pi: \mathfrak{W} \to \mathfrak{B}^1$ is an injection. Hence, the map $(-l)\circ\pi$ is identical on $\mathfrak{W}$.

It is clear from the above considerations that the $\mathfrak{A}$-linear map $\alpha: \mathfrak{B} \to \mathfrak{A}^1$ is a bijection whose inverse map is given by α^{-1} and the diagram is commutative.

3. Infinitesimal contact transformations of a contact manifold. A $(2n+1)$-dimensional manifold M is said to be a *contact manifold*, if a global 1-form $\omega \in \mathfrak{A}^1(M)$ is given so that it satisfies the condition $\omega \wedge d\omega^n \neq 0$ at every point of M, where the form ω is called the *contact form* of the contact structure. Then, the forms ω, $d\omega$ give an almost contact structure on M, and so we have the canonical field E, Lagrange brackets l, L and the R-linear map K defined in the preceding section. They are characterized by the following properties.

1° $X = E \in \mathfrak{B}(M)$, *if and only if*

(i) $i(X)\omega = 1$, (ii) $i(X)d\omega = 0$.

2° $X = l(\varphi) \in \mathfrak{B}$ *for* $\varphi \in \mathfrak{A}^1$ *if and only if*

(i) $i(X)\omega = 0$, (ii) $i(X)d\omega = \varphi(E)\omega - \varphi$.

3° $X = L(f) \in \mathfrak{B}$ *for* $f \in \mathfrak{A}$, *if and only if*

(i) $i(X)\omega = 0$, (ii) $i(X)d\omega = (Ef)\omega - df$.

4° $X = K(f) \in \mathfrak{B}$ *for* $f \in \mathfrak{A}$, *if and only if*

(i) $i(X)\omega = f$, (ii) $i(X)d\omega = (Ef)\omega - df$.

Let ω be the contact form on M. A vector field $X \in \mathfrak{B}$ is called an *infinitesimal contact transformation*, if there exists a function $k \in \mathfrak{A}$ such that $\mathfrak{L}(X)\omega = k\omega$, where $\mathfrak{L}$ denotes the Lie derivative. Moreoiver, a vector field $X \in \mathfrak{B}$ is called an *infinitesimal automorphism* of the contact structure, if $\mathfrak{L}(X)\omega = 0$. Let $\mathfrak{C}, \mathfrak{C}_0$ denote respectively, the R-module of all infinitesimal contact transformations, and the R-module of all infinitesimal automorphisms of the contact structure. Then $\mathfrak{C}_0$ becomes an R-submodule of $\mathfrak{C}$.

PROPOSITION 4. *For an infinitesimal contact transformation $X \in \mathfrak{C}$, the function $k \in \mathfrak{A}$ such that $\mathfrak{L}(X)\omega = k\omega$, is given by $k = E\omega(X)$.*

PROOF. Assume that $X \in \mathfrak{C}$. Then, by definitions,

$$\mathfrak{L}(X)\omega = d\omega(X) + i(X)d\omega = k\omega.$$

Applying the inner product $i(E)$ to this, and taking account of the property 1°, we have $E\omega(X) = k$.

THEOREM 3. *Let ω be the contact form on M. The R-linear map $K: \mathfrak{A} \to \mathfrak{B}$ gives a bijection between the R-modules $\mathfrak{A}$ and $\mathfrak{C}$, and its inverse map is given by the form $\omega: \mathfrak{B} \to \mathfrak{A}$ restricted on $\mathfrak{C}$.*

PROOF. First, we shall show that a vector field $X = K(f)$ for any function $f \in \mathfrak{A}$ is an infinitesimal contact transformation. From the property 4°, we have

$$\mathfrak{L}(X)\omega = di(X)\omega + i(X)d\omega = df + (Ef)\omega - df = (Ef)\omega.$$

Hence, $X \in \mathfrak{C}$ and $\omega(K(f)) = \omega(X) = i(X)\omega = f$. This proves that the R-linear map $\omega \circ K$ is identical on $\mathfrak{A}$.

Next, let us consider a function $f = \omega(X) \in \mathfrak{A}$ for any $X \in \mathfrak{C}$. Then we have

$$i(X)\omega = \omega(X) = f,$$

and $\mathfrak{L}(X)\omega = k\omega$, where $k = E\omega(X) = Ef$ by the Proposition 4. Therefore,

$$i(X)d\omega = \mathfrak{L}(X)\omega - di(X)\,\omega = (Ef)\omega - df$$

Hence, it follows from the property 4° that

$$X = K(f) = K(\omega(X)).$$

This proves that the R-linear map $K \circ \omega$ is identical on $\mathfrak{C}$.

COROLLARY. $E = K(1) \in \mathfrak{C}_0.$

The Theorem 3 and the Proposition 4 show that an exact and commutative diagram of R-modules

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{A}_0 \arrow[r] \arrow[d, shift left, "K"] & \mathfrak{A} \arrow[r, "E"] \arrow[d, shift left, "K"] & \mathfrak{A} \arrow[d, equal] \\ 0 \arrow[r] & \mathfrak{C}_0 \arrow[r] \arrow[u, shift left, "\omega"] & \mathfrak{C} \arrow[r, "\mu"] \arrow[u, shift left, "\omega"] & \mathfrak{A} \end{tikzcd}$$

holds, where $\mathfrak{A}_0$ denotes the R-module of all first integrals of the vector field E, and μ denotes the R-linear map which maps each vector field $X \in \mathfrak{C}$ to its multiple factor $k \in \mathfrak{A}$ such that $\mathfrak{L}(X)\omega = k\omega$.

Let us assume always that the manifold M is paracompact. Taking the sheaves of germs of the above R-modules, we have an exact and commutative diagram of sheaves on M

$$\begin{tikzcd} 0 \arrow[r] & \mathbf{A}_0 \arrow[r] \arrow[d, shift left] & \mathbf{A} \arrow[r, "E"] \arrow[d, shift left, "K"] & \mathbf{A} \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathbf{C}_0 \arrow[r] \arrow[u, shift left] & \mathbf{C} \arrow[r] \arrow[u, shift left, "\omega"] & \mathbf{A} \arrow[r] & 0, \end{tikzcd}$$

where the map $E\colon \mathbf{A} \to \mathbf{A}$ becomes a surjection, because the differential equation $Ef = g$ for a given function $g \in \mathfrak{A}$ has always a local solution $f \in \mathfrak{A}$. Since the sheaf $\mathbf{A}$ is fine, so is the sheaf $\mathbf{C}$. Therefore, from the cohomology sequence, we have the following result which was shown by Gray [4].

$$0 \to \mathfrak{C}_0(M) \to \mathfrak{C}(M) \to \mathfrak{A}(M) \to H^1(M, \mathbf{C}_0) \to 0,$$
$$H^q(M, \mathbf{C}_0) = 0, \quad q \geqq 2.$$

It is easy to see that the R-module $\mathfrak{C}$ is a Lie-algebra by the product $[X, Y]$ for $X, Y \in \mathfrak{C}$, and $\mathfrak{C}_0$ is a subalgebra of $\mathfrak{C}$. By the bijection ω, the R-module $\mathfrak{A}$ becomes also a Lie-algebra whose product we denote by $[f, g]$ for $f, g \in \mathfrak{A}$. Moreover, $\mathfrak{A}$ is a commutative ring. By the bijection K, the R-module $\mathfrak{C}$ becomes also a commutative ring, whose product we denote be $X \circ Y$ for $X, Y \in \mathfrak{C}$.

PROPOSITION 5. *Let $X = K(f)$, $Y = K(g)$ for $f, g \in \mathfrak{A}$ be infinitesimal contact transformations.*

1° $L(f)g = d\omega(X, Y)$.

2° $[f, g] = \omega([X, Y]) = L(f)g + fEg - gEf$.

3° $X \circ Y = K(fg) = fK(g) + gK(f) - fgE$
$= \omega(X)Y + \omega(Y)X - \omega(X)\omega(Y)E$.

PROOF. Since $\omega = K^{-1}$ on $\mathfrak{C}$, we have $f = \omega(X)$, $g = \omega(Y)$.

1° $d\omega(X, Y) = -i(X)i(Y)d\omega = -i(X)((Eg)\omega - dg)$
$= -fEg + K(f)g = L(f)g$.

2° By the formula at the end of §1, we have
$\omega([X, Y]) = i([X, Y])\omega = \mathfrak{L}(X) \circ i(Y)\omega - i(Y) \circ \mathfrak{L}(X)\omega$
$= Xg - (Ef)\omega(Y) = K(f)g - gEf = L(f)g + fEg - gEf$.

3° By the Proposition 2, it holds that
$K(fg) = fK(g) + gK(f) - fgE$.

4. The contact structure in the wide sense. In this section, we concern ourselves with the contact structure in the wide sense introduced by Spencer.

Let $\{U_i\}_{i \in I}$ be an open covering of a $(2n+1)$-dimensional manifold M. If a system of local contact forms

$$\{\omega_i\}_{i \in I}, \ \omega_i \in \mathfrak{A}^1(U_i), \ \omega_i \wedge d\omega_i^n \neq 0$$

is given and there exists a system of functions $\{g_{ij}\}_{i,j \in I}$ such that

$$\omega_i = g_{ij}\omega_j \quad \text{in } U_i \cap U_j, \ g_{ij} \in \mathfrak{A}(U_i \cap U_j),$$

then we call M a *contact manifold in the wide sense*. Of course, we suppose that two systems of contact forms $\{U_i, \omega_i\}$ and $\{U'_k, \omega'_k\}$ define the same contact structure on M, if their union gives also a contact structure on M.

Let R^*, R^+ denote respectively the multiplicative group of all non-zero real numbers, and its subgroup of all positive numbers. Then, we have an exact sequence of abelian groups

$$0 \to R^+ \to R^* \xrightarrow{j} Z_2 \to 0,$$

where $Z_2 = \{0, 1\}$ denotes the cyclic group of order 2, which we represent as an additive group. Taking the sheaves on M of germs of C^∞-functions with values in R^+, R^*, Z_2 respectively, we get an exact sequence of sheaves

$$0 \to \mathbf{R}^+ \to \mathbf{R}^* \xrightarrow{j} Z_2 \to 0.$$

Since, the map $\log : \mathbf{R}^+ \to \mathbf{A}$ gives a bijection between the sheaves of abelian groups $\mathbf{R}^+$ and $\mathbf{A}$, the sheaf $\mathbf{R}^+$ is fine like the sheaf $\mathbf{A}$ of germs of C^∞-functions. Therefore, by the cohomology sequence, we have a bijection

$$\begin{array}{c} 0 \to H^1(M, \mathbf{R}^*) \xrightarrow{j^*} H^1(M, Z_2) \to 0 \\ \xi \longrightarrow w_1(\xi), \end{array}$$

where a cohomology class $\xi \in H^1\,(M, \mathbf{R}^*)$ gives a C^∞ R^*-bundle structure, that is, the associated principal bundle of a C^∞ line bundle on M, and the class $w_1(\xi)$ denotes the Stiefel-Whitney class of the R^*-bundle ξ.

Whenever a contact structure $\{U_i, \omega_i\}_{i \in I}$, $\omega_i = g_{ij}\omega_j$, is defined on M, we get uniquely a C^∞ R^*-bundle $\eta = \{g_{ij}\} \in H^1(M, \mathbf{R}^*)$, since $g_{ij} \neq 0$. On the other hand, it holds that

$$\omega_i \wedge d\omega_i^n = g_{ij}^{n+1}\omega_j \wedge d\omega_j^n,$$

which shows that the cohomology class $\eta^{-(n+1)} \in H^1(M, \mathbf{R})$ gives the canonical line bundle, that is, the line bundle on M consisting of the $(2n + 1)$-forms. Because, each non-zero $(2n + 1)$-form $\omega_i \wedge d\omega_i^n$ on U_i can be regarded as a local cross-section of the principal bundle associated to the canonical line bundle, and so the functions $\{g_{ij}^{-(n+1)}\}$ give the transition functions of the bundle structure. Hence, the class $j^*\eta^{n+1} \in H^1(M, Z_2)$ gives the 1-st Stiefel-Whitney class $w_1(M)$ of the manifold M. Therefore, setting $\sigma = j^*(\eta) = w_1(\eta)$, we have the formula

$$(n + 1)\sigma = w_1(M),$$

which implies the following results derived by Gray [4].

1° *If n is odd, then M is orientable.*

2° *If n is even, then $\sigma = w_1(M)$.*

Moreover, taking into account that j^* is a bijection, we have clearly the following:

3° *If n is even and M is orientable, then any contact structure in the* *wide sense is given by a global contact form* $\omega \in \mathfrak{A}(M)$.

Similar results hold for the complex analytic contact structure. Let M be a complex analytic manifold of complex dimension $2n+1$. In this case, we consider an exact sequence of abelian groups

$$0 \to Z \to C \xrightarrow{\exp} C^* \to 0,$$

where Z, C, C^* denote respectively the ring of integers, the complex number field, and the multiplicative group of all non-zero complex numbers. Taking the sheaves on M of germs of holomorphic functions with values in Z, C, C^* respectively, we get an exact sequence of sheaves

$$0 \to Z \to \mathbf{C}_h \xrightarrow{\exp} \mathbf{C}_h^* \to 0.$$

Then, by the cohomology sequence, we have the homomorphism

$$\begin{aligned} \to H^1(M, \mathbf{C}_h^*) &\xrightarrow{\delta^*} H^2(M, Z) \to \\ \xi &\longrightarrow \alpha = c_1(\xi), \end{aligned}$$

where a cohomology class $\xi \in H^1(M, \mathbf{C}_h^*)$ gives an analytic C^*-bundle structure, that is, the associated principal bundle of a complex analytic line bundle on M, and the class $c_1(\xi)$ denotes the Chern class of C^*-bundle ξ.

If a complex analytic contact structure $\{U_i, \omega_i\}_{i\in I}$, $\omega_i = g_{ij}\omega_j$, is given, then we get an analytic C^*-bundle $\eta = \{g_{ij}\} \in H^1(M, \mathbf{C}_h^*)$. Setting $\alpha = c_1(\eta)$, we have easily the following formula derived by Kobayashi [5]:

$$(n+1)\alpha = c_1(M),$$

where the class $c_1(M) \in H^2(M, Z)$ denotes the 1-st Chern class of the manifold M.

5. The Poisson brackets of an almosts symplectic manifold. A C^∞-manifold M of even dimension $2n$ is said to be an *almost symplectic manifold*, if a global 2-form $\Omega \in \mathfrak{A}^2(M)$ is given so that it satisfies the condition $\Omega^n \neq 0$ at every point of M, where the form Ω is called the *almost symplectic form* on M.

If the almost symplectic form Ω is given on M, then there exist uniquely maps

$$\begin{aligned} p &: \mathfrak{A}^1 \to \mathfrak{B}, \quad \varphi \to p(\varphi), \\ P &: \mathfrak{A} \to \mathfrak{B}, \quad f \to P(f), \end{aligned}$$

which satisfy respectively the following formulas.

$$p(\varphi)g\Omega^n = n\varphi \wedge dg \wedge \Omega^{n-1}, \quad g \in \mathfrak{A}. \tag{4}$$

$$P(f)g\Omega^n = ndf \wedge dg \wedge \Omega^{n-1}, \quad g \in \mathfrak{A}. \tag{5}$$

In fact, it is trivial by the condition $\Omega^n \neq 0$ and the definitions that the maps $p(\varphi), P(f): \mathfrak{A} \to \mathfrak{A}$ are determined uniquely and they satisfy the axioms of vector field. Both the maps p, P are called the *Poisson brackets* of the almost symplectic structure. Obviously, we get the following.

PROPOSITION 6. *Let p, P be the Poisson brackets of the almost symplectic structure.*

1° $P(f) = p(df)$, $f \in \mathfrak{A}$.

2° *The map* $p: \mathfrak{A}^1 \to \mathfrak{B}$ *is* $\mathfrak{A}$*-linear.*

3° *The map* $P: \mathfrak{A} \to \mathfrak{B}$ *is* R*-linear.*

4° $P(f)g = -P(g)f$, $f, g \in \mathfrak{A}$.

Let us notice that each 1-form $\theta \in \mathfrak{A}^1$ can be written locally as

$$\theta = \sum_k h_k dg_k, \quad h_k, g_k \in \mathfrak{A},$$

and then the inner product $i(X)\theta$ is given by

$$i(X)\theta = \theta(X) = \sum_k h_k X g_k, \quad X \in \mathfrak{B}.$$

Then, we can obtain clearly the formulas:

$$(4)' \qquad \theta(p(\varphi))\Omega^n = n\varphi \wedge \theta \wedge \Omega^{n-1}, \quad \theta, \varphi \in \mathfrak{A}^1,$$

$$(5)' \qquad \theta(P(f))\Omega^n = n df \wedge \theta \wedge \Omega^{n-1}, \quad \theta \in \mathfrak{A}^1, f \in \mathfrak{A}.$$

THEOREM 4. *Let Ω be the almost symplectic form on a $2n$-dimensional manifold M. Then, the Poisson brackets p, P are characterized by the following properties.*

1° $X = p(\varphi)$ *for* $\varphi \in \mathfrak{A}^1$, *if and only if* $i(X)\Omega = -\varphi$.

2° $X = P(f)$ *for* $f \in \mathfrak{A}$, *if and only if* $i(X)\Omega = -df$.

PROOF. Let us notice that any $(2n+1)$-form on a $2n$-dimensional manifold is identically zero.

1° Assume that $X = p(\varphi)$. Putting $\theta = i(Y)\Omega$ for $Y \in \mathfrak{B}$ into $(4)'$, and considering the $(2n+1)$-form $\varphi \wedge \Omega^n = 0$, we have

$$\begin{aligned} &\Omega(Y, X)\Omega^n = n\varphi \wedge i(Y)\Omega \wedge \Omega^{n-1}, \\ &i(Y)(\varphi \wedge \Omega^n) = 0 \\ &= \varphi(Y)\Omega^n - \varphi \wedge n i(Y)\Omega \wedge \Omega^{n-1}. \end{aligned}$$

Therefore, we have $\Omega(Y, X)\Omega^n = \varphi(Y)\Omega^n$. It follows that

$$(i(X)\Omega)(Y) = -\Omega(Y, X) = -\varphi(Y)$$

for any vector field Y. This proves $i(X)\Omega = -\varphi$.

Conversely, assume that a vector field X has the property in 1°. Then, considering the $(2n+1)$-form $dg \wedge \Omega^n = 0$, we have

$$\begin{aligned} i(X)(dg \wedge \Omega^n) &= 0 \\ &= (Xg)\Omega^n - dg \wedge ni(X)\Omega \wedge \Omega^{n-1} \\ &= (Xg)\Omega^n + ndg \wedge \varphi \wedge \Omega^{n-1}. \end{aligned}$$

This proves

$$(Xg)\Omega^n + ndg \wedge \varphi \wedge \Omega^{n-1} = 0,$$

which is nothing but the formula (4). Hence $X = p(\varphi)$.

2° This is a special case of 1° where $\varphi = df$.

COROLLARY. *The $\mathfrak{A}$-linear map $p\colon \mathfrak{A}^1 \to \mathfrak{B}$ is a bijection, and its inverse map $p^{-1}\colon \mathfrak{B} \to \mathfrak{A}^1$ is given by $p^{-1}(X) = -i(X)\Omega$ for $X \in \mathfrak{B}$.*

6. Infinitesimal automorphisms of a symplectic manifold. A $2n$-dimensional manifold M is said to be a *symplectic manifold*, if a global 2-form Ω is given so that it satisfies the conditions $\Omega^n \neq 0$ and $d\Omega = 0$ at every point of M. Of course, the form Ω gives an almost symplectic structure on M, and so we have the Poisson brackets p, P defined in the preceding section.

PROPOSITION 7. *Let Ω be the symplectic form on M, If $X = p(\varphi)$ for $\varphi \in \mathfrak{A}^1$, then*

$$\mathfrak{L}(X)\Omega = -d\varphi.$$

PROOF. Since $d\Omega = 0$ and $i(X)\Omega = -\varphi$, we have

$$\mathfrak{L}(X)\Omega = i(X)d\Omega + di(X)\Omega = -d\varphi.$$

Let Ω be the symplectic form on M. A vector field $X \in \mathfrak{B}$ is called an *infinitesimal automorphism* of the symplectic structure, if $\mathfrak{L}(X)\Omega = 0$. On a symplectic manifold, a vector field $X \in \mathfrak{B}(M)$ such that $\mathfrak{L}(X)\Omega = k\Omega$, $k \in \mathfrak{A}(M)$, is trivial, because we can see easily when $n > 1$ that k is a constant if M is connected, and that $k = 0$ if M is compact [6].

THEOREM 5. *Let $\mathfrak{S}_0$ be the R-module of all infinitesimal automorphisms of the symplectic structure and let $\mathfrak{Z}^1$ denote the R-module of all closed 1-forms. Then the R-linear map p gives a bijection between the R-modules $\mathfrak{Z}^1$ and $\mathfrak{S}_0$.*

PROOF. It is known that the map $p\colon \mathfrak{A}^1 \to \mathfrak{B}$ is a bijection, and if

$X = p(\varphi)$, then $\mathfrak{L}(X)\Omega = - d\varphi$. Therefore, $X \in \mathfrak{S}_0$ if and only if $d\varphi = 0$.

7. Connections on principal bundles. Now, let us introduce some terminologies being useful in the next section.

We dente by $T(M)$, $T_x(M)$ respectively, the tangent vector bundle of a differentiable manifold M, and the tangent vector space at a point $x \in M$. Then, concerning the topological product $M \times N$ of two differentiable manifolds M, N, we can see that there exist natural bijections

$$T(M \times N) \cong T(M) \times T(N),\ T_{(x,y)}(M \times N) \cong T_x(M) + T_y(N).$$

A differentiable map $\alpha: M \to N$ induces a map of tangent bundles given by

$$\alpha: T(M) \to T(N),\ (\alpha X)f = X(f\circ\alpha), \quad \text{for } f \in \mathfrak{A} \text{ on } N,$$

which we designate by the same letter α. In particular, let us consider a differentiable map from a topological product $M \times N$ to another manifold K, expressed as a multiplication

$$M \times N \to K, \qquad (x, y) \to xy.$$

Taking a point $a \in M$ and a point $b \in N$, we have maps

$$a: N \to K,\ y \to ay, \quad \text{and } r(b): M \to K,\ x \to xb,$$

whose induced maps are expressed also as multiplications

$$a: T(N) \to T(K),\ Y \to aY, \quad \text{and } r(b): T(M) \to T(K), X \to Xb.$$

Then, the induced map of the multiplication is given by

$$T(M) \times T(N) \to T(K),\ (X, Y) \to Xy + xY,$$

where $X \in T_x(M)$, $Y \in T_y(N)$ and $x \in M$, $y \in N$.

Let $B(M, G)$ be a differentiable or complex analytic G-bundle over M, and let us denote the projection and the right translation for an element $g \in G$ by

$$\rho: B \to \mathrm{M}, \quad \text{and } r(g): B \to B, \quad b \to bg,$$

respectively. A point $b \in B$ can be regarded as a map so called an admissible map

$$b: G \to G_x, \quad g \to bg,$$

where G_x is a fibre of B over a point $x = \rho(b) \in M$. Moreover, let us denote the Lie-algebra of G by $\mathfrak{g}$ to be identified with the tangent vector space $T_e(G)$ at the unit element $e \in G$. A tangent vector $X \in T(B)$ is said to be *vertical* if $\rho X = 0$. Then, any vertical vector $X \in T_b(B)$ at a point $b \in B$ is given by uniquely in the form $X = bA$, where $A \in \mathfrak{g}$. Hence, we have an injection

$$\lambda: B \times \mathfrak{g} \to T(B), \quad (b, A) \to bA.$$

Since it holds that $bg(g^{-1}Ag) = (bA)g$ for $g \in G$, dividing the spaces $B \times \mathfrak{g}$ and $T(B)$ by the group G, we have an exact sequence of vector bundles over M so called the *fundamental sequence* of B

$$0 \to L(M) \xrightarrow{\lambda} Q(M) \xrightarrow{\rho} T(M) \to 0,$$

where $L(M) = B \times_{ad(G)} \mathfrak{g}$ is the associated vector bundle of B determined by the adjoint representation of G on $\mathfrak{g}$, and $Q(M) = T(M)/r(G)$ is the quotient space of $T(B)$ by the equivalent relation $X \sim Xg$ for $g \in G$.

A *connection* of a principal bundle $B(M, G)$ is defined by a global $\mathfrak{g}$-valued 1-form $\omega \in \mathfrak{A}^1(B, \mathfrak{g})$ on B satisfying the following conditions.

(i) $\omega \circ r(g) = ad(g^{-1})\omega, \quad g \in G.$

(ii) $\omega(bA) = A, \quad b \in B, \ A \in \mathfrak{g}.$

Then, the *curvature form* of the connection ω is a $\mathfrak{g}$-valued 2-form $\Omega \in \mathfrak{A}^2(B, \mathfrak{g})$ on B, given by

$$\Omega(X, Y) = (d\omega + \frac{1}{2}[\omega, \omega])(X, Y) = d\omega(X, Y) + [\omega(X), \omega(Y)]$$

for $X, Y \in T_b(B)$, and it satisfies the conditions:

(i) $\Omega \circ r(g) = ad(g^{-1})\Omega, \quad g \in G,$

(ii) $\Omega(X, Y) = 0,$ *if X is vertical.*

In the case of G being abelian, it is notable that

$$ad(g)A = A, \ [A, B] = 0, \text{ for all } g \in G, \ A, B \in \mathfrak{g}.$$

The conditions (i), (ii) of the connection form ω show that to give a connection on B means to give a splitting of the fundamental sequence of B, namely, a homomorphism of vector bundles $\omega: Q(M) \to L(M)$ such that $\omega \circ \lambda = \mathbf{1}$, where we denote by $\mathbf{1}$ the identical automorphism of the vector bundle $L(M)$. Applying a functor $\mathrm{Hom}(*, L)$ to the fundamental sequence, we have an exact sequence of vector bundles over M

$$0 \to \mathrm{Hom}\,(T, L) \xrightarrow{\rho} \mathrm{Hom}\,(Q, L) \xrightarrow{\lambda} \mathrm{End}(L) \to 0.$$

Taking the sheaves of germs of local cross-sections of these vector bundles, we obtain the cohomology sequence:

$$\begin{array}{ccccccc} \xrightarrow{\rho^*} H^0(M, \mathbf{Hom}(Q, L)) & \xrightarrow{\lambda^*} & H^0(M, \mathbf{End}(L)) & \xrightarrow{\delta^*} & H^1(M, \mathbf{Hom}(T, L)) \xrightarrow{\rho^*} \\ \omega & \longrightarrow & \mathbf{1} & \longrightarrow & \delta^*\mathbf{1}. \end{array}$$

Then, a connection can be regarded as a global section $\omega \in H^0(M, \mathbf{Hom}(Q, L))$ such that $\lambda^*\omega = \mathbf{1}$. Therefore, there exists a connection of B if and only if

$\delta^* \mathbf{1} = 0$, by the exactness of the cohomology sequence. Thus, we can suppose that the cohomology class $\delta^*\mathbf{1} \in H^1(M, \mathbf{Hom}(T, L))$ expresses the obstruction of the existence of connection on B. With the consideration of the class $\delta^*\mathbf{1}$, we can get the following well-known results [1].

On a C^∞ principal bundle there exists always a C^∞ connection, and on a complex analytic principal bundle there exists a C^∞ (1, 0)-connection, that is, a connection form ω of type (1, 0). Let ω be a (1, 0)-connection form of a complex analytic G-bundle $B(M, G)$, and let Ω be its curvature form. Then, the (1, 1)-component Ω^{11} of the 2-form Ω represents a cohomology class

$$\Omega^{11} \in H^1(M, \mathbf{Hom}(T, L)_h) \cong H^1(\mathfrak{A}^{1*}(M, L))$$

in the sense of the theorem of Dolbeault, and the class $-\Omega^{11}$ coincides with the obstruction class $\delta^*\mathbf{1}$ of the existence of analytic connection.

8. **The dynamic contact structures**. Now, let us consider a C^∞ circle bundle $B(M, S^1)$ over M, where S^1 is the circle regarded as a 1-dimensional abelian group. The Lie-algebra $\mathfrak{s}^1$ of the group S^1 can be identified with the additive group R of all real numbers, if we take a base $\mathbf{e} \in \mathfrak{s}^1$ to correspond to $1 \in R$. Then we have a global vector field $E \in \mathfrak{V}(B)$ called the *unit fundamental field* defined by

$$E: B \to T(B), \quad b \to b1.$$

It is clear that the vector field E is vertical and is invariant under the right translations. Therefore, a connection of the circle bundle $B(M, S^1)$ is given by a global real valued 1-form $\omega \in \mathfrak{A}^1(B)$ satisfying the following conditions.

(i) $\omega \circ r(s) = \omega$, $s \in S^1$,

(ii) $\omega(E) = 1$.

Moreover, the curvature form of the connection ω is given by $\Omega = d\omega \in \mathfrak{A}^2(B)$ and has properties:

(i) $\Omega \circ r(s) = \Omega$, $s \in S^1$,

(ii) $i(E)\Omega = 0$,

which show that the 2-form Ω on B may be regarded as a 2-form on M, namely, there exists a unique 2-form $\Omega' \in \mathfrak{A}^2(M)$ such that $\Omega = \Omega' \circ \rho$, and we can suppose Ω to be identified with Ω'. Since $d\Omega = 0$, the 2-form $\Omega \in \mathfrak{A}^2(M)$ represents a cohomology class $\Omega \in H^2(\mathfrak{A}^*(M))$, which does not depend on the choice of connection on B.

Let us consider an exact sequence of abelian groups

$$0 \to Z \to R \to S^1 \to 0.$$

Taking the sheaves on M of germs of C^∞-functions with values in Z, R, S^1 respectively, we get an exact sequence of sheaves of abelian groups

$$0 \to Z \to \mathbf{R} \to \mathbf{S}^1 \to 0.$$

Since the sheaf $\mathbf{R}$ is fine, by the cohomology sequence, we have the bijection

$$0 \to H^1(M, \mathbf{S}^1) \xrightarrow{\delta^*} H^2(M, Z) \to 0$$
$$\xi \longrightarrow \chi(\xi),$$

where a cohomology class $\xi \in H^1(M, \mathbf{S}^1)$ expresses an S^1-bundle structure and the class $\delta^*\xi = \chi(\xi)$ is the Euler-Poincaré class of the S^1-bundle ξ. On the other hand, by the cohomology sequence for the constant coefficients, we get homomorphisms

$$\to H^1(M, S^1) \to H^2(M, Z) \to H^2(M, R) \to$$
$$\chi(\xi) \longrightarrow \Omega,$$

where Ω is the curvature form of a connection on an S^1-bundle B of bundle structure ξ, and it represents a cohomology class

$$\Omega \in H^2(\mathfrak{A}^*(M)) \cong H^2(M, R),$$

in the sense of the theorem of de Rham.

A contact form $\omega \in \mathfrak{A}^1(B)$, $\omega \wedge d\omega^n \neq 0$, on a C^∞ S^1-bundle $B(M, S^1)$ over a $2n$-dimensional manifold M is said to be *dynamic*, if the 1-form ω defines also a connection on B. Then, the canonical field E of the contact structure coincides with the unit fundamental field of the S^1-bundle B, since $i(E)\omega = 1$ and $i(E)d\omega = 0$. In this case, the curvature form $\Omega = d\omega \in \mathfrak{A}^2(M)$ regarded as a 2-form on M becomes a symplectic form on M, since it satisfies the conditions $\Omega^n \neq 0$ and $d\Omega = 0$. Moreover, the closed 2-form Ω expresses an integral cocycle which represents the Euler-Poincaré class of the S^1-bundle B, in the sense of the theorem of de Rham.

Through the bijection $\delta^* : H^1(M, \mathbf{S}^1) \to H^2(M, Z)$, we get easily the following result derived by Boothby and Wang [2].

Let Ω be a symplectic form on a $2n$-dimensional C^∞ manifold M. Then, there exist a C^∞ circle bundle $B(M, S^1)$ over M and a C^∞ dynamic contact form ω on B such that $d\omega = \Omega$, if any only if the closed 2-form Ω represents an integral cohomology class of M.

Next, we make researches on the complex analytic C^*-bundle. Let M be a complex analytic manifold, and let us consider an exact sequence of abelian groups

$$0 \to Z \xrightarrow{\lambda} C \xrightarrow{\exp} C^* \to 0.$$

Taking the sheaves on M of germs of holomorphic functions with values in

Z, C, C^* respetively, we get an exact sequence of sheaves of abelian groups

$$0 \to Z \xrightarrow{\lambda} \mathbf{C}_h \xrightarrow{\exp} \mathbf{C}_h^* \to 0.$$

Then, we have the cohomology sequence

$$\xrightarrow{\exp^*} H^1(M, \mathbf{C}_h^*) \xrightarrow{\delta^*} H^2(M, Z) \xrightarrow{\lambda^*} H^2(M, \mathbf{C}_h) \to$$
$$\xi \longrightarrow c_1(\xi), \quad \Theta \longrightarrow \Theta^{02},$$

where the class $\delta^*\xi = c_1(\xi) \in H^2(M, Z)$ expresses the Chern class of an analytic C^*-bundle structure $\xi \in H^1(M, \mathbf{C}_h^*)$. Moreover, if we denote an integral 2-cocycle of M by a complex valued C^∞ closed 2-form $\Theta \in \mathfrak{A}^2(M, C)$ in the sense of the theorem of de Rham, then a cohomology class $\lambda^* \Theta \in H^2(M, \mathbf{C}_h)$ is represented by the (0, 2)-component $\Theta^{02} \in \mathfrak{A}^{02}(M, C)$ of the 2-form Θ in the sense of the theorem of Dolbeault, that is,

$$\Theta \in H^2(M, Z) \to H^2(M, C) \cong H^2(\mathfrak{A}^*(M, C)),$$
$$\Theta^{02} \in H^2(M, \mathbf{C}_h) \cong H^2(\mathfrak{A}^{0*}(M, C)).$$

Let Ω be a curvature form of any C^∞ connection on an analytic C^*-bundle $B(M, C^*)$. Then, the closed 2-form $(1/2\pi i)\Omega \in \mathfrak{A}^2(M, C)$ represents an integral cohomology class of M which expresses the Chern class of B, being independent on the choice of connection on B.

A complex valued C^∞ 1-form $\omega \in \mathfrak{A}^1(M, C)$ on a complex $(2n+1)$-dimensional analytic manifold M is called a C^∞ *contact form* on M, if it satisfies the condition

$$(\omega \wedge d\omega^n) \wedge (\bar{\omega} \wedge d\bar{\omega}^n) \neq 0$$

on M. It is clear that, if the 1-form ω is holomorphic, the condition of contact form is reduced to $\omega \wedge d\omega^n \neq 0$. In this case, the contact form ω is said to be *analytic*.

A complex valued C^∞ 2-form $\Omega \in \mathfrak{A}^2(M, C)$ on a complex $2n$-dimensional analytic manifold M is called a C^∞ *symplectic form* on M, if it satisfies the conditions $\Omega^n \wedge \bar{\Omega}^n \neq 0$ and $d\Omega = 0$ on M. It is clear that, if the 2-form Ω is holomorphic, the conditions of symplectic form are reduced to $\Omega^n \neq 0$ and $d\Omega = 0$. In this case, the symplectic form Ω is said to be *analytic*.

Let $B(M, C^*)$ be a complex analytic C^*-bundle over a complex $2n$-dimensional manifold M. A C^∞ contact form $\omega \in \mathfrak{A}^1(B, C)$ on B is said to be *dynamic* if ω defines also a C^∞ connection on B. Then the curvature form $\Omega = d\omega \in \mathfrak{A}^2(M, C)$ regarded as a 2-form on M becomes a C^∞ symplectic form on M. In particular, if ω is an analytic dynamic contact form on B, then the curvature form $\Omega = d\omega$ becomes an analytic symplectic form on M.

THEOREM 6. *Let M be a complex $2n$-dimensional analytic manifold,* *and let* Ω *be a* C^∞ *symplectic form on* M.

1° *There exist a complex analytic* C^**-bundle* $B(M, C^*)$ *over* M *and a* C^∞ *dynamic contact form* ω *on* B *such that* $d\omega = \Omega$, *if and only if the closed 2-form* $(1/2\pi i)\Omega$ *determines an integral cocycle on* M *and the* $(0, 2)$*-component* Ω^{02} *of the 2-form* Ω *is coboundary with respect to the* d''*-cohomology.*

2° *There exists locally a* $(1, 0)$ *dynamic contact form* ω *of* $B(M, C^*)$ *such that* $d\omega = \Omega$, *if and only if the closed 2-form* $(1/2\pi i)\Omega$ *determines an integral cocycle on* M *and* $\Omega^{02} = 0$. *Moreover, if* $H^1(M, \mathbf{C}_h) = 0$, *then the* $(1, 0)$ *dynamic contact form* ω *can be defined globally on* B.

PROOF. Let us consider the cohomology sequence

$$\to H^1(M, \mathbf{C}_h) \xrightarrow{\exp^*} H^1(M, \mathbf{C}_h^*) \xrightarrow{\delta^*} H^2(M, Z) \xrightarrow{\lambda^*} H^2(M, \mathbf{C}_h) \to .$$

A class $\xi \in H^1(M, \mathbf{C}_h^*)$ determines an analytic C^*-bundle $B(M, C^*)$. Let ω be a C^∞ dynamic contact form on B, and let $\Omega = d\omega$ be its curvature form. Then the closed 2-form $(1/2\pi i)\Omega$ represents the Chern class $\delta^*\xi = c_1(\xi)$ of B, which is an integral cohomology class of M. The $(0, 2)$-component Ω^{02} of the 2-form Ω represents the class $(2\pi i)\lambda^*\circ\delta^*\xi \in H^2(M, \mathbf{C}_h)$, which is null by the exactness of the cohomology sequence. Therefore, Ω^{02} is coboundary.

Conversely, if Ω^{02} is coboundary, by the exactness of the sequence, there exists an analytic C^*-bundle structure $\xi \in H^1(M, \mathbf{C}_h^*)$ such that

$$\delta^*\xi = (1/2\pi i)\Omega \in H^2(M, Z).$$

Taking a C^*-bundle $B(M, C^*)$ of the structure ξ and a C^∞ connection form θ on B, we can find a global 1-form $\varphi \in \mathfrak{A}^1(M, C)$ on M such that $d\varphi = \Omega - d\theta$, since both the closed 2-forms Ω and $d\theta$ represent the same cohomology class $2\pi i c_1(\xi) \in H^2(M, C)$. Then, the 1-form $\omega = \theta + \varphi \in \mathfrak{A}^1(B, C)$ becomes a C^∞ dynamic contact form on B such that $d\omega = \Omega$.

In particular, when $\Omega^{02} = 0$, if we take the connection form θ to be of type $(1, 0)$, then the 1-form $\varphi = \varphi^{10} + \varphi^{01}$ satisfies the condition $d''\varphi^{01} = 0$. Hence, there exists a local C^∞-function f defined on a neighborhood of any point $x \in M$ such that $d''f = \varphi^{01}$. Setting $\psi = \varphi^{10} - d'f$, we have

$$d\psi = d\varphi^{10} - d''d'f = d\varphi^{10} + d'\varphi^{01} = d\varphi = \Omega - d\theta.$$

Then, the $(1, 0)$-form $\omega = \theta + \psi$ becomes a local dynamic contact form such that $d\omega = \Omega$.

Moreover, if $H^1(M, \mathbf{C}_h) = 0$, we can take the C^∞-function f globally on M, through the Dolbeault isomorphism $H^1(M, \mathbf{C}_h) \cong H^1(\mathfrak{A}^{0*}(M, C))$. Then, the $(1, 0)$-form ω is also defined globally on B. For instance, if M is a Stein manifold, it holds that $H^1(M, \mathbf{C}_h) = 0$.

Here, if the 2-form Ω is holomorphic, that is, $\Omega^{11} = \Omega^{02} = 0$ and $d''\Omega = 0$, then the (1, 0)-form ω is also holomorphic, since $d''\omega = \Omega^{11} = 0$ Thus we get the following

COROLLARY. *Let Ω be an analytic symplectic form on M. Then, there exists locally an analytic dynamic contact form ω of $B(M, C^*)$ such that $d\omega = \Omega$, if and only if the closed 2-form $(1/2\pi i)\Omega$ determines an integral cocycle on M. Moreover, if $H^1(M, \mathbf{C}_h) = 0$, then the analytic dynamic contact form ω can be defined globally on B.*

In this case, it is assured that there exists an analytic connection on B by the condition $\Omega^{11} = 0$, because the form Ω^{11} represents the obstruction class of the existence of analytic connection.

9. Infinitesimal cosymplectic transformations of a cosymplectic manifold A real C^∞-manifold M of odd dimension $2n+1$ is said to be a *cosymplectic manifold*, if a global 1-form $\omega \in \mathfrak{A}^1(M)$ and a global 2-form $\pi \in \mathfrak{A}^2(M)$ are given so that they satisfy the conditions:

$$\omega \wedge \pi^n \neq 0, \ d\omega = 0, \ d\pi = 0$$

on M, where the forms ω, π are called the *cosymplectic forms* on M. Then, the forms ω, π give an almost contact structure on M, and so we have the canonical field E and the Lagrange brackets l, L defined in §2. By the Proposition 3, there exists a natural bijection α between the $\mathfrak{A}$-modules $\mathfrak{B}$ and $\mathfrak{A}^1$ given by

$$\alpha : \mathfrak{B} \to \mathfrak{A}^1, \qquad X \to \varphi,$$
$$\alpha(X) = i(X)\pi + \omega(X)\omega, \qquad \alpha^{-1}(\varphi) = -\,l(\varphi) + \varphi(E)E.$$

Since the cosymplectic forms ω, π are closed, it holds that

$$\mathfrak{L}(X)\omega = di(X)\omega, \ \mathfrak{L}(X)\pi = di(X)\pi, \quad X \in \mathfrak{B}.$$

Let ω, π be the cosymplectic forms on M. A vector field $X \in \mathfrak{B}$ is called an *infinitesimal cosymplectic transformation*, if there exists a function $k \in \mathfrak{A}$ such that

$$\mathfrak{L}(X)\omega = dk, \qquad \mathfrak{L}(X)\pi = \omega \wedge dk.$$

Moreover, a vector field $X \in \mathfrak{B}$ is called an *infinitesimal automorphism* of the cosymplectic structure, if $\mathfrak{L}(X)\omega = 0$ and $\mathfrak{L}(X)\pi = 0$. Let $\mathfrak{C}, \mathfrak{C}_0$ denote respectively, the R-module of all infinitesimal cosymplectic transformations, and the R-module of all infinitesimal automorphisms of the cosymplectic structure. Obviously, $\mathfrak{C}_0$ is an R-submodule of $\mathfrak{C}$.

PROPOSITION 8. *For an infinitesimal cosymplectic transformation $X \in \mathfrak{C}$, the differential $dk \in \mathfrak{A}^1$ such that $\mathfrak{L}(X)\omega = dk$, is given by $dk = d\omega\,(X)$.*

PROOF. If $X \in \mathfrak{C}$, then $\mathfrak{L}(X)\omega = di(X)\omega = dk$.

A p-form $\varphi \in \mathfrak{A}^p$ is said to be *E-invariant*, if $\mathfrak{L}(E)\varphi = 0$. For instance, the 1-form ω is E-invariant, since $\mathfrak{L}(E)\omega = d\omega(E) = 0$.

THEOREM 7. *Let ω, π be the cosymplectic forms on M. Let $\mathfrak{Z}^1$ denote the R-module of all closed 1-forms, and let $\mathfrak{G}^1$ be the R-modules of all closed E-invariant 1-forms. Then, the R-linear map $\alpha: \mathfrak{B} \to \mathfrak{A}^1$ gives a bijection between the R-modules $\mathfrak{C}$ and $\mathfrak{Z}^1$. In particular. the map α gives a bijection between the R-modules $\mathfrak{C}_0$ and $\mathfrak{G}^1$.*

Accordingly, we get an exact and commutative diagram of R-modules

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathfrak{C}_0 & \longrightarrow & \mathfrak{C} & \xrightarrow{d\circ\omega} & d\mathfrak{A} \\ & & \uparrow\downarrow & & \alpha^{-1}\uparrow\downarrow\alpha & & \| \\ 0 & \longrightarrow & \mathfrak{G}^1 & \longrightarrow & \mathfrak{Z}^1 & \xrightarrow{\mathfrak{L}(E)} & d\mathfrak{A} \end{array}$$

where $d\mathfrak{A}$ is an R-submodule of $\mathfrak{A}^1$ consisting of all differentials of functions in $\mathfrak{A}$.

PROOF. It is known that the map $\alpha: \mathfrak{B} \to \mathfrak{A}^1$ is a bijection.

Assume that $X \in \mathfrak{C}$. Setting $\varphi = \alpha(X)$, we have $\varphi = i(X)\pi + \omega(X)\omega$, and

$$\begin{aligned} d\varphi &= di(X)\pi + d\omega(X) \wedge \omega = \mathfrak{L}(X)\pi + \mathfrak{L}(X)\omega \wedge \omega \\ &= \omega \wedge dk + dk \wedge \omega = 0. \end{aligned}$$

It follows that φ is closed.

Conversely, assume that $\varphi \in \mathfrak{Z}^1$. Setting $X = \alpha^{-1}(\varphi)$, we have

$$\begin{aligned} X &= -\,l(\varphi) + \varphi(E)E, \\ \mathfrak{L}(X)\omega &= di(X)\omega = d\{\varphi(E)i(E)\omega\} = d(\varphi(E)), \\ \mathfrak{L}(X)\pi &= di(X)\pi = d\{-\,i(l(\varphi))\pi\} \\ &= d\{-\,\varphi(E)\omega + \varphi\} = \omega \wedge d(\varphi(E)). \end{aligned}$$

It follows that $X \in \mathfrak{C}$. Moreover, we can see that

$$dk = d(\varphi(E)) = \mathfrak{L}(E)\varphi, \; dk = d\omega(X).$$

Therefore, $X = \alpha^{-1}(\varphi) \in \mathfrak{C}_0$, if and only if $\mathfrak{L}(E)\varphi = 0$.

COROLLARY. $E = \alpha^{-1}(\omega) \in \mathfrak{C}_0$.

Let $X, Y \in \mathfrak{C}$ be two infinitesimal cosymplectic transformations. Setting $dk = d\omega(X)$, $dh = d\omega(Y)$, we have easily

$$\begin{aligned} \mathfrak{L}([X, Y])\omega &= d(Xh - Yk), \\ \mathfrak{L}([X, Y])\pi &= 2dk \wedge dh + \omega \wedge d(Xh - Yk). \end{aligned}$$

This shows that $[X, Y] \in \mathfrak{C}$ if and only if $dk \wedge dh = 0$. Hence, the R-module

$\mathfrak{C}$ does not constitute in general a Lie-algebra with respect to the prouct $[X, Y]$. However, the R-module $\mathfrak{C}_0$ becomes a Lie-algebra.

10. Infinitesimal conformal transformations of a cosymplectic manifold. Now, we make some remarks cn the infinitesimal conformal transformations of the cosymplectic structure.

Let ω, π be the cosymplectic forms on a $(2n+1)$-dimensional C^∞-manifold M. A vector field $X \in \mathfrak{B}$ is called an *infinitesimal conformal transformation* of π, if there exists a function $h \in \mathfrak{A}$ such that $\mathfrak{L}(X)\pi = h\pi$. Then, we get the following result same as the case of symplectic structure in §6.

PROPOSITION 9. *Assume that $n > 1$. Let $X \in \mathfrak{B}(M)$ be a vector field such that $\mathfrak{L}(X)\pi = h\pi$.*

1° *If M is connected, then h is a constant.*

2° *If M is compact, then $h = 0$.*

PROOF. The manifold M is orientable, since there exists a global non-zero $(2n+1)$-form $\omega \wedge \pi^n \in \mathfrak{A}^{2n+1}(M)$. By assumption, $\mathfrak{L}(X)\pi = di(X)\pi = h\pi$. Applying the exterior derivation d, we have $dh \wedge \pi = 0$, and hence $dh \wedge \pi^2 = 0$. Therefore, applying an inner prcduct $i(Y)$ for a vector field $Y \in \mathfrak{B}$, we get

$$\begin{aligned} &i(Y)(dh \wedge \pi^2) = 0 \\ &= (Yh)\pi^2 - dh \wedge 2i(Y)\pi \wedge \pi = (Yh)\pi^2. \end{aligned}$$

Since $\pi^2 \neq 0$, we have $Yh = 0$ for any vector field $Y \in \mathfrak{B}$. This proves that h is constant on a connected component of M. Moreover, if $h \neq 0$, then it holds that

$$d\left(\frac{-1}{h}\omega \wedge i(X)\pi \wedge \pi^{n-1}\right) = \frac{1}{h}\omega \wedge h\pi \wedge \pi^{n-1} = \omega \wedge \pi^n.$$

This shows that the form $\omega \wedge \pi^n$ becomes a coboundary with respect to the d-cohomology, On the other hand, the non-zero $(2n+1)$-form $\omega \wedge \pi^n$ represents a base of the real cohomology group $H^{2n+1}(M, R) \cong R$ of the orientable manifold M, provided M to be connected and compact. This is impossible. Hence $h = 0$.

Let ω, π be the cosymplectic forms on M. A vector field $X \in \mathfrak{B}$ is called an *infinitesimal conformal transformation* of ω, if there exists a function $k \in \mathfrak{A}$ such that $\mathfrak{L}(X)\omega = k\omega$. Then we can see clearly $k = E\omega(X)$. We denote by $\mathfrak{T}$, $\mathfrak{T}_0$ respectively, the R-mcdule of all infinitesimal conformal transformations of ω, and an R-submcdule of $\mathfrak{T}$ ccnsisting of all vector fields $X \in \mathfrak{B}$ such that $\mathfrak{L}(X)\omega = 0$.

THEOREM 8. *Let ω, π be the cosymplectic forms on M, and let $\mathfrak{D}^1$, $\mathfrak{D}^1_0$*

denote respectively the R-module of all 1-forms $\varphi \in \mathfrak{A}^1$ *such that* $d\varphi(E) \wedge \omega = 0$, *and an R-submodule of* $\mathfrak{D}^1$ *consisting of all 1-forms* $\varphi \in \mathfrak{A}^1$ *such that* $d\varphi(E) = 0$. *Then, the R-linear map* $\alpha : \mathfrak{X} \to \mathfrak{A}^1$, *defined by*

$$\alpha(X) = i(X)\pi + \omega(X)\omega, \ X \in \mathfrak{X},$$

gives a bijection between the R-modules $\mathfrak{T}$ *and* $\mathfrak{D}^1$. *In particular, the map* α *gives a bijection between the R-modules* $\mathfrak{T}_0$ *and* $\mathfrak{D}^1_0$.

Accordingly, we get an exact and commutative diagram of R-modules

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{T}_0 \arrow[r] \arrow[d, shift left] & \mathfrak{T} \arrow[r, "E\circ\omega"] \arrow[d, shift left, "\alpha"] & \mathfrak{A} \arrow[d, equal] \\
0 \arrow[r] & \mathfrak{D}^1_0 \arrow[r] \arrow[u, shift left] & \mathfrak{D}^1 \arrow[r, "E\circ i(E)"] \arrow[u, shift left, "\alpha^{-1}"] & \mathfrak{A}.
\end{tikzcd}$$

PROOF. It is known that the map $\alpha : \mathfrak{X} \to \mathfrak{A}^1$ is a bijection whose inverse map is given by

$$X = \alpha^{-1}(\varphi) = -\, l(\varphi) + \varphi(E)E$$

for $\varphi \in \mathfrak{A}^1$. This implies $\omega(X) = \varphi(E)$, and hence

$$\mathfrak{L}(X)\omega = di(X)\omega = d\varphi(E).$$

Therefore, it holds that $d\varphi(E) = \mathfrak{L}(X)\omega = k\omega$ if and only if $d\varphi(E) \wedge \omega = 0$. Moreover, we have $k = E\omega(X) = E\varphi(E)$.

If $\varphi \in D^1_0$, then clearly $E\varphi(E) = i(E)d\varphi(E) = 0$. Conversely, if $\varphi \in \mathfrak{D}^1$ and $E\varphi(E) = 0$, then

$$d\varphi(E) = -\, E\varphi(E)\omega + d\varphi(E) = -\, i(E)(d\varphi(E) \wedge \omega) = 0.$$

This shows that $\varphi \in \mathfrak{D}^1_0$.

PROPOSITION 10. *If* $X \in \mathfrak{T}$, *then* $\mathfrak{L}(X)\pi = d\varphi$, *where* $\varphi = \alpha(X)$.

PROOF. Setting $\varphi = \alpha(X) \in \mathfrak{D}^1$ for $X \in \mathfrak{T}$, we have

$$X = -\, l(\varphi) + \varphi(E)E, \ i(X)\pi = -\, \varphi(E)\omega + \varphi.$$

Since $\varphi \in \mathfrak{D}^1$, we can see that

$$\mathfrak{L}(X)\pi = di(X)\pi = -\, d\varphi(E) \wedge \omega + d\varphi = d\varphi.$$

By this proposition, if $X \in \mathfrak{T}_0$ and the 1-form $\varphi = \alpha(X)$ is closed, then X becomes an infinitesimal automorphism of the cosymplectic structure.

Moreover, it is obvious that the R-module $\mathfrak{T}$ constructs a Lie-algebra with respect to the product $[X, Y]$, and $\mathfrak{T}_0$ becomes a subalgebra of $\mathfrak{T}$.

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