MODULUS OF RINGS IN SPACE

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1. Let R^n be a ring in the n-dimensional Euclidean space E^n, which is defined as a bounded domain homeomorphic to the domain between two concentric spheres in E^n. The complement of R^n consists of two components C_1^n, C_2^n, where C_1^n is bounded and C_2^n is unbounded, and $B_1^n = C_1^n \cap \overline{R^n}$ is called the inner boundary and $B_2^n = C_2^n \cap \overline{R^n}$ the outer one of R^n. By an arc γ we mean a subset in E^n homeomorphic to the unit interval $[0,1]$. Let $\{\gamma\}$ be the family of all rectifiable arcs in R^n joining B_1^n, B_2^n, and let P be the family of all non-negative lower semi-continuous functions $\rho(x)$ in R^n.

Put

$$L_\rho(\gamma) = \inf_{\gamma\in\{\gamma\}} \int_\gamma \rho(x(s))\, ds,$$

$$V_\rho(R^n) = \iint \cdots \int_{R^n} \rho(x)^n \, d\tau_n,$$

where $x = x(s)$ $(0 \leqq s \leqq l)$ is the equation by arc-length s of γ, and $d\tau_n$ is the n-dimensional volume element, then by following Väisälä [4], the quantity

$$\frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} \sup_{\rho\in P} \frac{(L_\rho(\gamma))^n}{V_\rho(R^n)}$$

is called the modulus of R^n, which is denoted by $\operatorname{mod} R^n$.

Now, assume that C_1^n contains the origin $x = 0$, and perform the following transformation of coordinates:

$$x_1 = r \cos\theta_1,$$
$$x_2 = r \sin\theta_1 \cos\theta_2,$$
$$x_3 = r \sin\theta_1 \sin\theta_2 \cos\theta_3,$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$x_{n-1} = r \sin\theta_1 \sin\theta_2 \sin\theta_3 \cdots \sin\theta_{n-2} \cos\theta_{n-1},$$
$$x_n = r\sin\theta_1 \sin\theta_2 \sin\theta_3 \cdots \sin\theta_{n-2} \sin\theta_{n-1}.$$

*) Dedicated to Professor Kunugui on his Sixtieth birthday.

We denote by $l_{\theta_1\theta_2\ldots\theta_{n-1}}$ the intersection of the half straight line determined by a pair of $(\theta_1, \theta_2, \cdots, \theta_{n-1})$ with R^n, and by $l(\theta_1, \theta_2, \cdots, \theta_{n-1})$ its logarithmic length:

$$l(\theta_1, \theta_2, \cdots, \theta_{n-1}) = \int_{l_{\theta_1\theta_2\ldots\theta_{n-1}}} \frac{dr}{r}.$$

2. Using Hölder's inequality, we have

$$L_\rho(\gamma) \leqq \int_{l_{\theta_1\theta_2\ldots\theta_{n-1}}} \rho\, dr \leqq l(\theta_1, \cdots, \theta_{n-1})^{\frac{n-1}{n}} \left(\int_{l_{\theta_1\theta_2\ldots\theta_{n-1}}} \rho^n r^{n-1} dr \right)^{\frac{1}{n}},$$

so that

$$\frac{(L_\rho(\gamma))^n}{l(\theta_1, \cdots, \theta_{n-1})^{n-1}} \leqq \int_{l_{\theta_1\theta_2\ldots\theta_{n-1}}} \rho^n r^{n-1} dr.$$

Multiply both sides by $(\sin\theta_1)^{n-2}(\sin\theta_2)^{n-3}\cdots(\sin\theta_{n-2})$ and integrate them with respect to $\theta_1, \theta_2, \cdots, \theta_{n-2}, \theta_{n-1}$, then we have

$$(L_\rho(\gamma))^n \int_0^{2\pi}\int_0^{\pi}\cdots\int_0^{\pi} \frac{(\sin\theta_1)^{n-2}(\sin\theta_2)^{n-3}\cdots(\sin\theta_{n-2})}{l(\theta_1, \cdots, \theta_{n-1})^{n-1}}\, d\theta_1 \cdots d\theta_{n-2}\, d\theta_{n-1}$$

$$\leqq \int_0^{2\pi}\int_0^{\pi}\cdots\int_0^{\pi}\int_{l_{\theta_1\theta_2\ldots\theta_{n-1}}} \rho^n r^{n-1}(\sin\theta_1)^{n-2}(\sin\theta_2)^{n-3}\cdots$$
$$\cdots(\sin\theta_{n-2})\, dr d\theta_1 \cdots d\theta_{n-2}\, d\theta_{n-1},$$

so that

$$\sup_{\rho\in P} \frac{(L_\rho(\gamma))^n}{V_\rho(R^n)} \leqq$$
$$1 \Big/ \int_0^{2\pi}\int_0^{\pi}\cdots\int_0^{\pi} \frac{(\sin\theta_1)^{n-2}(\sin\theta_2)^{n-3}\cdots(\sin\theta_{n-2})}{l(\theta_1, \cdots, \theta_{n-1})^{n-1}}\, d\theta_1 \cdots d\theta_{n-2}\, d\theta_{n-1}.$$

Hence we have the following space form of Akaza-Kuroda's inequality [1] of plane rings.

THEOREM 1. *The modulus of R^n satisfies the inequality*

$$\operatorname{mod} R^n \leqq \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)\int_0^{2\pi}\int_0^{\pi}\cdots\int_0^{\pi} \frac{(\sin\theta_1)^{n-2}(\sin\theta_2)^{n-3}\cdots(\sin\theta_{n-2})}{l(\theta_1, \cdots, \theta_{n-1})^{n-1}}\, d\theta_1\cdots d\theta_{n-2} d\theta_{n-1}}.$$

3. Further, as also remarked in [1], using Schwarz's inequality, we have

$$\left(\frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}\right)^2 = \left(\int_0^{2\pi}\int_0^{\pi}\cdots\int_0^{\pi}(\sin\theta_1)^{n-2}(\sin\theta_2)^{n-3}\cdots(\sin\theta_{n-2})d\theta_1\cdots d\theta_{n-2}\,d\theta_{n-1}\right)^2$$

$$\leqq \int_0^{2\pi}\int_0^{\pi}\cdots\int_0^{\pi}\frac{(\sin\theta_1)^{n-2}(\sin\theta_2)^{n-3}\cdots(\sin\theta_{n-2})}{l(\theta_1,\cdots,\theta_{n-1})^{n-1}}d\theta_1\cdots d\theta_{n-2}\,d\theta_{n-1}$$

$$\times\int_0^{2\pi}\int_0^{\pi}\cdots\int_0^{\pi}(\sin\theta_1)^{n-2}(\sin\theta_2)^{n-3}\cdots(\sin\theta_{n-2})l(\theta_1,\cdots,\theta_{n-1})^{n-1}\,d\theta_1\cdots d\theta_{n-2}\,d\theta_{n-1}.$$

Hence we have the following space form of Rengel's inequality.

COROLLARY 1.

$$\text{mod } R^n \leqq \frac{\Gamma\left(\frac{n}{2}\right)}{2\pi^{\frac{n}{2}}}\int_0^{2\pi}\int_0^{\pi}\cdots\int_0^{\pi}(\sin\theta_1)^{n-2}(\sin\theta_2)^{n-3}\cdots$$
$$\cdots(\sin\theta_{n-2})\times l(\theta_1,\cdots,\theta_{n-1})^{n-1}\,d\theta_1\cdots d\theta_{n-2}\,d\theta_{n-1}.$$

4. Next, assume in particular that the inner boundary is the unit sphere $|x| = 1$ and that the outer is starlike with respect to $x = 0$, in other words, each half-line starting from $x = 0$ has a single point common with the outer boundary.

Denote by $r = f(\theta_1,\cdots,\theta_{n-1})$ the value of r at such a single common point, then

$$l(\theta_1,\cdots,\theta_{n-1}) = \int_{l_{\theta_1\ldots\theta_{n-1}}}\frac{dr}{r} = \int_1^{f(\theta_1,\ldots,\theta_{n-1})}\frac{dr}{r} = \log f(\theta_1,\cdots,\theta_{n-1}).$$

Using Hölder's inequality, we have

$$\frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} = \left(\int_0^{2\pi}\int_0^{\pi}\cdots\int_0^{\pi}\frac{(\sin\theta_1)^{n-2}(\sin\theta_2)^{n-3}\cdots(\sin\theta_{n-2})}{(\log f(\theta_1,\cdots,\theta_{n-1}))^{n-1}}d\theta_1\cdots d\theta_{n-2}\,d\theta_{n-1}\right)^{\frac{1}{n}}$$

$$\times\left(\int_0^{2\pi}\int_0^{\pi}\cdots\int(\log f(\theta_1,\cdots,\theta_{n-1}))(\sin\theta_1)^{n-2}(\sin\theta_2)^{n-3}\cdots\right.$$
$$\left.\cdots(\sin\theta_{n-2})\,d\theta_1\cdots d\theta_{n-2}\,d\theta_{n-1}\right)^{\frac{n-1}{n}}.$$

Hence we have

$$2\pi^{\frac{n}{2}}\Big/\Gamma\left(\frac{n}{2}\right)\int_0^{2\pi}\int_0^{\pi}\cdots\int_0^{\pi}\frac{(\sin\theta_1)^{n-2}(\sin\theta_2)^{n-3}\cdots(\sin\theta_{n-2})}{(\log f(\theta_1,\cdots,\theta_{n-1}))^{n-1}}d\theta_1\cdots d\theta_{n-2}\,d\theta_{n-1}$$

$$\leqq \left(\frac{\Gamma\left(\frac{n}{2}\right)}{2\pi^{\frac{n}{2}}} \int_0^{2\pi}\int_0^{\pi}\cdots\int_0^{\pi} (\log f(\theta_1, \cdots, \theta_{n-1}))(\sin\theta_1)^{n-2}(\sin\theta_2)^{n-3}\cdots \right.$$

$$\left. \cdots(\sin\theta_{n-2})\, d\theta_1\cdots d\theta_{n-2}\, d\theta_{n-1} \right)^{n-1}$$

$$= \left(\frac{\Gamma\left(\frac{n}{2}\right)}{2\pi^{\frac{n}{2}}} \int\int\cdots\int_{S^{n-1}} \log f(\theta_1, \cdots, \theta_{n-1})\, d\sigma_{n-1} \right)^{n-1},$$

where S^{n-1} denotes the unit sphere $|x| = \sqrt{x_1^2 + \cdots + x_n^2} = 1$ and $d\sigma_{n-1}$ means the surface element on $S^{n}.^{-1}$ Since $-\log f$ is a convex function of f, we can see

$$\frac{\Gamma\left(\frac{n}{2}\right)}{2\pi^{\frac{n}{2}}} \int\int\cdots\int_{S^{n-1}} \log f(\theta_1, \cdots, \theta_{n-1})\, d\sigma_{n-1}$$

$$\leqq \log\left(\frac{\Gamma\left(\frac{n}{2}\right)}{2\pi^{\frac{n}{2}}} \int\int\cdots\int_{S^{n-1}} f(\theta_1, \cdots, \theta_{n-1})\, d\sigma_{n-1} \right).$$

Using again Hölder's inequality, we get

$$\int\int\cdots\int_{S^{n-1}} f(\theta_1, \cdots, \theta_{n-1})\, d\sigma_{n-1}$$

$$\leqq \left(\int_0^{2\pi}\int_0^{\pi}\cdots\int_0^{\pi} (f(\theta_1, \cdots, \theta_{n-1}))^n (\sin\theta_1)^{n-2}(\sin\theta_2)^{n-3}\cdots \right.$$

$$\left. \cdots(\sin\theta_{n-2})\, d\theta_1\cdots d\theta_{n-2}\, d\theta_{n-1} \right)^{\frac{1}{n}}$$

$$\times \left(\int_0^{2\pi}\int_0^{\pi}\cdots\int_0^{\pi} (\sin\theta_1)^{n-2}(\sin\theta_2)^{n-3}\cdots(\sin\theta_{n-2})d\theta_1\cdots d\theta_{n-2}d\theta_{n-1} \right)^{\frac{n-1}{n}}$$

$$= \left(\frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} \right)^{\frac{n-1}{n}} \left(\int_0^{2\pi}\int_0^{\pi}\cdots\int_0^{\pi} (f(\theta_1, \cdots, \theta_{n-1})^n (\sin\theta_1)^{n-2}(\sin\theta_2)^{n-3}\cdots \right.$$

$$\left. \cdots(\sin\theta_{n-2})\, d\theta_1\cdots d\theta_{n-2}\, d\theta_{n-1} \right)^{\frac{1}{n}},$$

so that

$$\frac{\Gamma\left(\frac{n}{2}\right)}{2\pi^{\frac{n}{2}}}\iint\cdots\int_{S^{n-1}} f(\theta_1,\cdots,\theta_{n-1})\,d\sigma_{n-1}$$

$$\leqq\left(\frac{\Gamma\left(\frac{n}{2}\right)}{2\pi^{\frac{n}{2}}}\int_0^{2\pi}\int_0^{\pi}\cdots\int_0^{\pi}(f(\theta_1,\cdots,\theta_{n-1}))^n(\sin\theta_1)^{n-2}(\sin\theta_2)^{n-3}\cdots\right.$$

$$\left.\cdots(\sin\theta_{n-2})\,d\theta_1\cdots d\theta_{n-2}\,d\theta_{n-1}\right)^{\frac{1}{n}}.$$

Combining in turn the above relations, we have finally

$$2\pi^{\frac{n}{2}}\Big/\Gamma\left(\frac{n}{2}\right)\int_0^{2\pi}\int_0^{\pi}\cdots\int_0^{\pi}\frac{(\sin\theta_1)^{n-2}(\sin\theta_2)^{n-3}\cdots(\sin\theta_{n-2})}{(\log f(\theta_1,\cdots,\theta_{n-1}))^{n-1}}\,d\theta_1\cdots d\theta_{n-2}\,d\theta_{n-1}$$

$$\leqq\left[\log\left\{\frac{\Gamma\left(\frac{n}{2}\right)}{2\pi^{\frac{n}{2}}}\int_0^{2\pi}\int_0^{\pi}\cdots\int_0^{\pi}(f(\theta_1,\cdots,\theta_{n-1}))^n(\sin\theta_1)^{n-2}(\sin\theta_2)^{n-3}\cdots\right.\right.$$

$$\left.\left.\cdots(\sin\theta_{n-2})\,d\theta_1\cdots d\theta_{n-2}\,d\theta_{n-1}\right\}^{\frac{1}{n}}\right]^{n-1}$$

$$=\left[\log\left(\frac{1}{n}\int_0^{2\pi}\int_0^{\pi}\cdots\int_0^{\pi}(f(\theta_1,\cdots,\theta_{n-1}))^n(\sin\theta_1)^{n-2}(\sin\theta_2)^{n-3}\cdots\right.\right.$$

$$\left.\left.\cdots(\sin\theta_{n-2})\,d\theta_1\cdots d\theta_{n-2}d\theta_{n-1}\Big/\frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)}\right)^{\frac{1}{n}}\right]^{n-1}.$$

Here, the denominator and the numerator inside the above parenthesis denote the volumes V_1, V_2 bounded by the inner boundary and the outer respectively.

We have assumed that the inner boundary of R^n is the unit sphere, but it may be taken without loss of generality that the inner boundary of R^n is a sphere $|x| = a$ (const. $\neq 0$), since the similarity transformation $x_j' = \frac{1}{a}x_j$ ($j = 1, 2, \cdots, n$) preserves the modulus of R^n and the ratio V_2/V_1.
Hence we enunciate

COROLLARY 2.*) *Let the inner boundary of R^n be a sphere with the origin as its center, and let the outer be starlike with respect to the origin, then it holds*

*) F. W. Gehring [2] defined the modulus of ring R^n amounting to $n-1$st root of the one by our definition and proved the above Corollary 2 for $n=3$ by means of point symmetrization.

$$\operatorname{mod} R^n \leqq \left(\log \sqrt[n]{\frac{V_2}{V_1}} \right)^{n-1}.$$

We shall state in the final section 6 that through this Corollary 2, a geometric meaning can be given to the last Theorem 5 in Ozawa-Kuroda [3].

5. Now, we first introduce, for completeness' sake a necessary notion analogously to the 2-dimensional case in [3].

Let E be a totally disconnected and compact set in the $(n+1)$-dimensional Euclidean space E^{n+1}, and let D be the domain with E as its complement in E^{n+1}.

A set $\{R_m^{n+1(j)}\}$ $(j = 1, 2, \cdots, \nu(m) < \infty;\ m = 1, 2, \cdots)$ of rings $R_m^{n+1(j)}$ will be referred a system inducing an exhaustion of D if it satisfies the following conditions:

(i) the closure $\overline{R_m^{n+1(j)}}$ of $R_m^{n+1(j)}$ is connected in D,
(ii) the boundary component of $R_m^{n+1(j)}$ consists of the inner boundary sphere $C_{m,1}^{n(j)}$ and the outer one $C_{m,2}^{n(j)}$, these being n-dimensional spheres,
(iii) the complement of $\overline{R_m^{n+1(j)}}$ consists of two domains, of which the one $F_m^{n+1(j)}$ is unbounded and the other $G_m^{n+1(j)}$ has at least one point common with E,
(iv) any point of E is contained in a certain $G_m^{n+1(j)}$,
(v) $R_m^{n+1(k)}$ lies in $F_m^{n+1(j)}$ if $k \neq j$,
(vi) each $R_{m+1}^{n+1(k)}$ is contained in a certain $G_m^{n+1(j)}$,
(vii) $\{D_m^{n+1}\}_{m=1}^{\infty}$ is an exhaustion of D, where

$$D_m^{n+1} = \bigcap_{j=1}^{\nu(m)} \left(F_m^{n+1(j)} \bigcup R_m^{n+1(j)} \right).$$

6. In particular, assume that E lies on a hyperplane H^n of E^{n+1} and the boundary spheres of $R_m^{n+1(j)}$ are symmetric with respect to H^n, then the intersection of H^n and $R_m^{n+1(j)}$ is the ring $R_m^{n(j)}$ bounded by two $(n-1)$-dimensional spheres. We denote by $r_{m,1}^{(j)}$, $V_{m,1}^{(j)}$ $(r_{m,2}^{(j)}, V_{m,2}^{(j)})$ the radius and the volume of the ball bounded by the inner (outer) boundary sphere of $R_m^{n(j)}$ respectively. Then, there holds by Corollary 2,

$$\operatorname{mod} R_m^{n(j)} \leqq \left(\log \sqrt[n]{\frac{V_{m,2}^{(j)}}{V_{m,1}^{(j)}}} \right)^{n-1}.$$

Now, put $\operatorname{mod} R_m^{n(j)} = (\log \mu_m^{(j)})^{n-1}$ and $\min\limits_{1 \leqq j \leqq \nu(m)} \mu_m^{(j)} = \mu_m$, then it becomes:

$$\mu_m \leqq \sqrt[n]{\frac{V_{m,2}^{(j)}}{V_{m,1}^{(j)}}}.$$

Since $V_{m,1}^{(j)}=\pi^{\frac{n}{2}}(r_{m,1}^{(j)})^n/\Gamma(\frac{n}{2}+1)$, this inequality is written as

$$\delta^n(\mu_m)^n(r_{m,1}^{(j)})^n\leqq V_{m,2}^{(j)},$$

where $\delta^n=\pi^{\frac{n}{2}}/\Gamma\left(\frac{n}{2}+1\right)$.

Hereafter, proceed similarly to Ozawa-Kuroda [3] using Hölder's inequality, the symmetry of $R_m^{n+1(j)}$ with respect to H^n and the above Corollary 2, then we obtain finally for $0<\alpha\leqq n$,

$$\delta^\alpha\sum_{j=1}^{\nu(m)}(r_{m,1}^{(j)})^\alpha\leqq\frac{(\nu(m))^{1-\frac{\alpha}{n}}}{\prod_{h=1}^{m}(\mu_h)^\alpha}\left(\sum_{l=1}^{\nu(1)}V_{1,2}^{(l)}\right)^{\frac{\alpha}{n}}.$$

Consequently we have

THEOREM 2. *Let E be a compact set on a hyperplane H^n in E^{n+1}, and let D be the domain with E as its complement. If there exists a system $\{R_m^{n+1(j)}\}$ $(j=1,2,\cdots,\nu(m);\ m=1,2,\cdots)$ inducing an exhaustion of D such that each $R_m^{n+1(j)}$ is symmertic with respect to H^n and the condition*

$$\limsup_{m\to\infty}\left(\alpha\sum_{h=1}^{m}\log\mu_h-\left(1-\frac{\alpha}{n}\right)\log\nu(m)\right)=+\infty$$

is valid for any α $(0<\alpha\leqq n)$, where $\mu_m=\min_{1\leqq j\leqq\nu(m)}\mu_m^{(j)}$, and $(\log\mu_m^{(j)})^{n-1}$ denotes the modulus of the ring $R_m^{n(j)}$ being the intersection of H^n and $R_m^{n+1(j)}$, then the α-dimensional measure of E is equal to zero.

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