ON COMPLEX MANIFOLDS WITH CERTAIN STRUCTURES WHICH ARE RELATED TO COMPLEX CONTACT STRUCTURES

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M. Obata and H. Wakakuwa studied the $4n$-dimensional differentiable manifolds with the structure group (of the tangent bundle) $Sp(n)$ ([1], [2], [3], [4]), while S. Hashimoto and C. J. Hsu studied the $(4n+1)$-dimensional cases ([5], [6]). The present paper is devoted to the study of some $(4n+2)$-dimensional manifolds. We restricted to these manifolds to get closer connection with complex contact manifolds ([7], [8], [9]). Hence this paper is an analogous work to that of S. Sasaki ([10], [11]), on complex manifolds.

In §1 we review complex tensors, in §2 we study the naturally arised Hermitian metric to the given complex (ϕ, ξ, η)-structure and define the analytic (ϕ, ξ, η)-structure or complex almost contact structure. In §3 we prove that complex contact manifolds whose first Chern class vanishes are complex manifolds with analytic (ϕ, ξ, η)-structure, this justifies the terminology "complex almost contact structure". In §4 we first give a criterion for the reduction of the structure group of fibre bundle which is an immediate consequence of a known theorem, but due to its good applications it deserves an explicit formulation. We reduced the group of the tangent bundle of complex manifolds with complex (ϕ, ξ, η)-structure as an application of the lemma.

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1. Tensors on complex manifold. A complex manifold M^m of complex dimension m is a Hausdorff space to each point p of which there is associated a neighbourhood $N(p)$ which is mapped topologically onto subdomain of the Euclidean space of complex variables $z^1, \cdots, z^m$. If $q \in N(p)$, the coordinates of q will be denoted by $z^i(q)$, $i = 1, 2, \cdots, m$. Wherever two neighbourhoods intersect, the coordinates are connected by a pseudo-conformal mapping.

Following [13] we introduce a conjugate manifold $\overline{M}^m$ which is a homeomorphic image of M^m in which the point $\bar{p}$ of $\overline{M}^m$ corresponds to the point p

of M^m and the neighbourhood $\overline{N}(\overline{p})$ to $N(p)$. Let Latin indices run from 1 to $2m$, and let

$$\bar{i} = i + m \pmod{2m}. \tag{1.1}$$

If $\bar{q} \in \overline{N}(\bar{p})$, we define

$$z^{\bar{i}}(\bar{q}) = (z^i(q))^-, \tag{1.2}$$

where $(z)^-$ denote the complex conjugate of the quantity z. By means of (1. 2) the neighbourhood $\overline{N}(\bar{p})$ is mapped onto a domain in the space of the variables $z^{\bar{i}} = \bar{z}^i (i = 1, 2, \cdots, m)$.

Now consider the product manifold $M^m \times \overline{M}^m$ whose points are ordered pair $(p, \bar{q})$, and let

$$z^i(p, \bar{q}) = \begin{cases} z^i(p), \; i = 1, 2, \cdots, m, \\ z^i(\bar{q}) = (z^{\bar{i}}(q))^-, \; i = m + 1, \cdots, 2m. \end{cases} \tag{1.3}$$

Then

$$z^i(p, \bar{q}) = (z^{\bar{i}}(q, \bar{p}))^-, \; i = 1, 2, \cdots, 2m. \tag{1.4}$$

The product manifold $M^m \times \overline{M}^m$ is covered by the coordinates $z^i(p, \bar{q})$ $i = 1, 2, \cdots, 2m$. Introduce coordinate $x^i(p, \bar{q})$ by formulas

$$x^i(p, \bar{q}) = \begin{cases} \frac{1}{2}(z^i(p, \bar{q}) + \bar{z}^i(p, \bar{q})), \; i = 1, \cdots, m, \\ \frac{1}{2}\sqrt{-1}(z^i(p, \bar{q}) - z^{\bar{i}}(p, \bar{q})), \; i = m + 1, \cdots, 2m, \end{cases} \tag{1.5}$$

$$z^i(p, \bar{q}) = \begin{cases} x^i(p, \bar{q}) + \sqrt{-1}\, x^{\bar{i}}(p, \bar{q}), \; i = 1, \cdots, m, \\ x^{\bar{i}}(p, \bar{q}) - \sqrt{-1}\, x^i(p, \bar{q}), \; i = m + 1, \cdots, 2m. \end{cases}$$

Then

$$x^i(p, \bar{q}) = (x^i(q, \bar{p}))^-, \; i = 1, \cdots, 2m. \tag{1.6}$$

On the diagonal manifold D^m of $M^m \times \overline{M}^m$ where $p = q$, we have

$$z^i = z^i(p, \bar{p}) = (z^{\bar{i}})^-, \; x^i = x^i(p, \bar{p}) = (x^i)^-. \tag{1.7}$$

Thus D^m is covered either by self-conjugate coordinate $(z^i, z^{\bar{i}} = \bar{z}^i)$ or by real coordinate x^i and we identify M^m with D^m.

A tensor on M^m (precisely speak on D^m) whose components are real when they are expressed in the real coordinates x^i will be called a real tensor. A real tensor T when expressed in self-conjugate coordinates z^i satisfies

$$T_{\alpha\bar{\beta}\ldots\gamma}{}^{\bar{\lambda}\varepsilon\ldots\delta} = (T_{\bar{\alpha}\beta\ldots\bar{\gamma}}{}^{\lambda\bar{\varepsilon}\ldots\bar{\delta}})^{-}, \tag{1. 8}$$

where $\alpha, \beta, \gamma, \lambda, \epsilon, \delta, \cdots = 1, \cdots, m,\ \bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{\lambda}, \bar{\epsilon}, \bar{\delta} \cdots = m+1, \cdots, 2m$.

Throughout this paper, if we use a Latin letter as an index for a tensor e. g., T^i, we mean $i = 1, \cdots, 2m$. If we use a Greek letter as an index for a tensor e.g., T^α, we mean $\alpha = 1, \cdots, m$. In this paper we shall be concerned only with real tensor of class C^ω (i.e., real analytic) and shall make the convention that the conponents of a tensor with Greek indices are expressed by self-conjugate coordinate and the components of a tensor with Latin indices are expressed by real coordinate.

2. Complex (ϕ, ξ, η)-structure and associated Hermitian metric.

Definition: A complex manifold M^{2n+1} of complex dimension $2n+1$ is said to have a complex (ϕ, ξ, η)-structure if there are a C^ω-differentiable tensor field ϕ^α_β and C^ω-differentiable vector fields ξ^α and η_β over M^{2n+1} such that

$$\xi^\alpha \eta_\alpha = 1, \tag{2. 1}$$

$$\text{rank } (\phi^\alpha_\beta) = 2n, \tag{2. 2}$$

$$\phi^\alpha_\beta \xi^\beta = 0, \tag{2. 3}$$

$$\phi^\alpha_\beta \eta_\alpha = 0, \tag{2. 4}$$

$$\phi^\alpha_\beta \phi^\beta_\gamma = -\delta^\alpha_\gamma + \xi^\alpha \eta_\gamma. \tag{2. 5}$$

THEOREM 1. *Let M^{2n+1} be a complex manifold with (ϕ, ξ, η)-structure. Then there exists a positive definite Hermitian metric g such that*

$$\eta_\alpha = g_{\alpha\bar{\beta}} \xi^{\bar{\beta}}, \tag{2. 6}$$

$$g_{\alpha\bar{\beta}} \phi^\alpha_\gamma \phi^{\bar{\beta}}_{\bar{\epsilon}} = g_{\epsilon\bar{\gamma}} - \eta_\epsilon \eta_{\bar{\gamma}}. \tag{2. 7}$$

We first construct a lemma which is already known.

LEMMA 1. *Suppose ξ^α and η_β be C^ω-differentiable or complex analytic contravariant and covariant vector fields on a complex manifold M^{2n+1} such that*

$$\xi^\alpha \eta_\alpha = 1. \tag{2. 8}$$

Then M^{2n+1} admits a positive definite Hermitian metric h of class C^ω such that

$$\eta_\alpha = h_{\alpha\bar{\beta}} \xi^{\bar{\beta}}. \tag{2. 9}$$

PROOF[1]: Let $f_{\alpha\bar{\beta}}$ be an arbitrary Hermitian metric on M^{2n+1}. And if we put

$$h_{\alpha\beta} = f_{\gamma\bar{\delta}}(\delta^\gamma_\alpha - \xi^\gamma \eta_\alpha)(\delta^{\bar{\delta}}_{\bar{\beta}} - \xi^{\bar{\delta}} \eta_{\bar{\beta}}) + \eta_\alpha \eta_{\bar{\beta}}.$$

1) This nice proof was given by Y. Hatakeyama.

Then $h_{\alpha\bar{\beta}}$ also defines a positive definite Hermitian metric on M^{2n+1}, for if

$$h_{\alpha\bar{\beta}}X^{\alpha}X^{\bar{\beta}}=0,$$

by virtue of the fact that $f_{\alpha\bar{\beta}}$ is a positive definite Hermitian metric, we get

$$(\delta_{\alpha}^{\gamma}-\xi^{\gamma}\eta_{\alpha})X^{\alpha}=0 \quad \text{and} \quad \eta_{\alpha}X^{\alpha}=0$$

which show $X^{\alpha}=0$.
Moreover we have

$$h_{\alpha\bar{\beta}}\xi^{\bar{\beta}}=f_{\gamma\bar{\delta}}(\delta_{\alpha}^{\gamma}-\xi^{\gamma}\eta_{\alpha})(\delta_{\bar{\beta}}^{\bar{\delta}}-\xi^{\bar{\delta}}\eta_{\bar{\beta}})\xi^{\bar{\beta}}+\eta_{\alpha}\eta_{\bar{\beta}}\xi^{\bar{\beta}}=\eta_{\alpha}.$$

Thus $h_{\alpha\bar{\beta}}$ defines the required Hermitian metric.

PROOF OF THEOREM 1. Let h be a Hermitian metric which has the property stated in Lemma 1 and put

$$g_{\alpha\bar{\beta}}=\frac{1}{2}(h_{\alpha\bar{\beta}}+h_{\gamma\bar{\delta}}\phi_{\alpha}^{\bar{\delta}}\phi_{\bar{\beta}}^{\gamma}+\eta_{\alpha}\eta_{\bar{\beta}}).$$

Then we can easily see that

$$g_{\alpha\bar{\beta}}\xi^{\bar{\beta}}=\eta_{\alpha},$$
$$g_{\alpha\bar{\beta}}\xi^{\alpha}\xi^{\bar{\beta}}=1.$$

In the next place we see that

$$\frac{1}{2}(h_{\alpha\bar{\beta}}+h_{\lambda\bar{\delta}}\phi_{\alpha}^{\bar{\delta}}\phi_{\bar{\beta}}^{\lambda}+\eta_{\alpha}\eta_{\bar{\beta}})\phi_{\epsilon}^{\bar{\beta}}\phi_{\bar{\gamma}}^{\alpha}$$
$$=\frac{1}{2}\{h_{\alpha\bar{\beta}}\phi_{\epsilon}^{\bar{\beta}}\phi_{\bar{\gamma}}^{\alpha}+h_{\lambda\bar{\delta}}(-\delta_{\bar{\gamma}}^{\bar{\delta}}+\xi^{\bar{\delta}}\eta_{\bar{\gamma}})(-\delta_{\epsilon}^{\lambda}+\xi^{\lambda}\eta_{\epsilon})\}$$
$$=\frac{1}{2}(h_{\epsilon\bar{\gamma}}+h_{\alpha\bar{\beta}}\phi_{\epsilon}^{\bar{\beta}}\phi_{\bar{\gamma}}^{\alpha}-\eta_{\epsilon}\eta_{\bar{\gamma}})$$

that is

$$g_{\alpha\bar{\beta}}\phi_{\bar{\gamma}}^{\alpha}\phi_{\epsilon}^{\bar{\beta}}=g_{\epsilon\bar{\gamma}}-\eta_{\epsilon}\eta_{\bar{\gamma}}.$$

Hence, the theorem is proved.

We shall say that the metric which has the property stated in Theorem 1 an associated Hermitian metric to the given complex (ϕ,ξ,η)-structure. And if a complex manifold admits tensor fields ϕ,ξ,η,g such that g is an associated Hermitian metric of the complex (ϕ,ξ,η)-structure, then we say this manifold has (ϕ,ξ,η,g)-structure.

Put

$$\phi_{\alpha\beta} = g_{\alpha\bar{\gamma}}\phi^{\bar{\gamma}}_{\beta}. \tag{2.10}$$

Then, the tensor $\phi_{\alpha\beta}$ is skew symmetric with respect to α and β. In fact,

$$(g_{\alpha\bar{\gamma}}\phi^{\bar{\gamma}}_{\beta}\phi^{\alpha}_{\bar{\delta}})\,\phi^{\bar{\delta}}_{\epsilon} = g_{\alpha\bar{\gamma}}\phi^{\bar{\gamma}}_{\beta}(\phi^{\alpha}_{\bar{\delta}}\phi^{\bar{\delta}}_{\epsilon}).$$

Putting (2. 7) and (2. 5) into the last equation, we get

$$(g_{\beta\bar{\delta}} - \eta_{\beta}\eta_{\bar{\delta}})\phi^{\bar{\delta}}_{\epsilon} = g_{\alpha\bar{\gamma}}\phi^{\bar{\gamma}}_{\beta}(-\delta^{\alpha}_{\epsilon} + \xi^{\alpha}\eta_{\epsilon})$$

or

$$\phi_{\beta\epsilon} = -\phi_{\epsilon\beta}.$$

Of course the rank $(\phi_{\alpha\beta})$ is $2n$. We call $\phi_{\alpha\beta}$ the associated tensor and $\phi = \frac{1}{2}\phi_{\alpha\beta}dz^{\alpha} \wedge dz^{\beta}$ the associated form.

Definition: A complex (ϕ, ξ, η)-structure is called analytic (ϕ, ξ, η)-structure or complex almost contact structure if ξ^{α} and η_{β} are complex analytic and there exists an associated tensor $\phi_{\alpha\beta}$ which is complex analytic.

3. Complex contact manifolds and complex almost contact manifolds.

DEFINITION: Let M^{2n+1} be a complex manifold of complex dimension $2n+1$. Let $\{U_i\}$ be an open covering of M^{2n+1}. We call M^{2n+1} a complex contact manifold if the following conditions are satisfied

(1) On each U_i there exists a complex analytic 1-form such that $\omega_i \wedge (d\omega_i)^n$ is different from zero at every point of U_i.

(2) If $U_i \cap U_j$ is nonempty, then there exists a nonvanishing complex analytic function f_{ij} on $U_i \cap U_j$ such that $\omega_i = f_{ij}\omega_j$ on $U_i \cap U_j$.

If $f_{ij} = 1$ for each i and j, in other words, there exists on M^{2n+1} a globally defined complex analytic 1-form η shuch that

$$\eta \wedge (d\eta)^n \neq 0,$$

at every point of the manifold, then M^{2n+1} is called a restricted complex contact manifold.

For completeness we restate a theorem in [8].

THEOREM 2. *A complex contact manifold M^{2n+1} is a restricted complex contact manifold if and only if its first Chern class $C_1(M^{2n+1})$ vanishes.*

PROOF: The characteristic class of the line bundle k which is defined by transition functions $\{f_{ij}^{-(n+1)}\}$ is $C_1(M^{2n+1})$. If $C_1(M^{2n+1}) = 0$, then k is equivalent to a product bundle k'. The form ω_i', which is the image of ω_i under the mapping induced by the bundle equivalence map of k to k', satisfies

$$\omega_i' \wedge (d\omega_i')^n = \omega_j' \wedge (d\omega_j')^n.$$

Thus M^{2n+1} is a restricted contact manifold.

Converse is easily seen to be true from [7].

The main result of this section is that a restricted complex contact manifold naturally induces an analytic (ϕ, ξ, η)-structure. This justifies the definition given at the end of §2.

To prove our main result we need a lemma which is an example of the following theorem: If G is a connected Lie group and K is the maximal compact subgroup of G, then G is real analytically homeomorphic with $K \times R^n$. But we shall prove it directly. The lemma is:

LEMMA 2. *Let $GL(n,C)$ be the complex general linear group of degree n. Let $U(n)$ be the unitary subgroup and $H(n)$ be the set of all positive definite Hermitian matrices. Then the mapping*

$$c: GL(n,C) \rightarrow U(n) \times H(n)$$

defined by the decomposition (i.e., any $A \in GL(n,C)$ can be written in one and only one way as the product $A = UH$ of a unitary matrix U and a Hermitian matrix H) gives a real analytic homeomorphism of these two manifolds with respect to the usual real analytic structure.

PROOF: Let ${}^R GL(n,C)$ be the real representation of $GL(n,C)$. Then we see that

$${}^R GL(n,C) = \{A \in GL(2n,R): A^{-1}JA = J\}$$

is an isotropy group, and therefore it is a regular Lie subgroup of $GL(2n,R)$(i.e., the underlying submanifold is regular) where

$$J = \begin{pmatrix} 0 & -E_n \\ E_n & 0 \end{pmatrix}$$

and E_n is the identity matrix of degree n.

Let ${}^R H(n)$ be the real representation of $H(n)$ and let $S(2n)$ be the set of all positive definite symmetric matrices of $GL(2n,R)$. It is easily seen that

$${}^R H(n) = \{A \in S(2n): A^{-1}JA = J\}$$

is a regular submanifold of $S(2n)$.

Let ${}^R U(n)$ be the real representation of $U(n)$. Then we see that

$${}^R U(n) = \{U \in O(2n): {}^t UJU = J\}$$

is a regular submanifold of $O(2n)$.

Let us consider the following commutative diagram

$$\begin{array}{ccc} {}^R GL(n,C) & \xrightarrow{i} & GL(2n,R) \\ \downarrow c & & \downarrow d \\ {}^R U(n) \times {}^R H(n) & \xrightarrow{j} & O(2n) \times S(2n) \end{array}$$

where d is a real analytic homeomorphism [12], c is a topological decomposition [15] and i, j are injections which are real analytic mappings. It is easily seen that $d \circ i$ and $d^{-1} \circ j$ are real analytic mappings.

Since ${}^R U(n)$ and ${}^R H(n)$ are regular submanifolds of $O(2n)$ and $S(2n)$ respectively, ${}^R U(n) \times {}^R H(n)$ is a regular submanifold of $O(2n) \times S(2n)$. Moreover the image of ${}^R GL(n, C)$ under the mapping $c = d \circ i$ is contained in ${}^R U(n) \times {}^R H(n)$. Hence c is a real analytic mapping[2]. Similarly, ${}^R GL(n, C)$ is a regular submanifold of $GL(2n, R)$ and the image of ${}^R U(n) \times {}^R H(n)$ under the mapping $c^{-1} = d^{-1} \circ j$ is contained in ${}^R GL(n, C)$. Hence the mapping c^{-1} is a real analytic mapping. Therefore $GL(n, C)$ is real analytically homeomorphic onto $U(n) \times H(n)$.

THEOREM 3. *Let M^{2n+1} be a complex manifold of complex dimension $2n+1$. Let η be a complex analytic 1-form over M^{2n+1} such that*

(3. 1) $$\eta \wedge (d\eta)^n \neq 0, \text{ at each point.}$$

Then the form η induces an analytic (ϕ, ξ, η)-structure.

PROOF: Let us express η by local coordinate, i. e.,

(3. 2) $$\eta = \eta_\alpha dz^\alpha$$

Then,

(3. 3) $$d\eta = \frac{1}{2} \phi_{\alpha\beta} dz^\alpha \wedge dz^\beta, \text{ where } \phi_{\alpha\beta} = \frac{\partial \eta_\beta}{\partial z^\alpha} - \frac{\partial \eta_\alpha}{\partial z^\beta}.$$

By virtue of the condition (3. 1), it follows that $d\eta$ is a 2-form of rank $2n$ everywhere over M^{2n+1} and $\phi_{\alpha\beta}$ is a matrix whose rank is everywhere $2n$ over M^{2n+1}. We can easily verify that (3. 1) is equivalent to

$$\eta_{[1}\phi_{23}\phi_{45}\dots\phi_{2n\ 2n+1]} \neq 0$$

where [ ] means a determinant divided by the factorial of the number of indices.

Now we define distributions in the following way: we set

2) Let F be a real analytic mapping of M^n into M^m and let N be a regular submanifold of M^m and $F(M^n) \subset N$, then $F: M^n \to N$ is a real analytic mapping.

$$D_p = \{X^\alpha : \phi_{\alpha\beta}X^\beta = 0\}, \text{ at every point } p \in M^{2n+1},$$

then the mapping $p \to D_p$ defines a distribution D of complex dimension 1 (D is analytic, because it is spanned by the analytic vector field

$$\xi^1 = \frac{1}{\lambda}\phi_{[23}\phi_{45}\ldots\phi_{2n\ 2n+1]},$$

$$\xi^2 = \frac{1}{\lambda}\phi_{[34}\phi_{56}\ldots\phi_{2n+1\ 1]},$$

$$\cdots\cdots\cdots\cdots$$

$$\xi^\alpha = \frac{1}{\lambda}\phi_{[\alpha+1\ \alpha+2}\phi\ldots_{2n+1\ 1}\ldots\phi_{\alpha-2\ \alpha-1]}, \qquad \alpha \text{ is a residue modulo } 2n+1,$$

$$\cdots\cdots\cdots\cdots$$

$$\xi^{2n+1} = \frac{1}{\lambda}\phi_{[12}\phi_{34}\ldots\phi_{2n-1\ 2n]},$$

where $\lambda = (2n+1)\eta_{[1}\phi_{23}\phi_{45}\ldots\phi_{2n\ 2n+1]})$. We also set

$$\bar{D}_p = \{X^\alpha : X^\alpha\eta_\alpha = 0\}, \text{ at every point } p \in M^{2n+1},$$

then the mapping $p \to \bar{D}_p$ defines a complex analytic distribution $\bar{D}$ of complex dimension $2n$ (η is analytic). By virtue of Lemma 1 we can take a Hermitian metric h such that $h_{\alpha\bar{\beta}}\xi^{\bar{\beta}} = \eta_\alpha$. This means that D is orthogonally complementary to $\bar{D}$ with respect to the metric h.

Now let us consider the real coordinate systems. In the real coordinate systems, $(\phi_{\alpha\beta},\phi_{\bar{\alpha}\bar{\beta}})$ becomes ϕ_{ij} and D becomes a distribution of real dimension 2 and $\bar{D}$ becomes a distribution of real dimension $4n$. We denote them by the same letters D and $\bar{D}$ respectively. Let J^i_j be the induced almost complex structure. It is easily seen that ξ^i which is the real components of $(\xi^\alpha,\xi^{\bar{\alpha}})$ and $J^i_k\xi^k$ which is the real components of $(i\xi^\alpha,\ -i\xi^{\bar{\alpha}})$ constitute a local base of D. By virtue of h_{ij} being Hermitian with respect to J^i_j, we see that ξ^i is orthogonal to $J^i_k\xi^k$.

Let $\{U\}$ be a sufficiently fine open covering of M^{2n+1} by coordinate neighborhoods. In every U we take $e^i_\Delta = \xi^i$, $e^i_{\Delta^*} = J^i_k\xi^k$ and take unit vector field e^i_1 such that e^i_1 is orthogonal to e^i_Δ, $e^i_{\Delta^*}$. It is easily seen that $e^i_1 \in \bar{D}$ on U and hence $e^i_{1^*} = J^i_k e^k_1 \in \bar{D}$. In such a way we construct an orthonormal frame e^i_1, $e^i_2,\cdots, e^i_{2n}$, $e^i_{1^*}$, $e^i_{2^*},\cdots$, $e^i_{2n^*}$, e^i_Δ, $e^i_{\Delta^*}$ on U, where e^i_Δ, $e^i_{\Delta^*} \in D$ and e^i_1, $e^i_2,\cdots$, e^i_{2n}, $e^i_{1^*}$, $e^i_{2^*},\cdots$, $e^i_{2n^*} \in \bar{D}$. We call such a frame adapted frame.

Then if $U \cap V$ is nonempty, the matrix of the transformation of components of the same vector relative to adapted frames on U and V is of the following form

$$U = \begin{pmatrix} U_{4n} & 0 \\ 0 & E_2 \end{pmatrix}$$

where $U_{4n} \in {}^RU(2n)$ and E_2 is the identity matrix of degree 2.

Now, let ϕ_u be a matrix whose elements are the components of ϕ_{ij} relative to adapted frame over U. Then ϕ_u is of the following form

$$\phi_u = \begin{pmatrix} \phi'_u & 0 \\ 0 & 0 \end{pmatrix} \begin{matrix} \}\, 4n \\ \}\, 2 \end{matrix}$$

where $\phi'_u \in {}^RGL(2n, C)$ and is skew symmetric. Then, by virtue of Lemma 2 we can write

$$\phi'_u = A'_u \cdot B'_u$$

where $A'_u \in {}^RU(2n)$, $B'_u \in {}^RH(2n)$. If we set

$$A_u = \begin{pmatrix} A'_u & 0 \\ 0 & 0 \end{pmatrix} \begin{matrix} \}\, 4n \\ \}\, 2, \end{matrix}$$

$$B_u = \begin{pmatrix} B'_u & 0 \\ 0 & E_2 \end{pmatrix} \begin{matrix} \}\, 4n \\ \}\, 2, \end{matrix}$$

then A_u and B_u define real analytic tensor fields on U. Since ϕ'_u is skew symmetric, we have

$${}^t\phi'_u = -\phi'_u$$

or

$${}^tB'_u \cdot {}^tA'_u = -A'_u \cdot B'_u$$

that is,

$$B'_u \cdot {}^tA'_u = -A'_u \cdot B'_u.$$

Multiplying A'_u to the right of both sides of the last equation we get

$$B'_u = -(A'_u)^2 \cdot {}^tA'_u \cdot B'_u \cdot A'_u.$$

As is easily seen, $-(A'_u)^2 \in {}^RU(2n)$, ${}^tA'_u \cdot B'_u \cdot A'_u \in {}^RH(2n)$. So, by virtue of the uniqueness of decomposition we get

$$-(A'_u)^2 = E_{4n},\ B'_u = {}^tA'_u \cdot B'_u \cdot A'_u,$$

i. e.,

$$(A'_u)^2 = -E_{4n},\ A'_u \cdot B'_u = B'_u \cdot A'_u.$$

Hence we have

$$A_u^2 = \begin{pmatrix} -E_{4n} & 0 \\ 0 & 0 \end{pmatrix},$$

$$\phi_u = B_u \cdot A_u.$$

Next if we consider the relation between ϕ_u and ϕ_v on $U \cap V$, we get

$$\phi'_v = {}^t U_{4n} \cdot \phi'_u \cdot U_{4n}$$

and so

$$\begin{aligned} B'_v \cdot A'_v &= {}^t U_{4n} \cdot B'_u \cdot A'_u \cdot U_{4n} \\ &= ({}^t U_{4n} \cdot B'_u \cdot U_{4n})({}^t U_{4n} \cdot A'_u \cdot U_{4n}). \end{aligned}$$

By virtue of the uniqueness of the decomposition, we have

$$\begin{aligned} B'_v &= {}^t U_{4n} \cdot B'_u \cdot U_{4n}, \\ A'_v &= {}^t U_{4n} \cdot A'_u \cdot U_{4n}. \end{aligned}$$

Hence we get

$$\begin{aligned} B_v &= {}^t U \cdot B_u \cdot U, \\ A_v &= {}^t U \cdot A_u \cdot U. \end{aligned}$$

By virtue of the last two relations, we see that the sets $\{A_u\}$ and $\{B_u\}$ define global tensor fields A and B of class C^ω on manifold.

From $B_u J = J B_u$ we get $B_u J = - {}^t J B_u$. Or if we express it in components (with respect to natural frames) we have

$$B_{ij} J^j_k = - J^j_i B_{jk},$$

in words, B_{ij} is hybrid with respect to i and j. Moreover B_u is positive definite, hence it defines a Hermitian metric g_{ij}.

Now let us go back to the complex coordinates and take frames

$$\epsilon_j = \frac{1}{\sqrt{2}}(e_j - \sqrt{-1}\, e_{j^*}).$$

Then, we have

$$B = \begin{pmatrix} {}^c B'_u & 0 \\ 0 & 1 \end{pmatrix} \tag{3. 4}$$

and

$$A^2 = \begin{pmatrix} -E_{2n} & 0 \\ 0 & 0 \end{pmatrix} \begin{matrix} \} 2n \\ \} 1 \end{matrix} \tag{3. 5}$$

with respect to the frames ϵ_j, where ${}^c B'_u \in H(2n)$ (i. e., B'_u is real representation of ${}^{c}B'_u$).

Since A is hybrid ($\phi_u = B_u \cdot A_u$. Or expressed in components relative to real coordinates $\phi_{ij} = g_{ik}A^k_j$. ϕ_{ij} is pure and g_{ij} is hybrid, hence A is hybrid). If we express A with respect to natural complex frames we have

$$A = \begin{pmatrix} 0 & \phi^{\alpha}_{\bar\beta} \\ \phi^{\bar\alpha}_{\beta} & 0 \end{pmatrix},$$

and

$$\phi_{\alpha\beta} = g_{\alpha\bar\gamma}\phi^{\bar\gamma}_{\beta}.$$

Write (3. 5) in components with respect to natural complex frame, we have

$$(3.\ 6) \qquad \phi^{\alpha}_{\bar\beta}\phi^{\bar\beta}_{\gamma} = -\delta^{\alpha}_{\gamma} + \xi^{\alpha}\eta_{\gamma}, \qquad \phi^{\bar\alpha}_{\beta}\phi^{\beta}_{\bar\gamma} = -\delta^{\bar\alpha}_{\bar\gamma} + \xi^{\bar\alpha}\eta_{\bar\gamma}.$$

From (3. 4) we see that the metric g defined by the tensor B coincides with h on D, i.e.,

$$g_{\alpha\bar\beta}X^{\bar\beta} = h_{\alpha\bar\beta}X^{\bar\beta}, \text{ if } X^{\alpha} \in \text{D}.$$

Therefore we have

$$(3.\ 7) \qquad g_{\alpha\bar\beta}\xi^{\bar\beta}\xi^{\alpha} = h_{\alpha\bar\beta}\xi^{\bar\beta}\xi^{\alpha} = \eta_{\alpha}\xi^{\alpha} = 1.$$

Hence

$$0 = \phi_{\alpha\beta}\xi^{\beta} = \phi_{\alpha\beta}h^{\beta\bar\gamma}\eta_{\bar\gamma} = \phi_{\alpha\beta}g^{\beta\bar\gamma}\eta_{\bar\gamma} = \phi^{\bar\gamma}_{\alpha}\eta_{\bar\gamma},$$

that is,

$$(3.\ 8) \qquad \phi^{\alpha}_{\bar\beta}\eta_{\alpha} = 0.$$

Moreover

$$\phi^{\alpha}_{\bar\beta}\xi^{\bar\beta} = g^{\alpha\bar\lambda}\phi_{\bar\lambda\bar\beta}\xi^{\bar\beta} = 0$$

i. e.,

$$(3.\ 9) \qquad \phi^{\alpha}_{\bar\beta}\xi^{\bar\beta} = 0.$$

Thus we have constructed (ϕ, ξ, η)-structure ((3. 6), (3. 7), (3. 8), (3. 9)). Moreover g satisfies

$$g_{\alpha\bar\beta}\phi^{\bar\beta}_{\gamma} = -g_{\gamma\bar\beta}\phi^{\bar\beta}_{\alpha} (= \phi_{\alpha\gamma})$$

or

$$(3.10) \qquad g_{\alpha\bar\beta}\phi^{\bar\beta}_{\gamma}\phi^{\alpha}_{\bar\epsilon} = -g_{\gamma\bar\beta}\phi^{\bar\beta}_{\alpha}\phi^{\alpha}_{\bar\epsilon} = g_{\gamma\bar\epsilon} - \eta_{\gamma}\eta_{\bar\epsilon}.$$

Hence g is an associated Hermitian metric.

Thus we have proved more than that stated in the theorem. We express the result in the following theorem.

THEOREM 4. *Let M^{2n+1} be a complex manifold and there are an analytic tensor field $\phi_{\alpha\beta}$ and analytic vector fields ξ^α and η_β over M^{2n+1} such that*

$$\operatorname{rank}(\phi_{\alpha\beta}) = 2n,$$
$$\phi_{\alpha\beta}\xi^\beta = 0,$$
$$\xi^\alpha\eta_\alpha = 1.$$

Then, there exists a complex (ϕ, ξ, η, g)-structure such that

$$\phi_{\alpha\beta} = g_{\alpha\bar{\gamma}}\phi^{\bar{\gamma}}_{\beta}$$

and ξ^α and η_β are the vector fields in (ϕ, ξ, η, g)-structure.

In fact, if we notice the following result, then clearly the proof of Theorem 3 is applicable to Theorem 4.

Suppose $\phi_{\alpha\beta}$, ξ^α, η_β be the tensor field and vector fields stated in Theorem 4 and put

$$\phi = \frac{1}{2}\phi_{\alpha\beta}dz^\alpha \wedge dz^\beta,$$

then

$$\eta \wedge \phi^n \neq 0, \text{ at every point of } M^{2n+1},$$

and

$$\xi^\alpha = \frac{1}{\lambda}\phi_{[\alpha+1\ \alpha+2}\phi_{\alpha+3\ \alpha+4}\ldots\phi_{2n+1\ 1}\ldots\phi_{\alpha-2\ \alpha-1]}$$

where $\lambda = (2n+1)\ \eta_{[1}\phi_{23}\phi_{45}\ldots\phi_{2n2n+1]}$, and α is a residue modulo $2n+1$.

REMARK: $\eta \wedge \phi^n \neq 0$ implies $\lambda \neq 0$.

In fact, η defines an analytic distribution. Hence there exist $2n$ locally defined analytic vector fields such that at every point they form a local base of this distribution. Let them be $e_1, e_2, \cdots, e_{2n}$ and we take $e_{2n+1} = \xi$. It is clear that $e_1, e_2, \cdots, e_{2n}, e_{2n+1}$ form a frame at a neighborhood of a point. Let $f_1, f_2, \cdots, f_{2n}, f_{2n+1} = \eta$ be the dual base. Consider the scalar product $((e_1 \wedge \cdots \wedge e_{2n+1}) \llcorner \phi^n\ \ \eta)$ of $(e_1 \wedge \cdots \wedge e_{2n+1}) \llcorner \phi^n$ and η where "$\llcorner$" denotes interior product (e. g., see [14]). From [14] pp. 43, we see that

$$((e_1 \wedge \cdots \wedge e_{2n+1} \llcorner \phi^n\ \ \eta)$$
$$= \left(\sum(-1)^{i-1}(e_1 \wedge \cdots \wedge e_{i-1} \wedge e_{i+1} \wedge \cdots \wedge e_{2n+1}\ \ \phi^n)e_i\ \ \eta\right)$$

$$= \sum(-1)^{i-1}(e_1 \wedge \cdots \wedge e_{i-1} \wedge e_{i+1} \wedge \cdots \wedge e_{2n+1} \quad \phi^n)\delta_i^{2n+1}$$
$$= (e_1 \wedge \cdots \wedge e_{2n} \ \phi^n)$$
$$= \det |A_{ab}| \neq 0$$

where δ_i^{2n+1} is Kronecker delta and A_{ab} are components of ϕ with respect to vectors $e_1, \cdots, e_{2n}$. Hence

$$((e_1 \wedge \cdots \wedge e_{2n+1}) \llcorner \phi^n \ \eta) = (e_1 \wedge \cdots \wedge e_{2n+1} \quad \phi^n \wedge \eta) \neq 0$$

and we get

$$\eta \wedge \phi^n \neq 0.$$

Moreover we shall easily get by calculation that

$$\sum_{\beta=1}^{2n+1} \phi_{\alpha\beta}\phi_{[\beta+1\ \beta+2}\ldots\phi_{2n+1\ 1}\ldots\phi_{\beta-2\ \beta-1]} = 0,$$
$$\sum_{\alpha=1}^{2n+1} \eta_\alpha \frac{1}{\lambda} \phi_{[\alpha+1\ \alpha+2}\ldots\phi_{2n+1\ 1}\ldots\phi_{\alpha-2\ \alpha-1]} = 1.$$

This proves

$$\xi^\alpha = \frac{1}{\lambda} \phi_{[\alpha+1\ \alpha+2}\ldots\phi_{2n+1\ 1}\ldots\phi_{\alpha-2\ \alpha-1]}.$$

Now let us consider the distributions defined by ξ and η as in Theorem 3, we denote them by D and $\bar{D}$ respectively. Then clearly the proof of Theorem 3 could apply to this theorem.

4. Reduction of the structure group of tangent bundle. Let G be a topological group, G' be its closed subgroup and $p: G \to G/G'$ be the natural projection. We assume in this section that $G \to G/G'$ has a local cross section. If G is a real or complex Lie group, then the assumption is satisfied [15].

LEMMA 3. *Let B be a principal bundle. The structure group G of B can be reduced to G' if and only if there exist an open covering $\{U_i\}$ and local cross sections C_i of B over each U_i such that if $U_i \cap U_j$ is nonempty, then $C_i(x) = R_{g(x)}C_j(x)$ on $U_i \cap U_j$, where $g(x) \in G'$ and $R_{g(x)}$ is a right translation associated to $g(x)$.*

PROOF: This follows from the following fact [16]: The structure group of B can be reduced to G' if and only if B/G' admits a cross section.

REMARK: This lemma is valid for "differentiable" and "complex analytic" fibre bundle by a trivial modification (i. e., replace each continuous function by differentiable or complex analytic function and topological group by real or complex Lie group) [17].

THEOREM 5. *Let M^{2n+1} be a complex manifold with analytic (ϕ, ξ, η)-structure, then the structure group of its complex analytic tangent bundle is reducible to $Sp(n, C) \times 1$. Conversely, if a complex manifold whose structure group of its complex analytic tangent bundle is reducible to $Sp(n, C) \times 1$, then we can endow to M^{2n+1} an analytic (ϕ, ξ, η)-structure.*

PROOF : Let $\{U\}$ be a sufficiently fine open covering of M^{2n+1} by coordinate neighborhoods. Then, the analytic associated form

$$\phi = \frac{1}{2}\phi_{\alpha\beta}dz^{\alpha} \wedge dz^{\beta}$$

of analytic (ϕ, ξ, η)-structure can be written as

$$\phi = \sum_{\alpha=1}^{n} f_u^{\alpha} \wedge f_u^{\alpha+n}$$

over U, where f_u are complex analytic 1-form over U([18] p. 28). If we take a suitable order of f_u^{α}'s, then the components of ϕ with respect to f_u^{α} and η is

$$\phi_u = \begin{pmatrix} 0 & -E_n & 0 \\ E_n & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

over U. It is easily seen that the coframe (f_u, η) defines a frame over U, hence a local cross section C_u of the associated principal bundle of the tangent bundle.

If $U \cap V$ is nonempty, we see that $C_u = R_g C_v$ where $g \in GL(2n+1, C)$ is a matrix. Then ϕ_u is transformed by

$${}^t g \phi_u g = \phi_v$$

and η is transformed by

$$\eta g = \eta.$$

Since the matrix $\phi_u = \phi_v$, $\eta = \eta$, g is of the following form

$$g = \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$$

where A is a complex sympletic matrix, i. e., $A \in Sp(n,C)$. Thus g is a matrix of $Sp(n, C) \times 1$. By virtue of Lemma 3 the structure group of the associated principal bundle of tangent bundle is reducible to $Sp(n, C) \times 1$. Hence the structure group of the complex analytic tangent bundle is reducible to $Sp(n, C) \times 1$.

Conversely, suppose the structure group of the complex analytic tangent bundle is reducible to $Sp(n, C) \times 1$. Then, by virtue of Lemma 3 on every coordinate neighborhood of a certain covering we can get a local cross section C_u of the associated principal bundle of the tangent bundle such that if $U \cap V$ is nonempty, then $C_u = R_g\ C_v$ where $g \in Sp(n,C) \times 1$. Since C_u is a frame over U, it defines a coframe (f_u, η_u) over U. It is clear that the local tensors ϕ_u, η_u in U, whose components (relative to coframe (f_u, η_u)) are

$$\phi_u = \begin{pmatrix} 0 & -E_n & 0 \\ E_n & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
$$\eta_u = (0, 0, \cdots, 0, 1)$$

constitute globally defined tensors (for ${}^t g\ \phi_u g = \phi_v$ and $\eta_u g = \eta_v$). Thus we get a 2-form ϕ, whose rank is clearly $2n$, and 1-form η (they are complex analytic, because the frames are complex analytic), namely,

$$\phi = \sum_{\alpha=1}^{n} \bar{\epsilon} f_u^\alpha \wedge f_u^{n+\alpha}, \qquad \bar{\epsilon} = 1 \qquad \text{or } -1,$$
$$\eta = \eta_u.$$

It is easily seen that

$$\phi^n = \epsilon \eta f_u^1 \wedge \cdots \wedge f_u^{2n}, \text{ where } \epsilon = 1 \text{ or } -1.$$

Hence we have

$$\epsilon \eta \wedge f_u^1 \wedge \cdots \wedge f_u^{2n} = \eta \wedge \phi^n \neq 0, \text{ at every point.}$$

Thus we could get an analytic vector field

$$\xi^\alpha = \frac{1}{\lambda} \phi_{[\alpha+1\ \alpha+2} \cdots \phi_{2n+1\ 1} \cdots \phi_{\alpha-2\ \alpha-1]}$$

where $\lambda = (2n+1)\eta_{[1}\phi_{23} \cdots \phi_{2n\ 2n+1]}$. Then by virtue of Theorem 4, we get an analytic (ϕ, ξ, η)-structure or complex almost contact structure.

By virtue of this theorem we could give another definition of complex almost contact manifold in analogous way to real contact manifold.

DEFINITION: A complex manifold of complex dimension $2n+1$ is called a complex almost contact manifold or is said to have a complex almost contact structure if and only if the structure group of its complex analytic tangent bundle is reducible to $Sp(n, C) \times 1$.

If we notice that Theorem 4 still holds good even if we take off analyticity of $\phi_{\alpha\beta}$, ξ^{α}, η_{β}, then we get the following theorem.

THEOREM 6. *Let M^{2n+1} be a complex manifold with complex (ϕ, ξ, η, g)-structure, then the structure group is reducible to $Sp(n) \times 1$. Conversely if the group is reducible to $Sp(n) \times 1$, then we can endow to M^{2n+1} a complex* (ϕ, ξ, η, g)-structure.

Proof of this theorem can be easily gotten from modification of the proof of Theorem 5 (e. g., we need not take complex analytic frames but take orthonormal frames).

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