CORRECTION AND REMARK TO "CESÀRO SUMMABILITY OF FOURIER SERIES."

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In my article, "Cesàro summability of Fourier series" this Journal, 5 (1953), 198-210, Lemma 1 (p. 204) and Lemma 2 (p. 208) are wrongly stated.[1] Hence though Theorem 2 (p. 204) is true, its proof has to be modified. Correct lemmas are as follows.

LEMMA 1. *If* $0 < \alpha \leqq 1$ *and* $\beta + 1 \geqq 0$, *then*

$$\int_0^t u^{\beta+1}(t-u)^{\alpha-1}e^{inu}du = O(t^{\beta+1}/n^{\alpha}).$$

Proof of Lemma 1 of the original is valid in this from.

LEMMA 2. *If* $\beta + 1 > \alpha > 0$, *then*

$$\int_0^t u^{\beta+1}(t^2-u^2)^{\alpha-1}\, e^{inu}du = O(t^{\alpha+\beta}/n^{\alpha}).$$

PROOF. (1°) When $0 < \alpha \leqq 1$, $\beta + 1 > 0$, by the successive use of the second mean value theorem and M. Riesz's mean value theorem, we have

$$\int^t u^{\beta+1}(t^2-u^2)^{\alpha-1}\cos nu\, du$$

$$= t^{\beta+1}\int_h^t (t^2-u^2)^{\alpha-1}\cos nu\, du \quad (0 < h < t)$$

$$= t^{\beta+1}(t+h)^{\alpha-1}\int_h^k (t-u)^{\alpha-1}\cos nu\, du \quad (h < k < t)$$

$$= t^{\beta+1}(t+h)^{\alpha-1}\left\{\int_0^k (t-u)^{\alpha-1}\cos nu\, du - \int_0^h (t-u)^{\alpha-1}\cos nu\, du\right\}$$

$$= O(t^{\alpha+\beta}n^{-\alpha}).$$

(2°) When $k < \alpha \leqq k+1$ where k is a positive integer, then $\beta + 1 > \alpha > k$.

1) I am much indebted to Professor Boas for pointing out this to me.

Integrating by part k-times,

$$\int_0^t u^{\beta+1}(t^2-u^2)^{\alpha-1}\cos nu\,du$$
$$=\frac{1}{n^k}\int_0\left(\frac{\partial}{\partial u}\right)^k\left\{u^{\beta+1}(t^2-u^2)^{\alpha-1}\right\}\cos\left(nu+\frac{k\pi}{2}\right)du.$$

The first factor of integrand

$$\left(\frac{\partial}{\partial u}\right)^k\left\{u^{\beta+1}(t^2-u^2)^{\alpha-1}\right\}$$

is a linear combination of terms

$$u^{\beta+1-k}(t^2-u^2)^{\alpha-1},\ u^{\beta-k+3}(t^2-u^2)^{\alpha-2},\cdots,\ u^{\beta+1+k}(t^2-u^2)^{\alpha-k-1}.$$

The integral of the first term is

$$\frac{1}{n^k}\int_0 u^{\beta+1-k}(t^2-u^2)^{\alpha-1}\cos\left(nu+\frac{k\pi}{2}\right)du$$
$$=\frac{1}{n^k}\int_0^t u^{\beta+1-k}(t^2-u^2)^k(t^2-u^2)^{\alpha-k-1}\cos\left(nu+\frac{k\pi}{2}\right)du$$
$$=\frac{(t^2)^k}{n^k}\int^h u^{\beta+1-k}(t^2-u^2)^{\alpha-k-1}\cos\left(nu+\frac{k\pi}{2}\right)du,\quad (0<h<t)$$

by the second mean value theorem. Since $\beta+1-k>0$, $0<\alpha-k\leqq 1$, from M. Riesz's mean value theorem and case (1^0), the last term is

$$O\left(\frac{(t^2)^k}{n^k}\frac{t^{\alpha-k}t^{\beta-k}}{n^{\alpha-k}}\right)=O\left(\frac{t^{\alpha+\beta}}{n^\alpha}\right).$$

The estimation of integral of other terms is all the same, so we get lemma for cosine. For sine we can proceed the same way as cosine.

LEMMA 3. *If $\alpha>0$, then we have for any non-negative integer k*

$$\int_0^t u^{2k}(t^2-u^2)^{\alpha-1}\cos nu\,du=O\left(\frac{t^{\alpha+2k-1}}{n^\alpha}\right)$$
$$\int_0 u^{2k+1}(t^2-u^2)^{\alpha-1}\sin nu\,du=O\left(\frac{t^{\alpha+2k}}{n^\alpha}\right).$$

PROOF. We shall proceed by induction. When $0<\alpha\leqq 1$, these formulas are special cases of Lemma 2, (when $k=0$ and $0<\alpha\leqq 1$, we can prove it in the same way.)

When we suppose these formulas are true for some α, then

$$\int_0^t u^{2k}(t^2-u^2)^\alpha \cos nu\, du$$

$$= -\frac{1}{n}\int_0 \frac{\partial}{\partial u}\left\{u^{2k}(t^2-u^2)^\alpha\right\} \sin nu\, du$$

$$= \frac{-2kt^2}{n}\int_0^t u^{2k-1}(t^2-u^2)^{\alpha-1} \sin nu\, du$$

$$+\frac{2(k+\alpha)}{n}\int_0^t u^{2k+1}(t^2-u^2)^{\alpha-1} \sin nu\, du$$

$$= \frac{2kt^2}{n} O\left(\frac{t^{\alpha+2k-2}}{n^\alpha}\right) + \frac{2(k+\alpha)}{n} O\left(\frac{t^{\alpha+2k}}{n^\alpha}\right) = O\left(\frac{t^{(\alpha+1)+2k-1}}{n^{\alpha+1}}\right).$$

If $k=0$, the first term does not appear.

In the same way, we have

$$\int_0^t u^{2k+1}(t^2-u^2)^\alpha \sin nu\, du$$

$$= \frac{1}{n}\int_0 \frac{\partial}{\partial u}\left\{u^{2k+1}(t^2-u^2)^\alpha\right\} \cos nu\, du$$

$$= \frac{(2k+1)t^2}{n}\int_0 u^{2k}(t^2-u^2)^{\alpha-1} \cos nu\, du$$

$$-\frac{2\alpha+2k+1}{n}\int_0^t u^{2k+2}(t^2-u^2)^{\alpha-1} \cos nu\, du$$

$$= \frac{(2k+1)t^2}{n} O\left(\frac{t^{\alpha+2k-1}}{n^\alpha}\right) - \frac{2\alpha+2k+1}{n} O\left(\frac{t^{\alpha+2k+1}}{n^\alpha}\right) = O\left(\frac{t^{\alpha+1+2k}}{n^{\alpha+1}}\right).$$

Thus induction is completed and Lemma 3 is proved.

PROOF OF THEOREM 2. From the hypothesis $\beta > \gamma$, we have

$$0 < \alpha = \frac{\delta(\beta+1)}{\beta-\gamma+\delta} < \beta + 1.$$

(1^0) when $0 < \alpha \leqq 1$, we proceed the same way as the original paper (pp.205-207).

(2^0) when $\alpha > 1$, we proceed the same way as pp. 208-210.

However, we have to use Lemma 2 in the estimation of term P and use Lemma 3 in other terms.

REMARK. The condition

$$\sum_{\nu=n}^{\infty} |a_\nu|/\nu = O(n^{-(1-\delta)})$$

implies

$$\sum_{\nu=n}^{2n} |a_\nu|/\nu = O(n^{-(1-\delta)}) \quad \text{and} \quad \sum_{\nu=n}^{2n} (|a_\nu| - a_\nu) = o(n^\delta).$$

Hence our theorem is a special case of recent Theorem 4′ of K. Yano, "A remark on convexity theorems for Fourier series", Proc. Japan Acad., 38(1962), 245-247.

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