

ON THE TOPOLOGY OF POSITIVELY CURVED KAEHLER MANIFOLDS

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1. Introduction. In a recent paper [6], S. Kobayashi obtained the following improvement of a theorem due to W. Klingenberg [5]:

A complete holomorphically pinched Kaehler manifold with holomorphic pinching $> 11/13$ has the homotopy type of complex projective space.

The latter employed Morse theory in order to obtain his constant whereas the former used the Sphere Theorem (for odd dimensional Riemannian manifolds) due to M. Berger [1] and Klingenberg [4], together with the bounds on sectional curvature, expressed in terms of the holomorphic pinching, found by Berger [2].

It is the purpose of this paper to improve these bounds on curvature, thereby resulting in a corresponding improvement of the constant $11/13$ in the above theorem. In fact, it is shown that the bounds we obtain are the best possible that can be derived by considering only the algebra of the curvature tensor (at one point). We state our main result.

THEOREM. *A complete holomorphically pinched Kaehler manifold with holomorphic pinching $> 4/5$ has the homotopy type of complex projective space.*

Since our key statement (Proposition 5.1) is an improvement of proposition 11 of [6] due essentially to the sharp bounds on curvature obtained in Proposition 4.2, the material leading to that result will not be duplicated. The reader is therefore asked to refer to this paper both for the sake of completeness and notation.

2. Holomorphic curvature. A plane section is called *holomorphic* if it has a basis $\{X, JX\}$ for some X and *anti-holomorphic* if it has a basis $\{X, Y\}$ where X is perpendicular to both Y and JY . More generally, with each section we associate an acute angle θ which measures by how much the section fails to be holomorphic. If $\{X, Y\}$ is an orthonormal basis of the section, $\cos \theta = |(X, JY)|$; it is readily verified that this is independent of the choice of X and Y .

The *holomorphic curvature* $H(X)$ of a non-zero vector X is the curvature

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of the holomorphic section spanned by X and JX , i.e., $H(X) = k(X, JX)$.

In a Riemannian manifold, it is well-known that the curvature tensor is determined algebraically by the *biquadratic curvature form* B :

$$B(X, Y) = K(X, Y, X, Y).$$

In fact,

$$6K(X, Y, Z, W) = \frac{\partial^2}{\partial s \partial t} (B(X + sZ, Y + tW) - B(X + sW, Y + tZ))|_{s=t=0}.$$

Since sectional curvature $k(X, Y)$ is the quotient of $B(X, Y)$ and $(X, X)(Y, Y) - (X, Y)^2$, it follows that the curvature tensor is algebraically determined by the functions k and (\cdot, \cdot) .

In a Kaehler manifold, we define the *quartic holomorphic curvature form* Q :

$$Q(X) = K(X, JX, X, JX).$$

That the holomorphic sectional curvatures are of fundamental importance is given by

PROPOSITION 2.1. B is determined algebraically by Q .

Perhaps more interesting is the formula which reduces the proof to a verification:

$$(2.1) \quad B(X, Y) = \frac{1}{32} [3Q(X + JY) + 3Q(X - JY) - Q(X + Y) - Q(X - Y) - 4Q(X) - 4Q(Y)].$$

As an immediate consequence of this formula, we derive

COROLLARY 2.1. Let X and Y be orthonormal vectors, and $(X, JY) = \cos \theta \geq 0$. Then

$$(2.2) \quad k(X, Y) = \frac{1}{8} [3(1 + \cos \theta)^2 H(X + JY) + 3(1 - \cos \theta)^2 H(X - JY) - H(X + Y) - H(X - Y) - H(X) - H(Y)].$$

Moreover, if $(X, JY) = 0$, then

$$(2.3) \quad k(X, Y) + k(X, JY) = \frac{1}{4} [H(X + JY) + H(X - JY) + H(X + Y) + H(X - Y) - H(X) - H(Y)],$$

and, more generally,

$$(2.4) \quad k(X, Y) + k(X, JY) \sin^2 \theta = \frac{1}{4} [(1 + \cos \theta)^2 H(X + JY) + (1 - \cos \theta)^2 H(X - JY) + H(X + Y) + H(X - Y) - H(X) - H(Y)].$$

As a consequence, we obtain a well-known result:

If holomorphic curvature is a constant H , then curvature is given by

$$(2.5) \quad k(X,Y) = \frac{H}{4} (1 + 3\cos^2\theta).$$

Formulas (2.2)–(2.4) will be used in §4 to derive the inequalities between curvature and holomorphic curvature required for the proof of Theorem.

3. Curvature as an average. When holomorphic curvature is constant, anti-holomorphic curvature is also a constant $A = H/4$, and (2.5) may be re-written as

$$k(X,Y) = H - 3A\sin^2\theta.$$

For any two orthonormal vectors X and Y such that $(X, JY) > 0$, we say that the holomorphic sections generated by $X \cos \alpha + Y \sin \alpha$ are the *holomorphic sections associated* with the section spanned by the pair X, Y , and the sections spanned by the vectors $X \cos \alpha + Y \sin \alpha, -JX \sin \alpha + JY \cos \alpha$ the *anti-holomorphic sections associated* with X, Y . These ‘circles’ of sections depend only on the plane of X and Y , and not on the choice of the vectors X, Y . If the manifold has constant holomorphic curvature, then H may clearly be interpreted as the average associated holomorphic curvature, and A as the average associated anti-holomorphic curvature. Thus, the following result may be viewed as a generalization of formula (2.5).

PROPOSITION 3.1. *Let $H(X, Y)$ be the average associated holomorphic curvature and $A(X, Y)$ the average associated anti-holomorphic curvature to the plane of the vectors X and Y , i.e., when X and Y are orthonormal,*

$$H(X, Y) = \frac{1}{\pi} \int_0^\pi H(X \cos \alpha + Y \sin \alpha) d\alpha,$$

$$A(X, Y) = \frac{1}{\pi} \int_0^\pi k(X \cos \alpha + Y \sin \alpha, -JX \sin \alpha + JY \cos \alpha) d\alpha.$$

Then,

$$(3.1) \quad k(X, Y) = H(X, Y) - 3A(X, Y)\sin^2\theta.$$

To see this, observe that since $H(X \cos \alpha + Y \sin \alpha)$ and $k(X \cos \alpha + Y \sin \alpha, -JX \sin \alpha + JY \cos \alpha)$ are quartic polynomials in $\cos \alpha, \sin \alpha$, indeed, quadratic polynomials in $\cos 2\alpha, \sin 2\alpha$, their average may be obtained by averaging any four equally spaced values, i.e.,

$$H(X, Y) = \frac{1}{4} [H(X) + H(X + Y) + H(Y) + H(X - Y)],$$

$$A(X, Y) = \frac{1}{4} [k(X, JY) + k(X + Y, -JX + JY) + k(Y, JX) + k(X - Y, JX + JY)]$$

$$= \frac{1}{2} [k(X, JY) + k(X + Y, -JX + JY)].$$

4. Inequalities. In the sequel, assume that the metric of our Kaehler manifold has been normalized so that holomorphic curvature satisfies $\delta \leqq H(X) \leqq 1$. The Kaehler manifold is then said to be δ -holomorphically pinched.

To begin with, we consider anti-holomorphic curvature. From formula (2.2), with $\cos \theta = 0$, we obtain

PROPOSITION 4.1. *If X, Y span an anti-holomorphic section, then*

$$\frac{3\delta - 2}{4} \leqq k(X, Y) \leqq \frac{3 - 2\delta}{4}.$$

To get an upper bound for an arbitrary sectional curvature, we eliminate the function $H(X, Y)$ occurring in (2.2) and (3.1), thereby obtaining

$$k(X, Y) = \frac{1}{4} [(1 + \cos \theta)^2 H(X + JY) + (1 - \cos \theta)^2 H(X - JY)] - A(X, Y) \sin^2 \theta.$$

Using the lower bound for $A(X, Y)$ obtained from Proposition 4.1 results in the inequality

$$k(X, Y) \leqq 1 - \frac{3\delta \sin^2 \theta}{4}.$$

To obtain a lower bound we apply formula (2.2) directly. Thus,

$$k(X, Y) \geqq \frac{1}{8} [6(1 + \cos^2 \theta)\delta - 4].$$

PROPOSITION 4.2. *Let X and Y be orthonormal vectors on a δ -holomorphically pinched Kaehler manifold. Then, if $(X, JY) = \cos \theta$,*

$$\frac{1}{4} [3(1 + \cos^2 \theta)\delta - 2] \leqq k(X, Y) \leqq \frac{4 - 3\delta(1 - \cos^2 \theta)}{4}.$$

5. Proof of Theorem. To begin with, an analogue of proposition 11 in [6] is obtained by employing Proposition 4.2 above. Indeed, we prove [see 6, proposition 11 for notation]

PROPOSITION 5.1. *If*

$$\frac{1}{4} [3(1 + \cos^2 \theta)\delta - 2] \leqq k \leqq \frac{4 - 3\delta(1 - \cos^2 \theta)}{4}$$

[see Proposition 4.2], and if

$$3\delta - 2 \leqq 4a^2 \leqq \delta,$$

then,

$$\frac{3\delta - 2}{4} \leqq S(p) \leqq 1 - 3a^2.$$

$S(p)$ is the sectional curvature of the principal circle bundle over the Kaehler manifold whose pinching is given by Proposition 4.2 [see 6, §5].

By proposition 5 of [6], we have

$$\begin{aligned}
 S(p) &\leq \frac{1}{4}(1 - X^0X^0 - Y^0Y^0)[4 - 3\delta(1 - \cos^2\theta) - 12a^2\cos^2\theta] + a^2(X^0X^0 + Y^0Y^0) \\
 &\leq \frac{1}{4}(1 - X^0X^0 - Y^0Y^0)[4 - 3\delta + 3(\delta - 4a^2)] + a^2(X^0X^0 + Y^0Y^0) \\
 &= (1 - 3a^2) + (4a^2 - 1)(X^0X^0 + Y^0Y^0) \\
 &\leq 1 - 3a^2.
 \end{aligned}$$

Moreover,

$$\begin{aligned}
 S(p) &\geq \frac{1}{4}(1 - X^0X^0 - Y^0Y^0)[3(1 + \cos^2\theta)\delta - 2 - 12a^2\cos^2\theta] + a^2(X^0X^0 + Y^0Y^0) \\
 &\geq \frac{1}{4}(1 - X^0X^0 - Y^0Y^0)(3\delta - 2) + a^2(X^0X^0 + Y^0Y^0) \\
 &= \frac{1}{4}[(3\delta - 2) + (4a^2 + 2 - 3\delta)(X^0X^0 + Y^0Y^0)] \\
 &\geq \frac{1}{4}(3\delta - 2).
 \end{aligned}$$

COROLLARY 5.1. *If $\delta = 4a^2$, then*

$$\frac{3\delta - 2}{4} \leq S(p) \leq \frac{4 - 3\delta}{4}.$$

Hence, the method used in the proof of theorem 1 of [6] gives

Let M be a complete Kaehler manifold with holomorphic pinching $> \delta$. Then, there are a principal circle bundle P over M and a Riemannian metric on P with Riemannian pinching $> (3\delta - 2)/(4 - 3\delta)$.

Consequently, if $\delta = 4/5$, then $(3\delta - 2)/(4 - 3\delta) = 1/4$. The proof of Theorem is now a consequence of the Spherc Theorem.

6. Remarks: (a) Further improvement of Proposition 4.2 by the methods employed above (consideration of the curvature at one point) is precluded by the example given below where the curvature components $R_{ijkl} = K(X_i, X_j, X_k, X_l)$ are taken with respect to an orthonormal basis $X_1, X_2, X_3 = JX_1, X_4 = JX_2$. In this example, $\delta \leq H(X) \leq 1$.

Take $R_{1212} = (2-\delta)/4, R_{1213} = R_{1214} = R_{1224} = R_{1314} = R_{1424} = 0, R_{1313} = R_{2424} = 1$ and $R_{1414} = (3\delta - 2)/4$.

The other curvature components are determined by the usual curvature identities together with the identity $K(X_i, X_j, X_k, X_l) = K(JX_i, JX_j, X_k, X_l)$.

We therefore have $(3\delta - 2)/4 \leq k(X, Y) \leq 1$.

(b) Theorems 3 and 4 of [6] have been greatly improved by a different method. In fact, the best possible statements have been obtained [3]:

Let M be a complete Kaehler manifold of strictly positive curvature. Then

$$H^2(M, R) = R.$$

A homogeneous Kaehler manifold of strictly positive curvature is an Einstein space, i.e., a space of constant mean curvature.

Let M be a compact δ -holomorphically pinched Kaehler manifold with $\delta > 1/2$. Then,

$$H^2(M, R) = R.$$

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