ON THE PARALLELISABILITY UNDER RIEMANNIAN METRICS OF DIRECTION FIELDS OVER 3-DIMENSIONAL MANIFOLDS, II

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The present note is continued from the preceding paper [1]. The object is to prove Theorem 9, and to induce Theorem 10 that is the desired conclusion. First let us explain some terminologies, though they are already defined in [1] except few ones, so that the meaning of Theorems 9, 10 can be immediately grasped. As for other terminologies and notations etc., see [1].

An *S-manifold* is a connected differentiable manifold over which a differentiable field of directions (oriented) is given. This field and each of its maximal integral curves are called the *S-field* and an *S-orbit* respectively of the S-manifold. That an S-manifold is *S-diffeomorphic* onto an S-manifold means that there exists a diffeomorphism preserving S-orbit. An *RS-manifold* is a connected complete differentiable Riemannian manifold over which a parallel field of directions is given. We shall sometimes regard it as an S-manifold whose S-field is the parallel field. The field of tangent vector subspaces orthogonal and complementary to the S-field is called the *R-field* and each of its maximal integral manifolds with the induced metric an *R-orbit*. An *RS-torus* is a locally Euclidean RS-manifold whose underlying manifold is a torus. The notation "$\times$" means the operation of metric product.

Let E be the Euclidean 1-space $\{t \mid -\infty < t < \infty\}$ and dt denotes the infinitesimal distance. Let R be a 2-dimensional connected complete differentiable Riemannian manifold. Then we define

A_0-manifold: RS-manifold $R \times E$ where each S-orbit is defined by (x, E) for fixed $x \in R$.

Let E' be the part $\{t \mid 0 \leqq t < \infty\}$ of E. For a constant $L > 0$ let $[L]$ be the part $\{t \mid 0 \leqq t \leqq L\}$ of E. Let X be a 2-dimensional RS-torus whose *S-orbits* are all non-closed and let S_X be any one of its S-orbits. Then we define

A_1-manifold: RS-manifold $X \times E$ where each S-orbit is defined by (S_X, t) for fixed $t \in E$.

A_2-manifold: RS-manifold formed from $X \times [L]$ by identifying (x, L) with $(J(x), 0)$ for all $x \in X$, where J is an isometry of X leaving the S-field invariant. Each S-orbit is defined by (S_X, t) for fixed $t \in [L]$.

A_3-manifold: RS-manifold formed from $X \times E'$ by identifying $(x, 0)$ with

$(J_0(x), 0)$ for all $x \in X$, where J_0 is an involutive isometry of X having no fixed point and leaving the S-field invariant. Each S-orbit is defined by (S_x, t) for fixed $t \in E'$.

A_4-manifold: RS-manifold formed from $X \times [L]$ by identifying $(x,0)$ with $(J_1(x), 0)$ and (x, L) with $(J_2(x), L)$ for all $x \in X$, where J_1, J_2 are isometries of X having the same properties as J_0 above. Each S-orbit is defined by (S_x, t) for fixed $t \in [L]$.

A_5-manifold: RS-torus of dimension 3, where each S-orbit is dense there as subset.

Let Y be Euclidean, elliptic, or spherical 2-space. Take an isometry J of Y leaving a point $x_0 \in Y$ fixed, i. e., a rotation at x_0, whose rotation angle $\theta (0 \leqq \theta \leqq \pi)$ satisfies $\pi/\theta =$ irrational number. Let B be an RS-manifofld formed from $Y \times [L]$ by identifying (x, L) with $(J(x), 0)$ for all $x \in Y$, where each R-orbit is defined by $t =$ const. $(t \in [L])$. Then we define

B_1-manifold: RS-manifold B where Y is Euclidean.

B_2-manifold: RS-manifold B where Y is elliptic.

B_3-manifold: RS-manifold B where Y is spherical.

Suppose that Y is spherical. Let L_0 be the half length of a closed geodesic on Y. Let u be any tangent unit vector at a point $x_0 \in Y$. By (x_0, u, s) we denote the terminal point on the geodesic arc issuing from x_0 whose direction is of u and whose length is s. Then we define

B_4-manifold: RS-manifold formed from $Y \times [L]$ by identifying $((x_0, u, s), L)$ with $((x_0, J{\cdot}u, L_0 - s), 0)$ for all u and $s (0 \leqq s \leqq L_0)$, where each R-orbit is defined by $t =$const. $(t \in [L])$.

Now let us prove the following theorem more excellent than Theorem 3.

THEOREM 3'. *In a 3-dimensional RS-manifold M, suppose that all the S-orbits are non-closed and that M satisfies Hypotheses II and II_2. Then M is an A_5-manifold.*

By Lemma 4.6 and Theorems 1,2,4, it is verified that this theorem is equivalent to the following

THEOREM 9. *In a 3-dimensional RS-manifold M, suppose that there exists an S-orbit dense in M as subset. Then M is an A_5-manifold.*

Under this assumption, it follows that any S-orbit is dense in M as subset and that M and any R-orbit are Euclidean space forms, from Lemma 4.2 and Theorem 3. Accordingly it suffices to prove that M is homeomorphic onto a 3-dimensional torus.

Let G be the Lie group which consists of all the isometries of M. On the other hand the S-field induces a Killing (unit) vector field over M. Let H be the one-parameter subgroup of G generated from the Killing vector field. Let H^* be the subgroup of G which is the closure of H in G. Then H^* is a closed abelian subgroup and a connected Lie group. As H^* contains the one-parameter group H dense in H^*,H^* must be a toral group ([2], p.83).

Take a point $x_0 \in M$. The set $I(x_0)$ is a subset dense in $R(x_0)$. For any $x \in I(x_0)$ let J_x denote the R-map with respect to $S[x_0, x]$. Then we have

$$d_R(x_0,x) = d_R(y,J_x(y)),\ d_R(x_0,y) = d_R(x,J_x(y))$$

for any $y \in I(x_0)$, by Lemma 4.7. Hence it follows that J_x is a parallel translation on $R(x_0)$ a Euclidean space form. Take an R-frame F_0 at x_0 and put $F_x = J_x \cdot F_0$. Then the R-frames F_x, planted at all $x \in I(x_0)$, are parallel to each other on $R(x_0)$. So, $R(x_0)$ admits a parallel field of R-frames containing the R-frames F_x. By R-map, transplant this parallel field on each of the R-orbits. We obtain over M the parallel field of R-frames. From now on let F_x denote the element of this parallel field at $x \in M$. The 3-dimensional frames $(F_x,d(x))$, for all $x \in M$, form a parallel field of tangent frames over M. As is easily shown, for any $x \in M$ there is an isometry of M carrying $(F_0,d(x_0))$, where $F_0 = F_{x_0}$ to $(F_x,d(x))$. This isometry belongs to H^* and conversely an element of H^* is such one. So the map

$$f: H^* \to M \text{ defined by } f(J) = J(x_0),$$

where $J \in H^*$, is one-to-one and onto. As it is easily seen to be continuous, the map f is a homeomorphism from the compactness of H^*. Hence, M is homeomorphic onto a torus. Therefore our theorem has been proved.

Summing up Theorems 6,7 and 9, we have

THEOREM 10. *In a 3-dimensional S-manifold V suppose that there exists a non-closed S-orbit. Then a necessary and sufficient condition that V admit a complete differentiable Riemannian metric leaving its S-field to be a parallel field is that V be S-diffeomorphic onto an A_i-manifold* ($i = 0,1,2,3,4,$ *or* 5) *or a B_j-manifold* ($j = 1,2,3,$ *or* 4).

BIBLIOGRAPHY

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