STRONG AND ORDINARY SUMMABILITY

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1. Introduction. We consider infinite matrices $A = (a_{nk})$ and corresponding matrix transforms and summability methods (compare [5]). A sequence $\{s_k\}$ is said to be $\overline{A}$-*summable* to the value σ, if all sums

$$(1) \qquad \sigma_n = \sum_{k=0}^{\infty} a_{nk} s_k, \ n = 0, 1, \cdots$$

exist and converge to σ for $k \to \infty$. The sequence $\{s_k\}$ is *strongly A-summable* (shortly: $\overline{A}$-*summable*) to the value σ, if all sums

$$(2) \qquad \sigma_n = \sum_{k=0}^{\infty} a_{nk} \mid s_k - \sigma \mid, \ \ n = 0, 1, \cdots$$

exist and converge to zero. Strong summability is usually considered only for positive A (i.e., for $a_{nk} \geqq 0$). In this case the limit σ is uniquely determined [3] if A is *regular*, i.e., sums each convergent sequence to its ordinary limit. A *row-finite* matrix contains only a finite number of non-zero elements in each row; a *normal* matrix has non-zero elements on the main diagonal and zeros above it.

We compare here strong and ordinary summability methods. The basic question is the following. Given a matrix A, does there exist a matrix B, such that a sequence $\{s_k\}$ is B-summable if and only if it is strongly A-summable? (In this case B and $\bar{A}$ are called *equivalent*). For the Cesàro method of order one, $A = C_1$, the question has been answered positively in [4]. We generalize this result to arbitrary row-finite regular matrices A (Theorem 1). There exist, however, row-infinite regular matrices A for which no equivalent B exists (Theorem 4). Even for row-finite regular A it is not always possible to find a normal B equivalent to $\overline{A}$. We give (Theorem 2) necessary and sufficient conditions for the existence of a matrix B with these properties. As a simple special case of Theorem 2 we have: the method $\overline{A}$ is not equivalent to any normal ordinary method B if $\rho_k = \max_n a_{nk} \to 0$ as $k \to \infty$. A corollary (Theorem 3) of Theorems 1 and 2 concerns the question of equivalence of ordinary row-finite and normal methods.

We avoid the use of Functional Analysis (although its application could

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shorten some proofs). For simplicity we assume that the matrix A of the strong summability method is always regular and positive; generalizations are possible.

2. Row-finite matrices. A strong summability method $\bar{A}$, based on a row-finite regular matrix A, can always be replaced by an ordinary matrix method:

THEOREM 1. *For each row-finite regular positive matrix A there is a row-finite regular positive matrix B such that a sequence $\{s_k\}$ is $\bar{A}$-summable to the value σ if and only if it is B-summable to this value.*

The proof is based on

LEMMA 1. *Let K be a finite set of natural numbers, let x_k be complex and a_k positive ($a_k \geqq 0$) values defined for $k \in K$. Put*

$$(3) \qquad \sum_{k\in K} |x_k| = x, \qquad \sum_{k\in K} a_k = a.$$

Then there exists a subset K' of K such that

$$(4) \qquad \left|\sum_{k\in K'} x_k\right| \geqq \frac{1}{6}\, x, \qquad \sum_{k\in K'} a_k \geqq \frac{1}{2}\, a.$$

PROOF. One of the sums $\sum |\mathrm{Re}\, x_k|$ or $\sum |Im x_k|$ is not less than $\frac{1}{2}\, x$, hence it is sufficient to derive (4) from (3) for the case of real x_k but with $\frac{1}{6}\, x$ replaced by $\frac{1}{3}\, x$. For real x_k, we destinguish two cases. If $\left|\sum_{k\in K} x_k\right| \geqq \frac{1}{3}\, x$, we can take $K' = K$. If this absolute value is less than $\frac{1}{3}\, x$, let K^+, K^- denote the sets of k with $x_k \geqq 0$ or $x_k < 0$, respectively. Then we select K' equal to one of the sets K^+, K^-, so as to satisfy the second condition (4); we will also have $\left|\sum_{k\in K'} x_k\right| \geqq \frac{1}{3} x$

PROOF OF THEOREM 1. It is obviously sufficient to find a regular matrix B such that $\bar{A}$-lim $s_n = 0$ and B-lim $s_n = 0$ are equivalent.

For each $n = 0, 1, \cdots$, let $K = K_n$ be the finite set of integers k for which $a_{nk} > 0$, $k \in K$, $a_{nk} = 0$, $k \notin K$. If $\sum_k a_{nk} = a_n$, we consider all subsets $K' = K'_{n\nu}$, $\nu = 1, 2, \cdots, N(n)$ of K_n which have the property $\sum_{k\in K'} a_{nk} \geqq \frac{1}{2}\, a_n$. Corresponding to one row a_{nk} of A, let us define $N(n)$ rows of a matrix B (each corresponding to a set $K'_{n\nu}$) which consist of the numbers

(5) $b_{mk} = (\sum_{k \in K'} a_{nk})^{-1} a_{nk}$ if $k \in K'_{nv}, b_{mk} = 0$ if $k \notin K'_{nv}$.

Since A is regular, for some $M > 0$, $a_n \leqq M$, and hence $a_{nk} \leqq Mb_{mk}$. We order the rows of B in the following way: first $N(0)$ rows corresponding to the row a_{0k} of A; then $N(1)$ rows corresponding to the row a_{1k} of A; and so on.

It is easy to see that B is regular, and that $\sum b_{nk}s_k$ converges to zero whenever the sequence s_n has the property $\sum_{k \in K_n} a_{nk}|s_k| \to 0$. Conversely, if s_k is B-summable to zero, then taking $x_k = a_{nk}s_k$, $a_k = a_{nk}$ in Lemma 1, we see that for at least one m with the corresponding set K'_{nv},

$$(6) \qquad \sum_{k \in K_n} a_{nk}\,|s_k| \leqq 6\Big|\sum_{k \in K'_{nv}} a_{nk}s_k\Big| \leqq 6M\Big|\sum_k b_{mk}s_k\Big|.$$

Thus, s_n is $\overline{A}$-summable to zero, and the result follows.

3. Normal matrices. In contrast to Theorem 1, it is not always possible to replace a row-finite strong summability method by a *normal* matrix method. We prove more. We consider also row-infinite matrices, and give necessary and sufficient conditions when this replacement is possible.

For a regular positive matrix A we write

$$(7) \qquad \rho_k = \max_n a_{nk}, \ k = 0, 1, \cdots.$$

THEOREM 2. *Let A be a regular positive matrix. There exists a normal method B which is equivalent to $\overline{A}$ if and only if for some M and k_0,*

$$(8) \qquad \rho_k \neq 0, \qquad k \geqq k_0,$$

$$(9) \qquad \sum_{k=k_0}^{\infty} a_{nk}\rho_k^{-1} \leqq M, n = 0, 1, \cdots.$$

If the conditions are satisfied, B may be taken to be regular and consistent with $\overline{A}$.

The following two lemmas will be needed:

LEMMA 2. *Let $\rho_k \geqq 0$, $k = 0,1, \cdots$ be an arbitrary sequence and B be an arbitrary matrix method. Then* (i) *B sums all sequences $\{s_k\}$ with $\rho_k s_k \to 0$ if and only if*

$$(10) \qquad \lim_{n\to\infty} b_{nk} \qquad \textit{exists for each } k = 0, 1, \cdots,$$

and there are M and k_0 such that

(11) $\rho_k = 0,\ k \geqq k_0$ *implies* $b_{nk}=0,\ n=0,1,\cdots,$

(12) $$\sum_{\rho_k \neq 0} |b_{nk}|\rho_k^{-1} \leqq M,\ n = 0, 1, \cdots.$$

(ii) *B sums all sequences* $\{s_k\}$ *with* $\sum \rho_k |s_k| < +\infty$ *if and only if B satisfies* (10) *and there are M and* k_0 *such that*

(13) $$|b_{nk}| \leqq M\rho_k, \quad k \geqq k_0.$$

PROOF. In both cases (i), (ii), it is easy to prove the necessity of the existence of a k_0 for which (11) is satisfied. If this holds, we can omit from B all columns b_{nk} for which $\rho_k = 0$. Thus it is sufficient to prove our lemma for the case when $\rho_k > 0,\ k = 0,1,\cdots$. But then the lemma follows from the well-known theorems about matrices which sum all null sequences, or all absolutely convergent series (compare for example [1, p. 63] for (i), and [2,p.29] for (ii)).

LEMMA 3. *Let A be a positive regular matrix, with the* ρ_k *defined by* (7). *Then* $\overline{A}$-lim $s_n = 0$ *implies* $\rho_k s_k \to 0$.

PROOF. Assume that $\{s_n\}$ is $\overline{A}$-summable to zero, we have to prove that $\rho_k s_k \to 0$. Let $\epsilon > 0$ be arbitrary. We take first N so large that $\sum_k a_{nk} |s_k| < \epsilon$ for $n \geqq N$ and then K so large that $a_{nk}|s_k| < \epsilon$ for $k \geqq K,\ n < N$. Then $a_{nk}|s_k| < \epsilon$ for $k \geqq K$ and all n, hence $\rho_k |s_k| < \epsilon,\ k \geqq K$.

PROOF OF THE SUFFICENCY OF THE CONDITIONS. Assume that the conditions of Theorem 2 are satisfied. We may suppose that $k_0 = 0$. Since A is regular, the sequence ρ_k is bounded. We cannot have $\rho_k \to 0$, since then (9) would imply that the sequence $\{1,1,\cdots\}$ is A-summable to zero, in contradiction to the regularity. Hence we can find a sequence k_j strictly increasing to infinity for which $\rho_{k_j} \to \rho \neq 0$. We define B as follows:

(14) $$B = \begin{pmatrix} 1 & & & & & & \\ 1-\rho_1 & \rho_1 & & & & & \\ \vdots & & \ddots & & & & \\ 1-\rho_{k_1-1} & & & \rho_{k_1-1} & & & \\ 0 & & & & 1 & & \\ & & & & 1-\rho_{k_1+1} & \rho_{k_1+1} & \\ & & & & \vdots & & \ddots \\ & & & & 1-\rho_{k_2-1} & & & \rho_{k_2-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

The matrix B is regular, and one easily sees that B-lim $s_n = 0$ is equivalent to $\rho_n s_n \to 0$. By Lemma 2 (i) applied to the matrix A, we have A-lim $s_n = 0$, and hence even $\bar{A}$-lim $s_n = 0$ for all sequences s_n with $\rho_n s_n \to 0$. By Lemma 3, $\bar{A}$-lim $s_n = 0$ is equivalent to $\rho_n s_n \to 0$. This, together with the regularity and the linearity of the methods B and A, implies that B-lim $s_n = \sigma$ is equivalent to $\bar{A}$-lim $s_n = \sigma$.

PROOF OF THE NECESSITY. We begin with the following

LEMMA 4. *If a normal matrix B and a sequence $\epsilon_k > 0$ are given, there exists a sequence $\{s_k\}$ with the properties*

$$|b_{kk}|\ |s_k| \geqq \epsilon_k,\ |\sigma_n| = \epsilon_n, \tag{15}$$

where σ_n is the B-transform of $\{s_k\}$.

Making $\epsilon_k \to 0$ slowly, we obtain a sequence s_k which is B-summable to zero, and whose terms in absolute value are close to $|b_{kk}|^{-1}$.

PROOF. We construct s_k by induction. Put $s_0 = \epsilon_0 b_{00}^{-1}$. If $s_0, \cdots, s_{k-1}$ are already determined, let $\tau_k = b_{k0}s_0 + \cdots + b_{k,k-1}\, s_{k-1}$. We choose s_k so that the modulus of $b_{kk}s_k$ is $|\tau_k| + \epsilon_k$, and the sign is opposite to that of τ_k; the sequence s_k satisfies (15).

Now we assume that there is a normal method B equivalent to $\bar{A}$. From Lemma 2 (ii) we derive that each sequence s_k with $\sum \rho_k |s_k| < +\infty$ is $\bar{A}$-summable to 0. This applies also to $|s_k|$, hence s_k is $\bar{A}$-summable to zero, and thus B-summable. Again from Lemma 2 (ii) we derive that

$$|b_{kk}| \leqq M\rho_k,\ k \geqq k_0. \tag{16}$$

Since $b_{kk} \neq 0$, we must have $\rho_k \neq 0$, $k \geqq k_0$, so that (8) is satisfied.

Applying Lemma 4 and (16), we find, for each null sequence $\epsilon_k > 0$, a sequence s_k, B-summable to zero, for which $M\rho_k|s_k| \geqq \sqrt{\epsilon_k}$. Hence s_k is $\bar{A}$-summable and the sequence $\epsilon_k\rho_k^{-1} = o(|s_k|)$ is $\bar{A}$-summable to zero. Applying Lemma 2 (i) to the matrix with the coefficients $a_{nk}\rho_k^{-1}$, we see that also the condition (9) is satisfied. This completes the proof of Theorem 2.

Theorems 1 and 2 contain the following corollary:

THEOREM 3. *There exists a row-finite regular matrix B which provides a 1-1 mapping and which is not equivalent to any normal matrix.*

PROOF. We take the matrix B which corresponds to the strong C_1-

summability according to the proof of Theorem 1. Since B contains all rows of C_1, it provides a 1-1 mapping as the latter matrix.

If the restriction to a 1-1 mapping is omitted, the construction of B becomes trivial. In this case one can take for B any row-finite regular matrix for which $b_{nk_i} = -b_{n,k_i+1}$, $n, i = 0, 1, \cdots$ for some sequence $k_i \to \infty$.

4. **Row-infinite matrices.** Another counterpart of Theorem 1 is the fact that a row-infinite strong summability method is in general not equivalent to an ordinary matrix method:

THEOREM 4. *There exists a row-infinite regular positive matrix A such that no ordinary matrix method B sums exactly the strongly A-summable sequences.*

PROOF. We put

$$A = \begin{pmatrix} 2^{-0} & 0 & 2^{-1} & 0 & 2^{-2} & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}.$$

The strongly A-summable sequences $\{s_k\}$ are exactly the sequences for which

$$\sum_{k=0}^{\infty} 2^{-k} |s_{2k}| < +\infty; \qquad \lim_{k\to\infty} s_{2k-1} \text{ exists.} \tag{17}$$

If B sums every such sequence, then the matrix C:

$$c_{nk} = 2^k b_{n,2k} \tag{18}$$

sums every sequence $\{w_k\}$ with $\sum |w_k| < +\infty$. The statement of the theorem is therefore a consequence of the following lemma:

LEMMA 5. *If a matrix C sums every sequence $\{w_k\}$ satisfying $\sum |w_k| < +\infty$, then it sums also a sequence $\{x_k\}$ with $\sum |x_k| = +\infty$.*

PROOF. If is easy to see (and is also the special case of Lemma 2(ii) when all $\rho_k = 1$) that the assumption about C of the lemma is equivalent to the following. There exists an $M \geqq 0$ and a (bounded) sequence $\{c_k\}$ such that

$$|c_{nk}| \leqq M, \quad n, k = 0, 1, \cdots, \tag{19}$$

$$\lim_{n\to\infty} c_{nk} = c_k, \quad k = 0, 1, \cdots. \tag{20}$$

By means of these conditions we construct the required sequence x_k. We define recursively integers $k_0 < l_0 < k_1 < l_1 < \cdots$ and $n_0 < n_1 < \cdots$ such that:

$$|c_{k_j} - c_{l_j}| \leqq 2^{-j},\ j = 0,1,\cdots; \tag{21}$$

$$|c_{nk_j} - c_{nl_j}| \leqq 2^{-j},\ n \leqq n_j; \tag{22}$$

$$|c_{nk_j} - c_{k_j}| \leqq 2^{-j},\ |c_{nl_j} - c_{l_j}| \leqq 2^{-j},\ n > n_{j+1}. \tag{23}$$

If k_{j-1}, l_{j-1}, n_j are already determined, we extract a convergent subsequence from the bounded vector sequence $\{c_k, c_{0k}, c_{1k}, \cdots c_{n_jk}\}_{k=0,k=0}^{\infty\ \ \infty}$ and hence are able to satisfy (21) and (22) with proper k_j, l_j; an integer n_{j+1}, suitable for (23), exists because of (20). From (23) and (21) we derive

$$|c_{nk_j} - c_{nl_j}| \leqq 3\cdot 2^{-j},\ n > n_{j+1}. \tag{24}$$

Now we put

$$x_{k_j} = -x_{l_j} = \frac{1}{j+1},\ j = 0, 1, \cdots; \qquad x_k = 0 \text{ for other } k. \tag{25}$$

The C-transform of x_k

$$\sum_k c_{nk}x_k = \sum_{j=0}^{\infty}(c_{nk_j} - c_{nl_j})\frac{1}{j+1} \tag{26}$$

exists because of (22). Also,

$$\sum_{j=0}^{\infty} |c_{nk_j} - c_{nl_j}| \leqq 3\sum_{j=0}^{\infty} 2^{-j} + 2M,$$

because of (22), (24) and (19). By a variant of Toeplitz' theorem ([1, p. 63]; this is the special case of Lemma 2(i) when all $\rho_k = 1$), the matrix $D = (d_{nk})$, $d_{nj} = c_{nk_j} - c_{nl_j}$ sums all null sequences. Hence $\{x_k\}$ is C-summable, while $\sum |x_k| = +\infty$.

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