ON THE CROSS-NORM OF THE DIRECT PRODUCT OF C^*-ALGEBRAS

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In [7] Turumaru introduced the notion of the direct product of C^*-algebras. Let $\mathfrak{A}_1 \odot \mathfrak{A}_2$ be the algebraic direct product of two C^*-algebras $\mathfrak{A}_1$ and $\mathfrak{A}_2$. Then $\mathfrak{A}_1 \odot \mathfrak{A}_2$ becomes a $*$-algebra under the natural algebraic operations. Turumaru's C^*-direct product $\mathfrak{A}_1 \widehat{\otimes}_\alpha \mathfrak{A}_2$ is given as the completion of $\mathfrak{A}_1 \odot \mathfrak{A}_2$ under the norm given by

$$\left\| \sum_{i=1}^n a_{1,i} \otimes a_{2,i} \right\| = \sup \frac{\varphi_1 \otimes \varphi_2 \left[\left(\sum_{j=1}^m x_{1,j} \otimes x_{2,j} \right)^* \left(\sum_{i=1}^n a_{1,i} \otimes a_{2,i} \right)^* \left(\sum_{i=1}^n a_{1,i} \otimes a_{2,i} \right) \left(\sum_{j=1}^m x_{1,j} \otimes x_{2,j} \right) \right]^{1/2}}{\varphi_1 \otimes \varphi_2 \left[\left(\sum_{j=1}^m x_{1,j} \otimes x_{2,j} \right)^* \left(\sum_{j=1}^m x_{1,j} \otimes x_{2,j} \right) \right]^{1/2}}$$

$\sum_{i=1}^n a_{1,i} \otimes a_{2,i} \in \mathfrak{A}_1 \odot \mathfrak{A}_2$, where φ_1, φ_2 run over the set of all states of $\mathfrak{A}_1$, $\mathfrak{A}_2$ and $\sum_{j=1}^m x_{1,j} \otimes x_{2,j}$ runs over $\mathfrak{A}_1 \odot \mathfrak{A}_2$. Let us call this norm T-cross norm. Then T-cross norm has the following property: If π_1 and π_2 are representations of $\mathfrak{A}_1$ and $\mathfrak{A}_2$ to Hilbert spaces $\mathfrak{H}_1$ and $\mathfrak{H}_2$ respectively, then the naturally defined product representation $\pi_1 \otimes \pi_2$ of $\mathfrak{A}_1 \odot \mathfrak{A}_2$ to the product Hilbert space $\mathfrak{H}_1 \otimes \mathfrak{H}_2$ is continuous with respect to T-cross norm, so that $\pi_1 \otimes \pi_2$ can be extented to the representation of $\mathfrak{A}_1 \widehat{\otimes}_\alpha \mathfrak{A}_2$ which is also denoted by $\pi_1 \otimes \pi_2$. Besides, if π_1 and π_2 are faithful then $\pi_1 \otimes \pi_2$ becomes faithful. Hence T-cross norm is very natural norm of $\mathfrak{A}_1 \odot \mathfrak{A}_2$. But it is another matter whether or not T-cross norm is unique compatible norm of $\mathfrak{A}_1 \odot \mathfrak{A}_2$, where a norm β of $\mathfrak{A}_1 \odot \mathfrak{A}_2$ is called compatible to the algebraic structure of $\mathfrak{A}_1 \odot \mathfrak{A}_2$ if the completion of $\mathfrak{A}_1 \odot \mathfrak{A}_2$ by β becomes a C^*-algebra and $\|x_1 \otimes x_2\|_\beta \leqq \|x_1\| \cdot \|x_2\|$ for $x_1 \in \mathfrak{A}_1$ and $x_2 \in \mathfrak{A}_2$. In the present note we shall answer for this question that T-cross norm is smallest among the compatible norms and that T-cross norm is unique in $\mathfrak{A}_1 \odot \mathfrak{A}_2$ for C^*-algebra $\mathfrak{A}_1$ of certain class but it is not so in general. So we say that C^*-algebra $\mathfrak{A}_1$ has the property (T) if the following is true;

(T): For every C^*-algebra $\mathfrak{A}_2$ T-cross norm of $\mathfrak{A}_1 \odot \mathfrak{A}_2$ is the unique compatible norm.

LEMMA 1. *If π is a $*$-representation of $\mathfrak{A}_1 \odot \mathfrak{A}_2$ to a Hilbert space $\mathfrak{H}$ which* *is continuous relative to a compatible norm, then there exist unique representations* π_1 *and* π_2 *of* $\mathfrak{A}_1$ *and* $\mathfrak{A}_2$ *to* $\mathfrak{H}$ *such that*

$$\pi(x_1 \otimes x_2) = \pi_1(x_1)\pi_2(x_2) = \pi_2(x_2)\pi_1(x_1)$$

for $x_1 \in \mathfrak{A}_1$ *and* $x_2 \in \mathfrak{A}_2$.

This is nothing but [3 : Prop. 1.] So we omit the proof. We call π_1 and π_2 the restrictions of π to $\mathfrak{A}_1$ and $\mathfrak{A}_2$ respectively. Let $\mathfrak{A}_1 \widehat{\otimes}_\beta \mathfrak{A}_2$ be the completion of $\mathfrak{A}_1 \odot \mathfrak{A}_2$ by a compatible norm β. For a state σ of $\mathfrak{A}_1 \widehat{\otimes}_\beta \mathfrak{A}_2$ let π_σ, ξ_σ and $\mathfrak{H}_\sigma$ be the cyclic representation, the cyclic vector and the cyclic Hilbert space respectively. Let $\pi_{\sigma,1}$ and $\pi_{\sigma,2}$ be the restrictions of π_σ to $\mathfrak{A}_1$ and $\mathfrak{A}_2$ respectively. By the equations

$$\sigma_1(x_1) = (\pi_{\sigma,1}(x_1)\ \xi_\sigma,\ \xi_\sigma) \text{ and } \sigma_2(x_2) = (\pi_{\sigma,2}(x_2)\xi_\sigma, \xi_\sigma)$$

for $x_1 \in \mathfrak{A}_1$ and $x_2 \in \mathfrak{A}_2$, we define the restrictions σ_1 and σ_2 of σ to $\mathfrak{A}_1$ and $\mathfrak{A}_2$ respectively. If $\sigma(x_1 \otimes x_2) = \sigma_1(x_1)\sigma_2(x_2)$ for $x_1 \in \mathfrak{A}_1$ and $x_2 \in \mathfrak{A}_2$, then we write $\sigma = \sigma_1 \otimes \sigma_2$. Conversely, if the functional on $\mathfrak{A}_1 \odot \mathfrak{A}_2$ defined by

$$\sigma\left(\sum_{i=1}^{n} x_{1,i} \otimes x_{2,i}\right) = \sum_{i=1}^{n} \sigma_1(x_{1,i})\sigma_2(x_{2,i})$$

is continuous under the β-norm for states σ_1 and σ_2 of $\mathfrak{A}_1$ and $\mathfrak{A}_2$, then σ can be extended to a state of $\mathfrak{A}_1 \widehat{\otimes}_\beta \mathfrak{A}_2$, which coincides with $\sigma_1 \otimes \sigma_2$.

LEMMA 2. *Let* $\widetilde{\mathfrak{A}}_1$ *and* $\widetilde{\mathfrak{A}}_2$ *be the* C^**-algebras obtained by adjoining units to* $\mathfrak{A}_1$ *and* $\mathfrak{A}_2$ *respectively. If* β *is a compatible norm of* $\mathfrak{A}_1 \odot \mathfrak{A}_2$, *then* β *can be extended to a compatible norm of* $\widetilde{\mathfrak{A}}_1 \odot \widetilde{\mathfrak{A}}_2$.

PROOF. Let π be a faithful representation of $\mathfrak{A}_1 \widehat{\otimes}_\beta \mathfrak{A}_2$ to a Hilbert space $\mathfrak{H}$ and π_1 and π_2 the restrictions of π to $\mathfrak{A}_1$ and $\mathfrak{A}_2$ respectively. Then π_1 and π_2 are faithful and can be naturally extended to the representations of $\widetilde{\mathfrak{A}}_1$ and $\widetilde{\mathfrak{A}}_2$ respectively, which are also denoted by π_1 and π_2. For each element $\sum_{i=1}^{n} (x_{1,i} + \lambda_{1,i}I) \otimes (x_{2,i} + \lambda_{2,i}I) \in \widetilde{\mathfrak{A}}_1 \odot \widetilde{\mathfrak{A}}_2$ putting

$$\left\| \sum_{i=1}^{n} (x_{1,i} + \lambda_{1,i}I) \otimes (x_{2,i} + \lambda_{2,i}I) \right\|_\beta = \left\| \sum_{i=1}^{n} (\pi_1(x_{1,i}) + \lambda_{1,i}I)(\pi_2(x_{2,i}) + \lambda_{2,i}I) \right\|,$$

the norm β coincides with the original one of $\mathfrak{A}_1 \odot \mathfrak{A}_2$. Let $\mathfrak{J}$ be the set of all elements $\sum_{i=1}^{n} (x_{1,i} + \lambda_{1,i}I) \otimes (x_{2,i} + \lambda_{2,i}I)$ with β-norm zero. Clearly, $\mathfrak{J}$ becomes an ideal of $\widetilde{\mathfrak{A}}_1 \odot \widetilde{\mathfrak{A}}_2$ and $\mathfrak{J} \cap \mathfrak{A}_1 \odot \mathfrak{A}_2 = \{0\}$. Since π_1 and π_2 are faithful, $\pi_1 \otimes \pi_2$

is so on $\tilde{\mathfrak{A}}_1\odot\tilde{\mathfrak{A}}_2$. Since $(\pi_1\otimes\pi_2)\,(\mathfrak{A}_1\odot\mathfrak{A}_2)\,(\mathfrak{H}\otimes\mathfrak{H})$ is dense in $\mathfrak{H}\otimes\mathfrak{H}$ and $\mathfrak{J}(\mathfrak{A}_1\odot\mathfrak{A}_2) \subset \mathfrak{J}\cap(\mathfrak{A}_1\odot\mathfrak{A}_2) = \{0\}$, we get $(\pi_1\otimes\pi_2)\,(\mathfrak{J}) = \{0\}$, that is, $\mathfrak{J} = \{0\}$. Hence the new β-norm of $\tilde{\mathfrak{A}}_1\odot\tilde{\mathfrak{A}}_2$ is a compatible norm.

In the following $\mathfrak{P}_0(\mathfrak{A})$ means the set of all pure states for any C^*-algebra $\mathfrak{A}$. For each state σ of $\mathfrak{P}_0(\mathfrak{A})$ and unitary $u \in \tilde{\mathfrak{A}}$ we define a state σ^u by

$$\sigma^u(x) = (uxu^{-1}) \quad \text{for } x \in \mathfrak{A}.$$

We remark that $\mathfrak{A}$ is an ideal of $\tilde{\mathfrak{A}}$ and so $x \to uxu^{-1}$ is an automorphism of $\mathfrak{A}$, so that $\sigma^u \in \mathfrak{P}_0(\mathfrak{A})$ if $\sigma \in \mathfrak{P}_0(\mathfrak{A})$. For any compatible norm β of $\mathfrak{A}_1\odot\mathfrak{A}_2$ put

$$S_\beta = \{(\sigma_1, \sigma_2) \in \mathfrak{P}_0(\mathfrak{A}_1)\times\mathfrak{P}_0(\mathfrak{A}_2) : \sigma_1\otimes\sigma_2 \text{ is } \beta\text{-continuous on } \mathfrak{A}_1\odot\mathfrak{A}_2\}.$$

Then we get

LEMMA 3. *S_β is w^*-closed and unitarily invariant. That is,* $(\sigma_1{}^u, \sigma_2{}^v) \in S_\beta$ *for each* $(\sigma_1, \sigma_2) \in S_\beta$ *and unitaries* $u \in \tilde{\mathfrak{A}}_1$ *and* $v \in \tilde{\mathfrak{A}}_2$.

PROOF. Closedness: Let $\{(\sigma_{1,\alpha}, \sigma_{2,\alpha})\}$ be a directed sequence in S_β converging to (σ_1, σ_2). Then we have

$$\sum_{i=1}^n \sigma_{1,\alpha}(x_{1,i})\,\sigma_{2,\alpha}(x_{2,i}) \to \sum_{i=1}^n \sigma_1(x_{1,i})\sigma_2(x_{2,i})$$

for each $\sum_{i=1}^n x_{1,i}\otimes x_{2,i} \in \mathfrak{A}_1\odot\mathfrak{A}_2$ and

$$\left|\sum_{i=1}^n \sigma_{1,\alpha}(x_{1,i})\sigma_{2,\alpha}(x_{2,i})\right| \leqq \left\|\sum_{i=1}^n x_{1,i}\otimes x_{2,i}\right\|_\beta,$$

so that we have

$$\left|\sum_{i=1}^n \sigma_1(x_{1,i})\sigma_2(x_{2,i})\right| \leqq \left\|\sum_{i=1}^n x_{1,i}\otimes x_{2,i}\right\|_\beta.$$

Hence $(\sigma_1,\sigma_2) \in S_\beta$, that is, S_β is closed.

Unitary invariance: By Lemma 2 we may consider β as the compatible norm of $\tilde{\mathfrak{A}}_1\odot\tilde{\mathfrak{A}}_2$. For each unitaries $u \in \tilde{\mathfrak{A}}_1$ and $v \in \tilde{\mathfrak{A}}_2$, we have

$$\begin{aligned}
\left|(\sigma_1{}^u\otimes\sigma_2{}^u)\left(\sum_{i=1}^n x_{1,i}\otimes x_{2,i}\right)\right| &= \left|\sigma_1\otimes\sigma_2\left(\sum_{i=1}^n ux_{1,i}u^{-1}\otimes vx_{2,i}v^{-1}\right)\right| \\
&= \left|\sigma_1\otimes\sigma_2(u\otimes v)\left(\sum_{i=1}^n x_{1,i}\otimes x_{2,i}\right)(u\otimes v)^{-1}\right| \\
&\leqq \left\|(u\otimes v)\left(\sum_{i=1}^n x_{1,i}\otimes x_{2,i}\right)(u\otimes v)^{-1}\right\|_\beta
\end{aligned}$$

$$= \left\| \sum_{i=1}^{n} x_{1,i} \otimes x_{2,i} \right\|_{\beta}$$

for each $(\sigma_1, \sigma_2) \in S_\beta$ and $\sum_{i=1}^{n} x_{1,i} \otimes x_{2,i} \in \mathfrak{A}_1 \odot \mathfrak{A}_2$. Hence S_β is unitarily invariant.

THEOREM 1. *Every commutative C^*-algebra has the property (T).*

PROOF. Let $\mathfrak{A}_1$ be an arbitrary commutative C^*-algebra and $\mathfrak{A}_2$ another C^*-algebra. Let β be arbitrary compatible norm of $\mathfrak{A}_1 \odot \mathfrak{A}_2$. Let σ be an arbitrary pure state of $\mathfrak{A}_1 \widehat{\otimes}_\beta \mathfrak{A}_2$ and σ_1 and σ_2 its restrictions to $\mathfrak{A}_1$ and $\mathfrak{A}_2$ respectively. Let π be the cyclic representation of $\mathfrak{A}_1 \widehat{\otimes}_\beta \mathfrak{A}_2$ to the Hilbert space $\mathfrak{H}_\sigma$ induced by σ and π_1 and π_2 its restrictions to $\mathfrak{A}_1$ and $\mathfrak{A}_2$ respectively. Since the commutant of $\pi(\mathfrak{A}_1 \widehat{\otimes}_\beta \mathfrak{A}_2)$ contains $\pi_1(\mathfrak{A}_1)$ by the commutativity of $\mathfrak{A}_1$, π_1 becomes the representation of $\mathfrak{A}_1$ to the scalar field over $\mathfrak{H}_\sigma$. Hence we have $\sigma_1(x_1)I = \pi_1(x_1)$ and

$$\begin{aligned} \sigma(x_1 \otimes x_2) &= (\pi_1(x_1)\pi_2(x_2)\xi_\sigma, \xi_\sigma) \\ &= \sigma_1(x_1)(\pi_2(x_2)\xi_\sigma, \xi_\sigma) \\ &= \sigma_1(x_1)\sigma_2(x_2) \end{aligned}$$

for each $x_1 \in \mathfrak{A}_1$ and $x_2 \in \mathfrak{A}_2$. Thus every irreducible representation π of $\mathfrak{A}_1 \widehat{\otimes}_\beta \mathfrak{A}_2$ is written in the form $\pi = \pi_1 \otimes \pi_2$ by some irreducible representations π_1 and π_2 of $\mathfrak{A}_1$ and $\mathfrak{A}_2$. Therefore we get

$$(*)\quad \begin{cases} \left\| \sum_{i=1}^{n} x_{1,i} \otimes x_{2,i} \right\|_\beta = \sup \left\{ \left\| \pi \left(\sum_{i=1}^{n} x_{1,i} \otimes x_{2,i} \right) \right\| ; \pi \text{ runs over all irreducible representations of } \mathfrak{A}_1 \widehat{\otimes}_\beta \mathfrak{A}_2 \right\} \\ \leqq \sup \left\{ \left\| \pi_1 \otimes \pi_2 \left(\sum_{i=1}^{n} x_{1,i} \otimes x_{2,i} \right) \right\| : \pi_1 \text{ and } \pi_2 \text{ run over all irreducible representations of } \mathfrak{A}_1 \text{ and } \mathfrak{A}_2 \right\} \\ = \left\| \sum_{i=1}^{n} x_{1,i} \otimes x_{2,i} \right\|_\alpha \end{cases}$$

for each $\sum_{i=1}^{n} x_{1,i} \otimes x_{2,i} \in \mathfrak{A}_1 \odot \mathfrak{A}_2$.

Suppose $S_\beta \neq \mathfrak{P}_0(\mathfrak{A}_1) \times \mathfrak{P}_0(\mathfrak{A}_2)$. By the closedness of S_β there exist non empty open sets U_1 and U_2 in $\mathfrak{P}_0(\mathfrak{A}_1)$ and $\mathfrak{P}_0(\mathfrak{A}_2)$ such that $U_1 \times U_2 \cap S_\beta = \phi$. Replacing U_1 and U_2 by $\bigcup_{u \in \tilde{\mathfrak{A}}_1;\text{unitary}} U_1^u$ and $\bigcup_{v \in \tilde{\mathfrak{A}}_2;\text{unitary}} U_2^v$, we may suppose that U_1 and U_2 are unitarily invariant and $U_1 \times U_2 \cap S_\beta = \phi$ by the unitary invariance of S_β. Putting $K_1 = \complement U_1$ and $K_2 = \complement U_2$, K_2 is a unitarily invariant closed subset of $\mathfrak{P}_0(\mathfrak{A}_2)$. By [2: Lemma 8] $K_1^+ = \mathfrak{J}_1$ and $K_2^+ = \mathfrak{J}_2$ are closed ideals of $\mathfrak{A}_1$ and $\mathfrak{A}_2$ such that $\mathfrak{J}_1^+ = K_1$ and $\mathfrak{J}_2^+ = K_2$ respectively. It follows from the non-emptiness of U_1 and U_2 that $\mathfrak{J}_1 \neq \{0\}$ and $\mathfrak{J}_2 \neq \{0\}$. Hence there exist non-zero positive elements $a_1 \in \mathfrak{J}_1$ and $a_2 \in \mathfrak{J}_2$ respectively. Since $S_\beta \subset \{K_1 \times \mathfrak{P}_0(\mathfrak{A}_2)\} \cup \{\mathfrak{P}_0(\mathfrak{A}_1) \times K_2\}$ we have $\sigma_1 \otimes \sigma_2\ (a_1 \otimes a_2) = 0$ for every $(\sigma_1, \sigma_2) \in S_\beta$. But every $\sigma \in \mathfrak{P}_0(\mathfrak{A}_1 \hat{\otimes}_\beta \mathfrak{A}_2)$ has the form $\sigma = \sigma_1 \otimes \sigma_2$ for some $(\sigma_1, \sigma_2) \in S_\beta$ as already shown, so that $a_1 \otimes a_2 = 0$. This is a contradiction. Hence we get $S_\beta = \mathfrak{P}_0(\mathfrak{A}_1) \times \mathfrak{P}_0(\mathfrak{A}_2)$, so that the inequality in (*) is replaced by the equality. This completes the proof.

LEMMA 4. *If the restriction of a pure state σ of $\mathfrak{A}_1 \hat{\otimes}_\beta \mathfrak{A}_2 = \mathfrak{A}_\beta$ to $\mathfrak{A}_1$ becomes a pure state σ_1 of $\mathfrak{A}_1$, then the restriction σ_2 of σ to $\mathfrak{A}_2$ becomes a pure state and σ is represented in the form $\sigma = \sigma_1 \otimes \sigma_2$.*

PROOF. Let π be the cyclic representation of $\mathfrak{A}_\beta$, induced by σ, to the cyclic Hilbert space $\mathfrak{H}$ with the cyclic vector ξ. Let π_1 and π_2 be the restrictions of π to $\mathfrak{A}_1$ and $\mathfrak{A}_2$ respectively. Since $\sigma_1(x) = (\pi_1(x)\xi, \xi)$, $x \in \mathfrak{A}_1$, is a pure state of $\mathfrak{A}_1$, $\pi_1^{\mathfrak{K}}$ becomes the irreducible cyclic representation of $\mathfrak{A}_1$ induced by σ_1, where $\mathfrak{K}$ is defined by $\mathfrak{K} = [\pi_1(\mathfrak{A}_1)\xi]$ and $\pi_1^{\mathfrak{K}}$ means the representation of $\mathfrak{A}_1$ to $\mathfrak{K}$ defined by $\pi_1^{\mathfrak{K}}(x)\eta = \pi_1(x)\eta$ for $\eta \in \mathfrak{K}$. Hence $\mathfrak{K}$ becomes a minimal subspace belonging to $\pi_1(\mathfrak{A}_1)'$, the commutant of $\pi_1(\mathfrak{A}_1)$.

On the other hand, the irreducibility of π implies $\pi(\mathfrak{A}_\beta)'' = \boldsymbol{B}(\mathfrak{H})$, so that $\boldsymbol{R}\ (\pi_1(\mathfrak{A}_1), \pi_2(\mathfrak{A}_2)) = \boldsymbol{B}(\mathfrak{H})$ where $\boldsymbol{B}(\mathfrak{H})$ means the full operator algebra on $\mathfrak{H}$ and $\boldsymbol{R}(S)$ means the von Neumann algebra generated by S for any subset S of $\boldsymbol{B}(\mathfrak{H})$. $\pi_1(\mathfrak{A}_1)' \supset \pi_2(\mathfrak{A}_2)$ implies $\boldsymbol{R}(\pi_1(\mathfrak{A}_1), \pi_1(\mathfrak{A}_1)') = \boldsymbol{B}(\mathfrak{H})$ and similarly $\boldsymbol{R}(\pi_2(\mathfrak{A}_2), \pi_2(\mathfrak{A}_2)') = \boldsymbol{B}(\mathfrak{H})$, so that both π_1 and π_2 are factor representations. Since $\pi_1(\mathfrak{A}_1)'$ has the minimal invariant subspace $\mathfrak{K}$, $\pi_1(\mathfrak{A}_1)''$ is a factor of type I.

Let e be the projection of $\mathfrak{H}$ onto $\mathfrak{K}$. Then e is a minimal projection of $\pi_1(\mathfrak{A}_1)'$, so that $e\pi_1(\mathfrak{A}_1)'e = \{\lambda e\colon \lambda \text{ is complex}\}$. Putting $exe = \lambda(x)e$ for $x \in \pi_1(\mathfrak{A}_1)'$, λ is a pure state of $\pi_1(\mathfrak{A}_1)'$. Since $\pi_2(\mathfrak{A}_2) \subset \pi_1(\mathfrak{A}_1)'$, we get

$$\begin{aligned}\sigma(x_1 \otimes x_2) &= (\pi(x_1 \otimes x_2)\xi, \xi) = (\pi_1(x_1)\pi_2(x_2)\xi, \xi)\\ &= (\pi_2(x_2)e\xi,\ e\pi_1(x_1)^*\xi)\\ &= (e\pi_2(x_2)e\xi,\ \pi_1(x_1)^*\xi) = \lambda(\pi_2(x_2))(\pi_1(x)\xi, \xi)\\ &= \sigma_1(x_1)\lambda(\pi_2(x_2))\end{aligned}$$

for every $x_1 \in \mathfrak{A}_1$ and $x_2 \in \mathfrak{A}_2$. Besides, we have

$$\lambda(\pi_2(x_2)) = (e\pi_2(x_2)e\xi, \xi) = (\pi_2(x_2)\xi, \xi) = \sigma_2(x_2)$$

for every $x_2 \in \mathfrak{A}$. Thus we get $\sigma = \sigma_1 \otimes \sigma_2$.

Finally we shall show that σ_2 is a pure state. By the equality

$$\left(\pi\left(\sum_{i=1}^{n} x_{1,i} \otimes x_{2,i}\right)\xi,\ \pi\left(\sum_{j=1}^{m} y_{1,j} \otimes y_{2,j})\xi\right)\right)$$

1) K_i^+ means the set of all elements a_i of $\mathfrak{A}_i$ such that $\sigma_i(a_i) = 0$ for every $\sigma_i \in K_i$ $(i = 1, 2)$.

$$= \sum_{i=1}^{n} \sum_{j=1}^{m} \sigma(y_{1,j}^{*} x_{1,i} \otimes y_{2,j}^{*} x_{2,i})$$
$$= \sum_{i=1}^{n} \sum_{j=1}^{m} \sigma_1(y_{1,j}^{*} x_{1,i}) \sigma_2(y_{2,j}^{*} x_{2,i})$$

for $\sum_{i=1}^{n} x_{1,i} \otimes x_{2,i}, \sum_{j=1}^{m} y_{1,j} \otimes y_{2,j} \in \mathfrak{A}_1 \odot \mathfrak{A}_2$, π becomes the product representation of the cyclic ones π_{σ_1} and π_{σ_2} of $\mathfrak{A}_1$ and $\mathfrak{A}_2$ induced by σ_1 and σ_2. Hence the irreducibility of π implies that of π_{σ_2}. Thus σ_2 becomes a pure state.

THEOREM 2. *T-cross norm in* $\mathfrak{A}_1 \odot \mathfrak{A}_2$ *is the smallest among compatible norms of* $\mathfrak{A}_1 \odot \mathfrak{A}_2$.

PROOF. Let β be another compatible norm of $\mathfrak{A}_1 \odot \mathfrak{A}_2$. Put $\mathfrak{A}_\beta = \mathfrak{A}_1 \widehat{\otimes}_\beta \mathfrak{A}_2$. If $S_\beta = \mathfrak{P}_0(\mathfrak{A}_1) \times \mathfrak{P}_0(\mathfrak{A}_2)$, then we have

$$\left\| \sum_{i=1}^{n} x_{1,i} \otimes x_{2,i} \right\|_\alpha$$
$$= \sup_{(\sigma_1 \sigma_2) \in \mathfrak{P}_0(\mathfrak{A}_1) \times \mathfrak{P}_0(\mathfrak{A}_2)} \frac{\sigma_1 \otimes \sigma_2 \left[\left(\sum_{j=1}^{m} a_{1,j} \otimes a_{2,j} \right)^* \left(\sum_{i=1}^{n} x_{1,i} \otimes x_{2,i} \right)^* \left(\sum_{i=1}^{n} x_{1,i} \otimes x_{2,i} \right) \left(\sum_{j=1}^{m} a_{1,j} \otimes a_{2,j} \right) \right]^{1/2}}{\sigma_1 \otimes \sigma_2 \left[\left(\sum_{j=1}^{m} a_{1,j} \otimes a_{2,j} \right)^* \left(\sum_{j=1}^{m} a_{1,j} \otimes a_{2,j} \right) \right]^{1/2}}$$
$$\leqq \left\| \sum_{i=1}^{n} x_{1,i} \otimes x_{2,i} \right\|_\beta .$$

Hence it suffices to prove only $S_\beta = \mathfrak{P}_0(\mathfrak{A}_1) \times \mathfrak{P}_0(\mathfrak{A}_2)$. Suppose $S_\beta \neq \mathfrak{P}_0(\mathfrak{A}_1) \times \mathfrak{P}_0(\mathfrak{A}_2)$. As in the last part of the proof of Theorem 1, there exist non-zero positive elements $a_1 \in \mathfrak{A}_1$ and $a_2 \in \mathfrak{A}_2$ such that $\sigma_1 \otimes \sigma_2(a_1 \otimes a_2) = 0$ for every $(\sigma_1, \sigma_2) \in S_\beta$. Let A be the commutative C^*-subalgebra of $\mathfrak{A}_2$ generated by a_2. By Theorem 1 the β-norm in $\mathfrak{A}_1 \odot A$ coincides with T-cross norm, so that $\mathfrak{A}_1 \widehat{\otimes}_\alpha A$ is naturally imbedded in $\mathfrak{A}_\beta$. Taking $\sigma_1 \in \mathfrak{P}_0(\mathfrak{A}_1)$ and $\rho_2 \in \mathfrak{P}_0(A)$ such that $\sigma_1(a_1) \neq 0$ and $\rho_2(a_2) \neq 0$, $\sigma_1 \otimes \rho_2$ is a pure state of $\mathfrak{A}_1 \widehat{\otimes}_\alpha A$. Let σ be a pure state extension of $\sigma_1 \otimes \rho_2$ to $\mathfrak{A}_\beta$. Then the restriction of σ to $\mathfrak{A}_1$ and A coincide with the original σ_1 and ρ_2 respectively. Hence σ is represented in the form $\sigma = \sigma_1 \otimes \sigma_2$ by Lemma 4. Besides, we have $\sigma(a_1 \otimes a_2) = \sigma_1(a_1)\, \rho_2(a_2) \neq 0$. But $\sigma = \sigma_1 \otimes \sigma_2 \in \mathfrak{A}_\beta^*$ implies $(\sigma_1, \sigma_2) \in S_\beta$. This is a contradiction. Hence $S_\beta = \mathfrak{P}_0(\mathfrak{A}_1) \times \mathfrak{P}_0(\mathfrak{A}_2)$. This completes the proof.

REMARK. In Theorem 2 we do not assume any relation between the compatible norm β and Schatten's λ-norm on $\mathfrak{A}_1 \odot \mathfrak{A}_2$, so that we can conclude that every compatible norm β is larger than λ-norm.

As a direct conclusion of Theorem 2, we can answer the open question proposed by Turumaru [8] in the following

COROLLARY. *The direct product of simple C^*-algebras, in the sense of Turumaru, is also simple.*

PROOF. Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be two simple C^*-algebra. Let $\mathfrak{A} = \mathfrak{A}_1 \hat{\otimes}_\alpha \mathfrak{A}_2$. Suppose that there exists a closed ideal $\mathfrak{J}$ of $\mathfrak{A}$. If π is an irreducible representation of $\mathfrak{A}$ vanishing on $\mathfrak{J}$, then the restriction π_1 and π_2 of π to $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are factor representations commuting each other. By the simplicity of $\mathfrak{A}_1$ and $\mathfrak{A}_2$ both π_1 and π_2 are faithful. For any $\sum_{i=1}^n x_{1,i} \otimes x_{2,i} \in \mathfrak{J}$ we have

$$0 = \pi\left(\sum_{i=1}^n x_{1,i} \otimes x_{2,i}\right) = \sum_{i=1}^n \pi_1(x_{1,i})\pi_2(x_{2,i}),$$

so that there exists a $n \times n$-matrix (λ_{ij}) by [4:Theorem III] such that

$$\sum_{i=1}^n \pi_{i,j}\pi_1(x_{1,i}) = 0 \qquad \text{and} \qquad \sum_{j=1}^n \lambda_{i,j}\pi_2(x_{2,j}) = \pi_2(x_{2,i}).$$

It follows that $\sum_{i=1}^n \lambda_{i,j} x_{1,i} = 0$ and $\sum_{j=1}^n \lambda_{i,j} x_{2,j} = x_{2,i}$ by the faithfulness of π_1 and π_2, so that $\sum_{i=1}^n x_{1,i} \otimes x_{2,i} = 0$. Thus we get $\mathfrak{J} \cap \mathfrak{A}_1 \odot \mathfrak{A}_2 = \{0\}$. Therefore the norm $\| \cdot \|_\beta$ defined by

$$\left\| \sum_{i=1}^n x_{1,i} \otimes x_{2,i} \right\|_\beta = \left\| \pi\left(\sum_{i=1}^n x_{1,i} \otimes x_{2,i}\right) \right\|$$

for $\sum_{i=1}^n x_{1,i} \otimes x_{2,i} \in \mathfrak{A}_1 \odot \mathfrak{A}_2$ is a compatible norm on $\mathfrak{A}_1 \odot \mathfrak{A}_2$, where π means the natural homomorphism of $\mathfrak{A}$ onto $\mathfrak{A}/\mathfrak{J}$. But Theorem 2 says that $\left\| \sum_{i=1}^n x_{1,i} \otimes x_{1,i} \right\|_\alpha \leqq \left\| \sum_{i=1}^n x_{1,i} \otimes x_{2,i} \right\|_\beta$ for every $\sum_{i=1}^n x_{1,i} \otimes x_{2,i} \in \mathfrak{A}_1 \odot \mathfrak{A}_2$, so that π becomes an isometry. Hence we have $\mathfrak{J} = \{0\}$. This completes the proof.

THEOREM 3. *Every C^*-algebra of type I has the property (T).*

PROOF. Let $\mathfrak{A}_1$ be a C^*-algebra of type I and $\mathfrak{A}_2$ an arbitrary C^*-algebra. Let β be an arbitrary compatible norm in $\mathfrak{A}_1 \odot \mathfrak{A}_2$. Let π be an irreducible representation of $\mathfrak{A}_1 \hat{\otimes}_\beta \mathfrak{A}_2$ to a Hilbert space $\mathfrak{H}$. Let π_1 and π_2 be the restrictions of π to $\mathfrak{A}_1$ and $\mathfrak{A}_2$. Let $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$ be the weak closure of $\pi_1(\mathfrak{A}_1)$ and $\pi_2(\mathfrak{A}_2)$ respectively. Then $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$ commute each other. By the irreducibility of π $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$ generates the full operator algebra $\boldsymbol{B}(\mathfrak{H})$ on $\mathfrak{H}$, so that both $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$ are factors. Since $\mathfrak{A}_1$ is a C^*-algebra of type I, $\boldsymbol{M}_1$ is a factor of type

I. By [4:Theorem IV] $\boldsymbol{B}(\mathfrak{H})$ is isomorphic to $\boldsymbol{M}_1\otimes\boldsymbol{M}_1'$ under the natural isomorphism $\sum_{i=1}^{n} x_i x'_i \rightarrow \sum_{i=1}^{n} x_i\otimes x'_i$, $x_i \in \boldsymbol{M}_1$, $x'_i \in \boldsymbol{M}_1' = 1, 2, \cdots, n$. Since $\boldsymbol{M}_2$ is contained in $\boldsymbol{M}_1'$, $\boldsymbol{R}(\boldsymbol{M}_1, \boldsymbol{M}_2) \cong \boldsymbol{M}_1\otimes\boldsymbol{M}_2$. Hence we get $\boldsymbol{M}_2 = \boldsymbol{M}_1'$, so that $\boldsymbol{M}_2$ is also of type I and $\pi = \pi_1{}^e\otimes\pi_2{}^f$ for minimal projections e and f of $\boldsymbol{M}'_1$ and $\boldsymbol{M}'_2$. After all, we get the following

$$\left\|\sum_{i=1}^{n} x_{1,i}\otimes x_{2,i}\right\|_\beta = \sup\left\{\left\|\pi\left(\sum_{i=1}^{n} x_{1,i}\otimes x_{2,i}\right)\right\| : \pi \text{ runs over all irreducible representations of } \mathfrak{A}_1\hat{\otimes}_\beta\mathfrak{A}_2\right\}$$

$$= \sup\left\{\left\|\sum_{i=1}^{n} \pi_1(x_{1,i})\otimes\pi_2(x_{2,i})\right\| : \pi_1 \text{ and } \pi_2 \text{ run over all irreducible representations of } \mathfrak{A}_1 \text{ and } \mathfrak{A}_2 \text{ respectively}\right\}$$

$$= \left\|\sum_{i=1}^{n} x_{1,i}\otimes \pi_{2,i}\right\|$$

for every $\sum_{i=1}^{n} x_{1,i}\otimes x_{2,i} \in \mathfrak{A}_1\odot\mathfrak{A}_2$. This completes the proof.

THEOREM 5. *If C^*-algebra $\mathfrak{A}_1$ is an inductive limit of C^*-subalgebras $\{\mathfrak{A}_\gamma\}$ with the property (T) in the sense of Takeda* [5], *then $\mathfrak{A}_1$ also has the property (T).*

PROOF. Let $\mathfrak{A}_2$ be an arbitrary C^*-algebra. Let β be another compatible norm in $\mathfrak{A}_1\odot\mathfrak{A}_2$. Let $\sum_{i=1}^{n} x_{1,i}\otimes x_{2,i}$ be a fixed element of $\mathfrak{A}_1\odot\mathfrak{A}_2$. For any $\varepsilon > 0$ there exist an index γ_0 and $x'_{1,1} \cdots, x'_{1,n} \in \mathfrak{A}_{\gamma_0}$ such that $\|x_{1,i} - x'_{1,i}\| < \varepsilon$ $i = 1, 2, \cdots, n$. Since $\mathfrak{A}_{\gamma_0}$ has the property (T), we have $\left\|\sum_{i=1}^{n} x'_{1,i}\otimes x_{2,i}\right\|_\beta = \left\|\sum_{i=1}^{n} x'_{1,i}\otimes x_{2,i}\right\|_\alpha$ Hence we get

$$\left\|\sum_{i=1}^{n} x_{1,i}\otimes x_{2,i}\right\|_\alpha \leqq \left\|\sum_{i=1}^{n} x_{1,i}\otimes x_{2,i}\right\|_\beta$$

$$\leqq \left\|\sum_{i=1}^{n} x'_{1,i}\otimes x_{2,i}\right\|_\beta + \left\|\sum_{i=1}^{n} (x_{1,i}-x'_{1,i})\otimes x_{2,i}\right\|_\beta$$

$$\leqq \left\|\sum_{i=1}^{n} x'_{1,i}\otimes x_{2,i}\right\|_\alpha + \sum_{i=1}^{n}\left\|x_{1,i} - x'_{1,i}\right\|\|x_{2,i}\|$$

$$\leqq \left\|\sum_{i=1}^{n} x'_{1,i}\otimes x_{2,i}\right\|_\alpha + \left(\sum_{i=1}^{n}\|x_{2,i}\|\right)\varepsilon$$

$$\leqq \left\| \sum_{i=1}^{n} x_{1,i} \otimes x_{2,i} \right\|_\alpha + \left\| \sum_{i=1}^{n} (x'_{1,i} - x_{i,1}) \otimes x_{2.i} \right\|_\alpha + \varepsilon \sum_{i=1}^{n} \|x_{2,i}\|$$

$$\leqq \left\| \sum_{i=1}^{n} x_{1,i} \otimes x_{2,i} \right\|_\alpha + 2\varepsilon \sum_{i=1}^{n} \| x_{2,i} \| ,$$

so that $\left\| \sum_{i=1}^{n} x_{1,j} \otimes x_{2,i} \right\|_\alpha = \left\| \sum_{i=1}^{n} x_{1,i} \otimes x_{2,i} \right\|_\beta$.

According to Theorem 3 and 4, the class of C^*-algebras with the property (T) is actually larger than that of C^*-algebras of type I. Indeed, the infinite C^*-direct product of finite dimensional matrix algebras in the sense of Takeda [5] has the property (T), but is not of type I. Finally we shall construct an example of C^*-algebra without the property (T).

Let G be the free group of two generators α, β. On the Hilbert space $\mathfrak{H} = l^2(G)$, space of square summable functions over G, we define unitary operators $u(g_0)$ and $v(g_0)$ for each $g_0 \in G$ by

$$[u(g_0)\xi](g) = \xi(g_0^{-1}g),\ [v(g_0)\xi](g) = \xi(gg_0) \text{ for } \xi \in \mathfrak{H}.$$

Putting $(w\xi)(g) = \xi(g^{-1})$ for $\xi \in \mathfrak{H}$, w becomes unitary on $\mathfrak{H}$ and we have

$$w^2 = I,\ wu(g)w = v(g) \text{ and } wv(g)w = u(g)$$

for each $g \in G$. Let $\mathfrak{A}$ be the C^*-algebra generated by $u(g)$'s. Let π_1 be identical representation of $\mathfrak{A}$ on $\mathfrak{H}$ and let $\pi_2(x) = w\pi_1(x)w$,

$$\pi_1(\mathfrak{A})' = \text{the weak closure of } \pi_2(\mathfrak{A}) = \pi_2(\mathfrak{A})^-,$$
$$\pi_2(\mathfrak{A})' = \text{the weak closure of } \pi_1(\mathfrak{A}) = \pi_1(\mathfrak{A})^-,$$

and $\pi_1(\mathfrak{A})^-$, $\pi_2(\mathfrak{A})^-$ are factors of type II_1.

Next we consider the C^*-algebra $\mathfrak{B}$ generated by $\{u(g_1)\otimes u(g_2);\ g_1, g_2 \in G\}$ on $\mathfrak{K} = \mathfrak{H}\otimes\mathfrak{H}$. Then $\mathfrak{B}$ is naturally isomorphic to the Turumau's direct product $\mathfrak{A}\widehat{\otimes}_\alpha\mathfrak{A}$. So we identity $\mathfrak{A}\widehat{\otimes}_\alpha\mathfrak{A}$ and $\mathfrak{B}$.

Then we can define a representation π of $\mathfrak{A}\odot\mathfrak{A}$ by

$$\pi\left(\sum_{i=1}^{n} x_{1,i}\otimes x_{2,i}\right) = \sum_{i=1}^{n} \pi_1(x_{1,i})\pi_2(x_{2,i}) \text{ for each } \sum_{i=1}^{n} x_{1,i}\otimes x_{2,i} \in \mathfrak{A}\odot\mathfrak{A}.$$

Suppose that π is continuous under T-cross-norm. Then π can be extended to the representation of $\mathfrak{B}$, denoted also by π. Since the weak closure $\widetilde{\mathfrak{B}}$ of $\mathfrak{B}$ has the coupling constant one, every normal state of $\widetilde{\mathfrak{B}}$ can be represented as a vector state. Hence the set of all vector states of $\widetilde{\mathfrak{B}}$ is w^*-dense in the state space of $\widetilde{\mathfrak{B}}$. As every state of $\mathfrak{B}$ can be extended to a state of $\widetilde{\mathfrak{B}}$, the set of all vector states of $\mathfrak{B}$ is w^*-dense in the state space of $\mathfrak{B}$. After all, for every unit vector ξ of $\mathfrak{H}$ the state φ_ξ of $\mathfrak{B}$, define by $\varphi_\xi(x) = (\pi(x)\xi, \xi)$, is weakly approximated by the vector states ω_η of $\mathfrak{B}$, where η is a unit vector of $\mathfrak{K}$ and ω_η is defined by $\omega_\eta(x) = (x\eta, \eta)$. Putting $\xi_0(g) = 1$ if $g = e$ and $\xi_0(g) = 0$ if $g \neq e$, $g \in G$, ξ_0 is a unit vector of $\mathfrak{H}$. Then we have

$$\begin{aligned}\varphi_{\xi_0}(u(g_1)\otimes u(g_2)) &= (\pi(u(g_1)\otimes u(g_2))\xi_0, \xi_0)\\ &= (u(g_1)\ v(g_2)\xi_0, \xi_0)\\ &= \sum_{g\in G}\xi_0(g_1{}^{-1}gg_2)\xi_0(g)\\ &= \xi_0(g_1{}^{-1}g_2) \qquad \text{for each } (g_1, g_2) \in G\times G.\end{aligned}$$

For any $\varepsilon > 0$, there exists a unit vector $\eta_0 \in \mathfrak{K}$ such that

$$(1) \qquad \begin{aligned}&|1-(u(\alpha)\otimes u(\alpha)\ \eta_0, \eta_0)| < \varepsilon^2/2,\\ &|1-(u(\beta)\otimes u(\beta)\eta_0, \eta_0)| < \varepsilon^2/2.\end{aligned}$$

From (1) it follows that

$$(2) \qquad \|\eta_0 - u(\alpha)\otimes u(\alpha)\eta_0\| < \varepsilon \text{ and } \|\eta_0 - u(\alpha)\otimes u(\alpha)\otimes u(\alpha)\eta_0\| < \varepsilon.$$

In fact, we have

$$\begin{aligned}\|\eta_0 - u(\alpha)\otimes u(\alpha)\eta_0\|^2 &= \|\eta_0\|^2 + \|u(\alpha)\otimes u(\alpha)\eta_0\|^2\\ &\quad - 2\Re e(u(\alpha)\otimes u(\alpha)\eta_0, \eta_0)\\ &= 2\Re e((1-(u(\alpha)\otimes u(\alpha)\ \eta_0, \eta_0)) < \varepsilon^2.\end{aligned}$$

For each subset S of $G\times G$ we define a projection p_S in $\mathfrak{K}$ by

$$(p_S\eta)(g_1, g_2) = \chi_S(g_1, g_2)\ \eta(g_1, g_2) \qquad \text{for } \eta \in \mathfrak{K},$$

where χ_S means the characteristic function of S. Then we have

$$\begin{aligned}(p_{(h_1,h_2)S}\,\eta)(g_1, g_2) &= \chi_{(h_1,h_2)S}(g_1, g_2)\eta(g_1, g_2)\\ &= \chi_S(h_1{}^{-1}g_1, h_2{}^{-1}g_2)\eta(g_1, g_2)\\ &= \chi_S(h_1{}^{-1}g_1, h_2{}^{-1}g_2)(u(h_1{}^{-1})\otimes u(h_2{}^{-1}))\eta(h_1{}^{-1}g_1, h_2{}^{-1}g_2)\\ &= [u(h_1)\otimes u(h_2)\ p_S(u(h_1)\otimes u(h_2))^{-1}\eta]\ (g_1, g_2)\end{aligned}$$

for each $(h_1, h_2) \in G\times G$ and $\eta \in \mathfrak{K}$, that is

$$P_{(h_1,h_2)S} = u(h_1)\otimes u(h_2)P_S u(h_1{}^{-1})\otimes u(h_2{}^{-1}).$$

Hence

$$\begin{aligned}&|(P_S\eta_0, \eta_0) - (P_{(\alpha,\alpha)^{-1}S}\eta_0, \eta_0)|\\ &\qquad = |(P_S\eta_0, \eta_0) - (u(\alpha)^{-1}\otimes(u(\alpha)^{-1}P_S u(\alpha)\otimes u(\alpha)\eta_0, \eta_0)|\end{aligned}$$

$$= |(P_S\eta_0, \eta_0) - (P_S u(\alpha)\otimes u(\alpha)\eta_0,\ u(\alpha)\otimes u(\alpha)\eta_0)|$$
$$\leqq |(P_S\eta_0,\ \eta_0) - (P_S\eta_0,\ u(\alpha)\otimes u(\alpha)\eta_0)|$$
$$+ |(P_S\eta_0,\ u(\alpha)\otimes u(\alpha)\eta_0) - (P_S u(\alpha)\otimes u(\alpha)\eta_0,\ u(\alpha)\otimes u(\alpha)\eta_0)|$$
$$\leqq \|P_S\eta_0\|\,\|\eta_0 - u(\alpha)\otimes u(\alpha)\eta_0\| + \|P_S(\eta_0 - u(\alpha)\otimes u(\alpha)\eta_0)\|\ \|u(\alpha)\otimes u(\alpha)\eta_0\|$$
$$< 2\varepsilon.$$

That is,

$$(P_{(\alpha^{-1},\alpha^{-1})S}\,\eta_0, \eta_0) > (P_S\eta_0, \eta_0) - 2\varepsilon.$$

Put $A = \{g \in G : g = \alpha^p\beta^q, \cdots,\ p \neq 0\}$ and $B = G - A$. Then the family $\{\alpha^{-n}B: n = 1, 2, \cdots\}$ is mutually disjoint, so that $\{(\alpha^{-n}, \alpha^{-n})(B \times G);\ n = 1, 2, \cdots\}$ is mutually disjoint. Hence

$$1 \geqq \left(P_{\bigcup_{k=0}^{n-1} (\alpha^{-k},\alpha^{-k})(B\times G)}\ \eta_0,\ \eta_0 \right) = \sum_{k=0}^{n-1} (P_{(\alpha^{-k},\alpha^{-k})(B\times G)}\eta_0, \eta_0)$$
$$> n(P_{(B\otimes G)}\,\eta_0, \eta_0) - n(n-1)\varepsilon.$$

That is, $(P_{(B\times G)}\,\eta_0, \eta_0) < \dfrac{1}{n} + (n-1)\varepsilon \qquad n = 1, 2\ \cdots.$

Similary we have $(P_{A\times G}\,\eta_0, \eta_0) < \dfrac{1}{n} + (n-1)\ \varepsilon \qquad n = 1, 2, \cdots.$

But we have

$$1 = \|\eta_0\|^2 = (P_{(A\times G)}\,\eta_0, \eta_0) + (P_{(B\times G)}\,\eta_0, \eta_0)$$
$$< 2\left(\frac{1}{n} + (n-1)\varepsilon\right) \qquad n = 1, 2, \cdots,$$

since $G = B \cup A$. This is impossible if $\varepsilon < \dfrac{1}{12}$ and $n = 3$. Hence φ_{ξ_0} can not be approximated by vector states. That is, π is discontinuous under T-cross-norm in $\mathfrak{A}_1 \odot \mathfrak{A}_2$. After all, we get the following

THEOREM 6. *There exists a C^*-algebra without the property* (T).

REFERENCES

[1] J. DIXMIER, Les algèbres d'opérateurs dans l'espace hilbertien, Gauthier-Villars, 1957.

[2] J. GLIMM. Type I C^*-algebras, Ann. Math., 73(1961), 572-612.

[3] A. GUICHARDET, Caractères et représentations des produits tensoriels de C^*-algèbres, the auther's manuscript.

[4] F. J. MURRAY AND J. VON NEUMANN, On rings of operators, Ann. Math., 37(1936), 116-229.

[5] Z. TAKEDA, Inductive limit and infinite direct product of operator algebras, Tôhoku Math. Journ., 7(1955), 67-86.

[6] M. TAKESAKI, A note on the cross-norm of the direct product of operator algebras, Kôdai Math. Sem. Rep., 10(1958), 137-140.

[7] T. TURUMARU, On the direct product of operator algebras I, Tôhoku Math. Journ.,

4(1952), 242–251.
[8] T. TURUMARU, On the direct product of operator algebras II, Tôhoku Math. Journ., 5(1953), 1–7.
[9] T. TURUMARU, On the direct product of operator algebras III, Tôhoku Math. Journ., 6(1954), 208–211.
[10] T. TURUMARU, On the direct product of operator algebras IV, Tôhoku Math. Journ., 8(1956), 281–285.
[11] H. YOSHIZAWA, Some remarks on unitary representations of the free group, Osaka Math. Journ., 3(1951), 55–63.

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