ON A DECOMPOSITION OF AN ALMOST-ANALYTIC VECTOR IN A K-SPACE WITH CONSTANT SCALAR CURVATURE

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Y. Matsushima [3][1)] proved the following

THEOREM. *In a compact Kähler-Einstein space* ($R > 0$), *any contravariant analytic vector* v^i *is uniquely decomposed in the form*

$$v^i = p^i + \phi_r{}^i q^r$$

where p^i *and* q^i *are both Killing vectors.*

As a generalization of this theorem, A.Lichnerowicz [2] proved that it holds good in a compact Kählerian space with constant scalar curvature. Recently, S.Sawaki [4] proved that the above theorem is valid for any contravariant almost-analytic vector in a compact Einstein K-space[2)].

In this paper we shall try to generalize these results to a compact K-space with constant scalar curvature.

MAIN THEOREM. *In a compact K-space with constant scalar curvature, any contravariant almost-analytic vector* v^i *is decomposed in the form*

$$v^i = p^i + \phi_r{}^i q^r$$

where p^i *and* q^i *are both Killing vectors.*

In §1 we shall give some definitions and propositions. In §2 we shall state some well known identities in a K-space. In §3 we shall deal with contravariant almost-analytic vectors in a K-space and prepare some lemmas which are useful for the proof of our main theorem. The last section will be devoted to the proof of the main theorem.

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1. Preliminaries. Let X_{2n} be a $2n$-dimensional almost-Hermitian space which admits an almost complex structure $\phi_j{}^i$ and a positive definite Riemannian metric tensor g_{ji} satisfying

$$\phi_l{}^i \phi_j{}^l = -\delta_j^i, \tag{1.1}$$

$$g_{lt}\phi_j{}^l \phi_i{}^t = g_{ji}. \tag{1.2}$$

Then from (1. 1) and (1. 2), we have

1) The number in brackets refers to Bibliography at the end of the paper.
2) For a compact almost-Kähler-Einstein space, see S. Sawaki [5].

(1. 3) $$\phi_{ji} = -\phi_{ij}$$

where $\phi_{ji} = \phi_j{}^l g_{li}$.

Now in an almost-Hermitian space X_{2n}, we define the following linear operators

$$O_{ih}^{ml} = \frac{1}{2}(\delta_i{}^m \delta_h{}^l - \phi_i{}^m \phi_h{}^l),\ {}^*O_{ih}^{ml} = \frac{1}{2}(\delta_i{}^m \delta_h{}^l + \phi_i{}^m \phi_h{}^l)$$

and a tensor is called pure (hybrid) in two indices, if it is annihilated by transvection of $^*O(O)$ on these indices. We have easily the following

PROPOSITION 1. *$^*O_{ih}^{ab}\nabla_j\phi_{ab} = 0$, $O_{ib}^{ah}\nabla_j\phi_a{}^b = 0$ where ∇_j denotes the operator of covariant derivative with respect to the Riemannian connection.*

PROPOSITION 2. *For two tensors T_{ji} and S^{ji}, if T_{ji} is pure in j,i and S^{ji} is hybrid in j,i, then $T_{ji}S^{ji}$ vanishes.*

PROPOSITION 3. *If $T_j{}^i$ is pure (hybrid) in j,i, then we have*

(1. 4) $$\phi_t{}^i T_j{}^t = \phi_j{}^t T_t{}^i \qquad (\phi_t{}^i T_j{}^t = -\phi_j{}^t T_t{}^i).$$

If S^{ji} is pure (hybrid) in j,i, then we have

(1. 5) $$\phi_t{}^j S^{ti} = \phi_t{}^i S^{jt} \qquad (\phi_t{}^j S^{ti} = -\phi_t{}^i S^{jt}).$$

2. *K*-spaces. An almost-Hermitian space X_{2n} is called a *K*-space, if it satisfies

(2. 1) $$\nabla_j\phi_{ih} + \nabla_i\phi_{jh} = 0,$$

from which we have easily

(2. 2) $$\nabla_j\phi_i{}^j = 0,$$

(2. 3) $$^*O_{ji}^{ab}\nabla_a\phi_{bh} = 0.$$ [3)]

Let $R_{kji}{}^h$ and $R_{ji} = R_{tji}{}^t$ be Riemannian curvature tensor and Ricci tensor respectively. Assuming we are in a *K*-space and putting

$$R^*{}_{ji} = \frac{1}{2}\phi^{ab}R_{abti}\phi_j{}^t,$$

then we get the following identities[4)]

(2. 4) $$\phi_{hk}\nabla^t\nabla_j\phi_t{}^h = R^*{}_{kj} - R_{jk},$$

(2. 5) $$O_{jh}^{ab}R_{ab} = 0,$$

(2. 6) $$R^*{}_{ji} = R^*{}_{ij},$$

(2. 7) $$(\nabla_j\phi_{tl})\nabla_i\phi^{tl} = R_{ji} - R^*{}_{ji}$$

where $\nabla^j = g^{tj}\nabla_t$ and $\phi^{ji} = \phi_t{}^i g^{tj}$,

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3) See S. Kotô [1].

4) S. Tachibana [7]

(2. 8) $$R - R^* = constant$$

where $R = g^{ji}R_{ji}$ and $R^* = g^{ji}R^*{}_{ji}$,

(2. 9) $$\nabla_k(N_{tl}{}^k \nabla^t v^l) = 0$$

where $N_{tl}{}^k$ is the Nijenhuis tensor:

$$N_{tl}{}^k = \phi_t{}^m(\nabla_m\phi_l{}^k - \nabla_l\phi_m{}^k) - \phi_l{}^m(\nabla_m\phi_t{}^k - \nabla_t\phi_m{}^k).$$

In a Riemannian space we know

(2.10) $$\frac{1}{2}\nabla_t R = \nabla^j R_{jt}$$ [5]

and in a K-space

(2.11) $$\frac{1}{2}\nabla_t R^* = \nabla^j R^*{}_{jt}.$$ [6]

Therefore from (2. 8), (2. 10) and (2. 11), we have

(2.12) $$\nabla^k(R_{tk} - R^*{}_{tk}) = \frac{1}{2}\nabla_t(R - R^*) = 0.$$

3. Contravariant almost-analytic vectors. In an almost-Hermitian space a contravariant vector v^i is called almost-analytic if it satisfies

$$\underset{v}{\pounds}\phi_j{}^i \equiv v^t\nabla_t\phi_j{}^i - \phi_j{}^t\nabla_t v^i + \phi_t{}^i\nabla_j v^t = 0$$ [7]

where $\underset{v}{\pounds}$ is the operator of Lie derivative. This is a generalization of the notion of contravariant analytic vectors in a Kählerian space. The above equation is equivalent to

(3. 1) $$v^t\nabla_t\phi_{ji} - \phi_j{}^t\nabla_t v_i - \phi_i{}^t\nabla_j v_t = 0$$

where $v_i = g_{it}v^t$.

In a K-space we know the following lemmas.

LEMMA 3. 1.[8] *In a compact K-space, a necessary and sufficient condition that a contravariant vector v^i be almost-analytic is that it satisfies*

(3. 2) $$\nabla^l\nabla_l v^i + R_t{}^i v^t = 0,$$

(3. 3) $$N_{tlk}\nabla^t v^l + 2v^t(R_{tk} - R^*{}_{tk}) = 0.$$

LEMMA 3. 2.[9] *When a contravariant vector v^i in a K-space is almost-*

5) For example see K. Yano and S. Bochner [8].
6) S. Sawaki [4].
7) S. Tachibana [6].
8) S. Tachibana [6].
9) S. Sawaki [4].

analytic, a necessary and sufficient condition that $\tilde{v}^i = \phi_t{}^i v^t$ *be almost-analytic is that it satisfies*

$$v^t \nabla_t \phi_{jk} = 0.$$

Next, we shall prove following lemmas.

LEMMA 3.3. *In an almost-Hermitian space, if a tensor* S_{jti} *is skew-symmetric, then we have*

$$\nabla^i \nabla^t S_{jti} = 0. \tag{3. 4}$$

PROOF. By virtue of the Ricci's identity, we obtain

$$\begin{aligned}\nabla^i \nabla^t S_{jti} &= \frac{1}{2}(\nabla^i \nabla^t S_{jti} - \nabla^t \nabla^i S_{jti}) \\ &= -\frac{1}{2}(R^{it}{}_j{}^a S_{ati} + R^{it}{}_t{}^a S_{jai} + R^{it}{}_i{}^a S_{jta}) \\ &= -\frac{1}{2}(R^{it}{}_j{}^a S_{ati} + R^{ia} S_{jai} - R^{ta} S_{jta}).\end{aligned}$$

In this equation, it is easy to see that the last three terms vanish respectively. Hence we have $\nabla^i \nabla^t S_{jti} = 0$.

LEMMA 3.4. *In a compact K-space, if* v^i *is an almost-analytic vector and* r^i *is a vector such that* $r^i = \nabla^i r$ *for a certain scalar* r, *then we have*

$$\int_{X_{2n}} r^j v^h (R_{hj} - R^*{}_{hj}) d\sigma = 0 \tag{3. 5}$$

where $d\sigma$ *means the volume element of the space* X_{2n}.

PROOF. From

$$\begin{aligned}\nabla^j \{r v^h (R_{hj} - R^*{}_{hj})\} = r^j v^h (R_{hj} - R^*{}_{hj}) + r \nabla^j v^h (R_{hj} - R^*{}_{hj}) \\ + r v^h \nabla^j (R_{hj} - R^*{}_{hj}),\end{aligned}$$

by Green's theorem, we have

$$\begin{aligned}\int_{X_{2n}} [r^j v^h (R_{hj} - R^*{}_{hj}) + r \nabla^j v^h (R_{hj} - R^*{}_{hj}) \\ + r v^h \nabla^j (R_{hj} - R^*{}_{hj})] d\sigma = 0.\end{aligned} \tag{3. 6}$$

On the other hand, operating ∇^k to (3. 3), we have

$$\nabla^k (N_{tlk} \nabla^t v^l) + 2\nabla^k v^t (R_{tk} - R^*{}_{tk}) + 2 v^t \nabla^k (R_{tk} - R^*{}_{tk}) = 0. \tag{3. 7}$$

In this place, using (2. 9) and (2. 12), (3. 7) turns to

$$\nabla^k v^t (R_{tk} - R^*{}_{tk}) = 0. \tag{3. 8}$$

Consequently from (3.6) we have

$$\int_{X_{2n}} r^j v^h (R_{hj} - R^*{}_{hj}) d\sigma = 0.$$

LEMMA 3.5. *In a compact K-space with constant scalar curvature, if* $\nabla_j p_i + \nabla_i p_j$ *is pure and* r_i *is a vector such that* $r_i = \nabla_i r$ *for a certain scalar* r, *then we have*

$$\int_{X_{2n}} p^i r^j R_{ji} d\sigma = 0. \tag{3.9}$$

PROOF. We consider the following equation:

$$\begin{aligned} \nabla^j (r p^i R_{ji}) &= p^i r^j R_{ji} + r(\nabla^j p^i) R_{ji} + r p^i \nabla^j R_{ji} \\ &= p^i r^j R_{ji} + \frac{1}{2} r(\nabla^j p^i + \nabla^i p^j) R_{ji} + r p^i \nabla^j R_{ji}. \end{aligned} \tag{3.10}$$

In this equation, since $\nabla^j p^i + \nabla^i p^j$ is pure in j,i and by (2.5) R_{ji} is hybrid in j,i, then by Proposition 2, we have $(\nabla^j p^i + \nabla^i p^j) R_{ji} = 0$. And by the assumption $\nabla^j R_{ji} = \frac{1}{2} \nabla_i R = 0$.

Accordingly, applying Green's theorem to (3.10), we have

$$\int_{X_{2n}} p^i r^j R_{ji} d\sigma = 0. \tag{3.11}$$

We conclude this section with the following lemma which is essential in this paper.

LEMMA 3.6. *In a compact K-space, any contravariant almost-analytic vector* v^i *can be decomposed as*

$$v^i = p^i + r^i \tag{3.12}$$

where $\nabla_i p^i = 0$ and r^i *is a vector such that* $r^i = \nabla^i r$ *for a certain scalar* r, *and*

$$*O^{ji}_{ab}(\nabla^a p^b + \nabla^b p^a) = 0, \tag{3.13}$$

$$r^t \nabla_t \phi_{ji} = 0. \tag{3.14}$$

PROOF. By the theory of harmonic integrals, (3.12) is the result that holds good for any vector v^i in a compact orientable Riemannian space. Next, putting

$$T_{ji} \equiv \nabla_j p_i + \nabla_i p_j + \phi_j{}^a \phi_i{}^b (\nabla_a p_b + \nabla_b p_a)$$

and writing out the square of T_{ji}, we get

$$\frac{1}{4} T_{ji} T^{ji} = (\nabla_j p_i) \nabla^j p^i + (\nabla_i p_j) \nabla^j p^i + \phi_j{}^a \phi_i{}^b \nabla^j p^i (\nabla_a p_b + \nabla_b p_a).$$

Therefore, from

$$\nabla^i(p^jT_{ji}) - p^j\nabla^iT_{ji} = (\nabla^ip^j)T_{ji} = \frac{1}{4}T_{ji}T^{ji},$$

it follows that

$$\begin{aligned}(3.15)\qquad \nabla^i(p^jT_{ji}) &= \frac{1}{4}T_{ji}T^{ji} + p^j\nabla^iT_{ji} \\ &= \frac{1}{4}T_{ji}T^{ji} + p^j\{\nabla^i(\nabla_jp_i + \nabla_ip_j) + \phi_j{}^a(\nabla^i\phi_i{}^b)(\nabla_ap_b+\nabla_bp_a) \\ &\quad + (\nabla^i\phi_j{}^a)\phi_i{}^b(\nabla_ap_b + \nabla_bp_a) + \phi_j{}^a\phi_i{}^b\nabla^i(\nabla_ap_b + \nabla_bp_a)\} \\ &= \frac{1}{4}T_{ji}T^{ji} + p^j\{\nabla^i(\nabla_jp_i + \nabla_ip_j) + \phi_j{}^a\phi_i{}^b\nabla^i(\nabla_ap_b + \nabla_bp_a)\}\end{aligned}$$

because, $\nabla^i\phi_i{}^b = 0$ since by (1. 5) and (2. 3), $\phi_i{}^b\nabla^i\phi_j{}^a = \phi_i{}^a\nabla^b\phi_j{}^i = -\phi_i{}^a\nabla^i\phi_j{}^b$, $\phi_i{}^b\nabla^i\phi_j{}^a$ is skew-symmetric with respect to a and b, and therefore $(\nabla^i\phi_j{}^a)\phi_i{}^b(\nabla_ap_b + \nabla_bp_a)$ vanishes.

On the other hand, if we interchange j and i in (3. 1) and subtract the equation thus obtained from (3. 1), then we get

$$(3.16)\qquad 2v^t\nabla_t\phi_{ji} - \phi_j{}^t(\nabla_tv_i - \nabla_iv_t) + \phi_{ti}(\nabla_jv^t - \nabla^tv_j) = 0.$$

Substituting (3. 12) into (3.16) and taking account of $\nabla_jr_i = \nabla_ir_j$, we have

$$2(p^t + r^t)\nabla_t\phi_{ji} - \phi_j{}^t(\nabla_tp_i - \nabla_ip_t) + \phi_{ti}(\nabla_jp^t - \nabla^tp_j) = 0.$$

Since $\nabla_i\phi_j{}^i = 0$ and $\nabla_ip^i = 0$, this equation can be easily written as

$$\begin{aligned}(3.17)\qquad &\phi_j{}^t(\nabla_ip_t + \nabla_tp_i) - \phi_i{}^t(\nabla_jp_t + \nabla_tp_j) \\ &= -2r^t\nabla_t\phi_{ji} + 2\nabla^t(\phi_{jt}p_i + \phi_{ti}p_j + \phi_{ij}p_t).\end{aligned}$$

Operating ∇^i to (3. 17) we have

$$\begin{aligned}(3.18)\qquad &\nabla^i\phi_j{}^t(\nabla_ip_t + \nabla_tp_i) + \phi_j{}^t\nabla^i(\nabla_ip_t + \nabla_tp_i) - \phi_i{}^t\nabla^i(\nabla_jp_t + \nabla_tp_j) \\ &= -2(\nabla^ir^t)\nabla_t\phi_{ji} - 2r^t\nabla^i\nabla_t\phi_{ji} + 2\nabla^i\nabla^tS_{jti}\end{aligned}$$

where $S_{jti} = \phi_{jt}p_i + \phi_{ti}p_j + \phi_{ij}p_t$.

In the above equation (3. 18), since $\nabla^i\phi_j{}^t$ is skew-symmetric with respect to i and t, $\nabla^i\phi_j{}^t(\nabla_ip_t + \nabla_tp_i) = 0$ and similarly $(\nabla^ir^t)\nabla_t\phi_{ji} = 0$. And, by Lemma 3.3 $\nabla^i\nabla^tS_{jti} = 0$.

Hence, (3. 18) turns to

$$\phi_j{}^t\nabla^i(\nabla_ip_t + \nabla_tp_i) - \phi_i{}^t\nabla^i(\nabla_jp_t + \nabla_tp_j) = -2r^t\nabla^i\nabla_t\phi_{ji}$$

or transvecting this equation with $p^h\phi_h{}^j$, we get

$$p^h\{\nabla^i(\nabla_ip_h + \nabla_hp_i) + \phi_h{}^j\phi_i{}^t\nabla^i(\nabla_jp_t + \nabla_tp_j)\} = 2p^h\phi_h{}^jr^t\nabla^i\nabla_t\phi_{ji}.$$

Moreover, by (2. 4), it can be written as

$$(3.19)\qquad p^h\{\nabla^i(\nabla_ip_h + \nabla_hp_i) + \phi_h{}^j\phi_i{}^t\nabla^i(\nabla_jp_t + \nabla_tp_j)\}$$

$$= 2p^h r^t (R^*{}_{th} - R_{th}).$$

Thus, substituting (3. 19) into (3. 15) and making use of Green's theorem, we have

$$\int_{X_{2n}} \left[\frac{1}{4} T_{ji} T^{ji} + 2p^h r^t (R^*{}_{ht} - R_{ht}) \right] d\sigma = 0. \tag{3.20}$$

Furthermore, substituting $p^h = v^h - r^h$ into (3.20), then (3.20) becomes

$$\int_{X_{2n}} \left[\frac{1}{4} T_{ji} T^{ji} + 2v^h r^t (R^*{}_{ht} - R_{ht}) + 2r^h r^t (R_{ht} - R^*{}_{ht}) \right] d\sigma = 0. \tag{3.21}$$

Hence, by Lemma 3. 4 and (2. 7), (3. 21) turns to

$$\int_{X_{2n}} \left[\frac{1}{4} T_{ji} T^{ji} + 2(r^h \nabla_h \phi_{ji}) r^t \nabla_t \phi^{ji} \right] d\sigma = 0$$

from which we can deduce $T_{ji} = 0$ and $r^t \nabla_t \phi_{ji} = 0$.

4. Proof of the main theorem. First of all, in order to prove that p^i in (3. 12) is a Killing vector, we put

$$U_{ji} \equiv \nabla_j p_i + \nabla_i p_j.$$

Operating ∇^i to $p^j U_{ji}$ and using $p_i = v_i - r_i$, we have

$$\begin{aligned} \nabla^i (p^j U_{ji}) &= \frac{1}{2} U_{ji} U^{ji} + p^j \nabla^i (\nabla_j p_i + \nabla_i p_j) \\ &= \frac{1}{2} U_{ji} U^{ji} + p^j (\nabla^i \nabla_j v_i + \nabla^i \nabla_i v_j - 2\nabla^i \nabla_j r_i). \end{aligned}$$

This equation can be written in the following form:

$$\begin{aligned} \nabla^i (p^j U_{ji}) = \frac{1}{2} U_{ji} U^{ji} + p^j (&\nabla^i \nabla_i v_j + \nabla^i \nabla_j v_i \\ &- \nabla_j \nabla^i v_i + \nabla_j \nabla^i v_i - 2\nabla^i \nabla_j r_i + 2\nabla_j \nabla^i r_i - 2\nabla_j \nabla^i r_i). \end{aligned} \tag{4. 1}$$

In this place, by the Ricci's identity and (3. 2), we have

$$\nabla^i \nabla_i v_j + \nabla^i \nabla_j v_i - \nabla_j \nabla^i v_i = \nabla^i \nabla_i v_j + R_{ji} v^i = 0 \tag{4. 2}$$

and

$$\nabla^i \nabla_j r_i - \nabla_j \nabla_i r^i = r^i R_{ji}. \tag{4. 3}$$

Hence, making use of (4. 2) and (4. 3), from (4. 1) by Green's theorem, we find

$$\int_{X_{2n}} \left[\frac{1}{2} U_{ji} U^{ji} - 2p^j r^i R_{ji} + p^j \nabla_j \alpha \right] d\sigma = 0 \tag{4. 4}$$

where $\alpha = \nabla^i v_i - 2\nabla^i r_i$.
And from $\nabla_i(\alpha p^i) = p^i\nabla_i\alpha + \alpha\nabla_i p^i = p^i\nabla_i\alpha$,
we have

$$\int_{X_{2n}} p^i\nabla_i\alpha d\sigma = 0. \tag{4.5}$$

Thus, by Lemma 3.5 and (4.5), (4.4) becomes

$$\int_{X_{2n}} \frac{1}{2} U_{ji}U^{ji}d\sigma = 0 \tag{4.6}$$

from which it follows

$$U_{ji} = \nabla_j p_i + \nabla_i p_j = 0, \tag{4.7}$$

that is, p^i is a Killing vector.

Secondly we shall show that r^i is almost-analytic.

Interchanging j and i in (3.1) and adding the equation thus obtained to (3.1), we get

$$\nabla_j v_i + \nabla_i v_j - \phi_j{}^a\phi_i{}^b(\nabla_a v_b + \nabla_b v_a) = 0.$$

Substituting $v_i = p_i + r_i$ into this equation and using (4.7), we have

$$\nabla_j r_i - \phi_j{}^a\phi_i{}^b\nabla_a r_b = 0 \qquad \text{i.e.} \qquad -\phi_j{}^t\nabla_t r_i - \phi_i{}^t\nabla_j r_t = 0. \tag{4.8}$$

Hence, adding this last equation to (3.14), we obtain

$$r^t\nabla_t\phi_{ji} - \phi_j{}^t\nabla_t r_i - \phi_i{}^t\nabla_j r_t = 0$$

which shows that r^i is almost-analytic.

Now, if we put

$$r^i = \phi_t{}^i q^t \qquad \text{i.e.} \qquad q^i = -\phi_t{}^i r^t,$$

then, $v^i = p^i + r^i$ can be written as

$$v^i = p^i + \phi_t{}^i q^t. \tag{4.9}$$

Lastly, we shall prove that q^i is a Killing vector. Taking account of (3.14), from Lemma 3.2 it follows that q^i is almost-analytic and therefore it satisfies

$$\nabla^t\nabla_t q^i + R_t{}^i q^t = 0. \tag{4.10}$$

On the other hand, by $\nabla^j r^i = \nabla^i r^j$ and $\nabla_i\phi_t{}^i = 0$,
we have

$$\nabla_i q^i = -\phi_t{}^i\nabla_i r^t = 0. \tag{4.11}$$

Thus, since our space is compact, (4.10) and (4.11) show that q^i is a Killing vector. q.e.d.

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