A CERTAIN SERIES OF MULTIPLICATIVELY ORTHOGONAL FUNCTIONS

KAZUAKI TAKEDA AND ZIRÔ TAKEDA

(Received September 2, 1963)

1. Introduction. The systems of ortho-normal functions introduced by Rademacher and Walsh have many remarkable properties. Especially Fine [2] has pointed out that the Walsh functions are characters of the dyadic group and hence it is naturally expected that a series of the Walsh functions has very similar properties to a trigonometric series. Many authors pursued this analogy and discussed it in detail. Some of them intended to generalize the Walsh functions. It may be said that the most efforts of them have done from the viewpoint of group character influenced by Fine's paper (see Lévy [4], Chrestenson [1] and Watari [6]).

Originally the Rademacher functions are closely related with probabilistic considerations. A connection between the Rademacher functions and the game of heads or tails are shown in below. The independent tosses of a coin give the simplest example of the binomial distribution ($n = 1$). In this paper we construct a multiplicatively orthogonal system of functions using the next simple binomial distribution ($n = 2$). These functions are real valued and so they are more convenient than the functions given by Lévy or others. We show the fundamental inequality of the Walsh functions given by Paley [5] remains true for our functions. That is to say, we know the abelian group character property of the Walsh functions is not essential for this inequality. Our viewpoint is very similar to that of Wiener [7] about Brownian motions. In fact, employing the normal distribution instead of binomial distributions we may arrive at the theory of Wiener integral. Analogous considerations are possible for general binomial distributions and for some another distributions but, since the calculations become considerably troublesome we want to reserve them with some allied topics for another occasion.

In this paper by *t-space* we mean the set of positive integers $1, 2, \cdots$. Let $\chi(t)$ be a discrete time series taking value 1 or -1 with probabilities $1/2$ for each $t = 1, 2, \cdots$. Then every time series satisfying these conditions may be identified with a point of the unit interval $[0, 1]$. Let us call this unit interval *α-space*. At first we divide the all time series into two classes according to $\chi(1) = 1$ or -1 and we make each class correspond to the semi-closed interval $\left[0, \frac{1}{2}\right)$ and $\left[\frac{1}{2}, 1\right)$ of α-space respectively. Next divide the set of time series into four classes according to $\chi(1)=1$, $\chi(2)=1$; $\chi(1)=1$, $\chi(2)=-1$; $\chi(1)=-1, \chi(2)=1$ and $\chi(1)=-1, \chi(2)=-1$ and associate them $\left[0, \frac{1}{4}\right)$, $\left[\frac{1}{4}, \frac{1}{2}\right)$, $\left[\frac{1}{2}, \frac{3}{4}\right)$ and $\left[\frac{3}{4}, 1\right)$ respectively. Continuing this process, to every interval $[m/2^n, (m+1)/2^n)$ (m is an integer such that $0 \leqq m \leqq 2^n-1$) of α-space we get a set of time series taking a specific value 1 or -1 at each time $t=1,2,\cdots,n$. Then we know ultimately every point α of α-space, except a set of enumerable points, shows a time series. Hence we denote this time series by $\chi(t,\alpha)$. We put $\varphi_n(\alpha)=\chi(n,\alpha)$. Then we know, by the construction of correspondence between time series and α-space, $\varphi_n(\alpha)$ $(n=1,2,\cdots)$ are nothing but the Rademacher functions. In the next section we consider time series taking values $\pm\sqrt{2}$ or 0 instead of ± 1 and get a system of multiplicatively orthogonal functions analogous to the Rademacher functions.

2. Time series $\chi(t,\alpha)$. Let n be a fixed positive integer and let $\chi_n(t)$ be a discrete time series defined for $t=1,2,3,\cdots$ and taking values

$$\frac{n-2k}{\sqrt{n}} \qquad (k=0,1,2,\cdots\ n) \tag{2. 1}$$

with probabilities

$$\frac{{}_nC_k}{2^n} \qquad (k=0,1,2,\cdots\ n) \tag{2. 2}$$

respectively for each time t. We assume always $\chi_n(0)=0$.

Putting $n=1$, we obtain a time series of random signs 1 and -1. This is just the case stated in introduction. For $n=2$, $\chi_2(t)$ has three possibilities, i.e. $\chi_2(t)=\sqrt{2}$, $=0$ and $=-\sqrt{2}$ with probabilities 1/4, 1/2 and 1/4 respectively. In the following we shall use only the time series $\chi_2(t)$ and we may write $\chi(t)$ instead of $\chi_2(t)$.

We begin to map the set of all time series $\chi(t)$ on the α-space [0,1]. Since the probabilities of $\chi(1)$ having the value $\sqrt{2}$, 0 and $-\sqrt{2}$ are 1/4, 1/2 and 1/4 respectively, we divide the α-space into three parts $[0,1/4)$, $[4/1,3/4)$ and $[3/4, 1)$. The length of these intervals are equal to the probabilities of having $\chi(1)=\sqrt{2}$, 0 and $-\sqrt{2}$ respectively. We make the interval $[0,1/4)$ correspond to the subset of time series such that $\chi(1)=\sqrt{2}$. Similarly the interval $[1/4,3/4)$ corresponds to the time series such that $\chi(1)=0$ and $[3/4,1)$ corresponds to the series such that $\chi(1)=-\sqrt{2}$.

Next we consider the class of time series which takes values

$$\chi(1) = \frac{2-2k}{\sqrt{2}} \qquad \text{and} \qquad \chi(2) = \frac{2-2l}{\sqrt{2}}$$

where $k = 0, 1$ or 2, $l = 0, 1$ or 2. The probability $P(k, l)$ that $\chi(t)$ should have these values at $t = 1$ and $t = 2$ is given by $({}_2C_k/4)({}_2C_l/4)$. That is

$$\begin{array}{lll} P(0,0) = 1/16, & P(0,1) = 2/16, & P(0,2) = 1/16 \\ P(1,0) = 2/16, & P(1,1) = 4/16, & P(1,2) = 2/16 \\ P(2,0) = 1/16, & P(2,1) = 2/16, & P(2,2) = 1/16\,; \end{array}$$

$$P(0,0) + P(0,1) + P(0,2) = 1/4$$
$$P(1,0) + P(1,1) + P(1,2) = 2/4$$
$$P(2,0) + P(2,1) + P(2,2) = 1/4.$$

Observing this relations, we divide each α-intervals $[0, 1/4)$, $[1/4, 3/4)$ and $[3/4, 1)$ into three sub-intervals having the ratio of length $1:2:1$ respectively, i.e.

$$\left[0, \frac{1}{4}\right) = \left[0, \frac{1}{16}\right) \cup \left[\frac{1}{16}, \frac{3}{16}\right) \cup \left[\frac{3}{16}, \frac{1}{4}\right)$$
$$\left[\frac{1}{4}, \frac{3}{4}\right) = \left[\frac{1}{4}, \frac{6}{16}\right) \cup \left[\frac{6}{16}, \frac{10}{16}\right) \cup \left[\frac{10}{16}, \frac{3}{4}\right)$$
$$\left[\frac{3}{4}, 1\right) = \left[\frac{3}{4}, \frac{13}{16}\right] \cup \left[\frac{13}{16}, \frac{15}{16}\right) \cup \left[\frac{15}{16}, 1\right).$$

Then we can map the classes of time series satisfying the above conditions to the finer α-subintervals than the first step preserving their probabilities.

Continuing this process for $t = 3, 4, \cdots$, we get finer and finer sub-intervals of α-space. Thus ultimately we are possible to assign for every point α of α-space (except for a set of measure zero) a time series, which we write $\chi(t,\alpha)$.

Now let us consider the function $\chi(n, \alpha)$ defined on α-space for a fixed n. $\chi(1, \alpha)$ is given as follows;

$$\chi(1,\alpha) = \begin{cases} \sqrt{2}, & \alpha \in \left[0, \frac{1}{4}\right) \\ 0, & \alpha \in \left[\frac{1}{4}, \frac{3}{4}\right) \\ -\sqrt{2}, & \alpha \in \left[\frac{3}{4}, 1\right). \end{cases} \tag{2. 3}$$

We denote the intevrals $\left[0, \frac{1}{4}\right)$, $\left[\frac{1}{4}, \frac{3}{4}\right)$ and $\left[\frac{3}{4}, 1\right)$ by $I(1, 1)$, $I(1, 2)$ and $I(1, 3)$ respectively. Similarly we have

$$\chi(2,\alpha)=\begin{cases}\sqrt{2}, & \alpha\in\left[0,\frac{1}{16}\right)\cup\left[\frac{4}{16},\frac{6}{16}\right)\cup\left[\frac{12}{16},\frac{13}{16}\right)\\ 0, & \alpha\in\left[\frac{1}{16},\frac{3}{16}\right)\cup\left[\frac{6}{16},\frac{10}{16}\right)\cup\left[\frac{13}{16},\frac{15}{16}\right)\\ -\sqrt{2}, & \alpha\in\left[\frac{3}{16},\frac{14}{16}\right)\cup\left[\frac{10}{16},\frac{12}{16}\right)\cup\left[\frac{15}{16},1\right).\end{cases} \tag{2. 4}$$

We also denote the intervals $\left[0,\frac{1}{16}\right),\left[\frac{1}{16},\frac{3}{16}\right),\left[\frac{3}{16},\frac{4}{16}\right),\cdots\cdots,\left[\frac{15}{16},1\right)$ by $I(2,1)$, $I(2,2)$, $I(2,3)$, $\cdots\cdots$, $I(2,3^2)$, respectively. We can define the functions $\chi(3,\alpha)$, $\chi(4,\alpha)$, $\cdots\cdots$ and intervals $I(3,1)$, $\cdots\cdots$, $I(3,3^3)$; $I(4,1)$, $\cdots\cdots$, $I(4,3^4)$; $\cdots\cdots$ analogously.

As is easily seen, the functions $\chi(n,\alpha)$ are normal and mutually orthogonal for different n : i. e.

$$\int_0^1 \chi(m,\alpha)\,\chi(n,\alpha)\,d\alpha=\delta_{mn} \tag{2. 5}$$

where δ_{mn} means Kronecker's delta. It is also easy to see that the system of functions is not complete, since for every n

$$\int_0^1 \chi(n,\alpha)\,\chi(1,\alpha)\,\chi(2,\alpha)\,d\alpha=0.$$

Analogous to the Rademacher functions, we can generate a system of multiplicatively orthogonal functions from these $\{\chi(n,\alpha)\}$, but the system obtained is not complete. This fact will be seen in the next section in detail.

3. Construction of ortho-normal functions. Let $L^2(t)$ and $L^2(\alpha)$ be real Hilbert spaces of real functions defined on t-space and α-space respectively. That is, $f(t)\in L^2(t)$ if and only if $\sum_{t=1}^{\infty}|f(t)|^2<\infty$ and $g(\alpha)\in L^2(\alpha)$ if and only if $\int_0^1|g(\alpha)|^2d\alpha<\infty$. Now using the time series $\chi(t,\alpha)$ stated in the preceding section, *we can map $L^2(t)$ linearly and isometrically into $L^2(\alpha)$.*

Let $e_n(t)$ be a characteristic function of a point n in t-space; that is,

$$e_n(t)=\begin{cases}1 & \text{for } t=n,\\ 0 & \text{for } t\neq n,\end{cases} \tag{3. 1}$$

where n is a fixed positive integer. It is easy to see that the system $\{e_n(t)\}$ $(n=1,2,\cdots\cdots)$ forms a complete orthonormal system in $L^2(t)$, i. e.

(3. 2) $$\sum_{t=1}^{\infty} g(t)\, e_m(t) = 0 \qquad \text{for all } m$$

implies $g(t) = 0$ and

(3. 3) $$\sum_{t=1}^{\infty} e_m(t)\, e_n(t) = \delta_{mn}.$$

Let $f(t)$ and $g(t)$ be normalized orthogonal functions in $L^2(t)$ having the following forms:

$$f(t) = \sum_{n=1}^{N} c_n e_n(t), \qquad g(t) = \sum_{n=1}^{N} d_n e_n(t).$$

By the assumption they satisfy

$$\sum_{t=1}^{\infty} f(t) g(t) = \sum_{n=1}^{N} c_n d_n = 0,$$

$$\sum_{t=1}^{\infty} f^2(t) = \sum_{n=1}^{N} c_n^2 = 1, \qquad \sum_{t=1}^{\infty} g^2(t) = \sum_{n=1}^{N} d_n^2 = 1.$$

Define the transform of $f(t)$ by $\sum f(t)\chi(t, \alpha)$, that is

$$\sum_{t=0}^{\infty} f(t)\chi(t, \alpha) = \sum_{t=0}^{\infty} \left[\sum_{n=1}^{N} c_n e_n(t)\right] \chi(t, \alpha)$$
$$= c_1\chi(1, \alpha) + c_2\chi(2, \alpha) + \cdots + c_N\chi(N, \alpha).$$

This is a function in $L^2(\alpha)$. Then we have

$$\int_0^1 \left[\sum_{t=1}^{\infty} f(t)\chi(t, \alpha)\right]\left[\sum_{t=1}^{\infty} g(t)\chi(t, \alpha)\right] d\alpha$$
$$= \int_0^1 \left[\sum_{n=1}^{N} c_n\chi(n, \alpha)\right]\left[\sum_{n=1}^{N} d_n\chi(n, \alpha)\right] d\alpha = \sum_{n=1}^{N} c_n d_n = 0$$

by (2. 5) and similarly

$$\int_0^1 \left[\sum_{t=1}^{\infty} f(t)\chi(t, \alpha)\right]^2 d\alpha = \sum_{n=1}^{N} c_n^2 = 1,$$
$$\int_0^1 \left[\sum_{t=1}^{\infty} g(t)\chi(t, \alpha)\right]^2 d\alpha = \sum_{n=1}^{N} d_n^2 = 1.$$

This proves the proposition.

Now we define on t-space homogeneous functions of zero degree, of first degree, of second degree etc. as follows. K_0 is a function of zero degree if it is a constant. $K_1(t)$ is homogeneous of the first degree if it is given by

$$K_1(t) = \sum_{n=1}^{N} a_n e_n(t). \tag{3. 4}$$

A homogeneous function of the second degree is given as

$$K_2(t_1, t_2) = \sum_{m=1}^{M} \sum_{n=1}^{M} a_{mn} e_m(t_1) e_n(t_2) \tag{3. 5}$$

(where $a_{mn} = a_{nm}$). This function $K_2(t_1, t_2)$ is clearly symmetric with respect to t_1, t_2. Similarly a homogeneous function of the third degree is given by

$$K_3(t_1, t_2, t_3) = \sum_{l=1}^{L} \sum_{m=1}^{L} \sum_{n=1}^{L} a_{lmn} e_l(t_1) e_m(t_2) e_n(t_3) \tag{3. 6}$$

where $a_{lmn} = a_{lnm} = a_{mln} = a_{mnl} = a_{nlm} = a_{nml}$. In general a homogeneous function of the n-th degree is given by the same manner. A linear combination of such homogeneous functions

$$K_n(t_1, t_2, \cdots, t_n) + K_{n-1}(t_1, t_2, \cdots, t_{n-1}) + \cdots + K_1(t_1) + K_0 \tag{3. 7}$$

is called a function of n-th degree. We show every function of an arbitrary degree can be transformed to a function belonging to $L^2(\alpha)$. By a function of the zero degree we mean the same constant. For a function of the first degree we take the transformation stated at the beginning of this section. For a function of the second degree

$$K_2(t_1, t_2) = \sum_{m=1}^{M} \sum_{n=1}^{M} a_{mn} e_m(t_1) e_n(t_2),$$

we put

$$\sum_{t_1=1}^{\infty} \sum_{t_2=1}^{\infty} K_2(t_1, t_2) \chi(t_1, \alpha) \chi(t_2, \alpha) = \sum_{t_1=1}^{\infty} \sum_{t_2=1}^{\infty} \left[\sum_{m=1}^{M} \sum_{n=1}^{M} a_{mn} e_m(t_1) e_n(t_2) \right] \chi(t_1, \alpha) \chi(t_2, \alpha)$$
$$= \sum_{m=1}^{M} \sum_{n=1}^{M} a_{mn} \chi(m, \alpha) \chi(n, \alpha)$$

In general, we define the transform of a homogeneous functions of n-th degree $K_n(t_1, t_2, \cdots, t_n)$ as

$$\sum_{l=1}^{L} \sum_{m=1}^{L} \cdots \sum_{n=1}^{L} a_{lm \ldots n} \chi(l, \alpha) \chi(m, \alpha) \cdots \chi(n, \alpha). \tag{3. 8}$$

By extending linearly, we can define the transforms of non-homogeneous functions of arbitrary degrees.

We may select up a complete system of normalized orthogonal functions from these transformed functions. Let us start with a zero degree function K_0. The normalization

$$\int_0^1 K_0^2 d\alpha = 1 \tag{3. 9}$$

implies $K_0 = 1$ or -1. By *the zero-th species of functions* we mean the constant 1. Let us consider the first degree functions in α-space:

$$\sum_{t=0}^{\infty} K_1(t)\chi(t, \alpha) + K_0. \tag{3.10}$$

Then, to be orthogonal to constants C,

$$\begin{aligned} &\int_0^1 \left\{\sum_{t=0}^{\infty} K_1(t)\chi(t, \alpha) + K_0\right\} C\, d\alpha \\ &= \int_0^1 \{a_1\chi(1, \alpha) + a_2\chi(2, \alpha) + \cdots + a_N\chi(N, \alpha) + K_0\}\, C\, d\alpha \\ &= K_0 C = 0 \end{aligned}$$

implies $K_0 = 0$. Thus we get a homogeneous first degree function of α. Now we normalize the first degree functions. It is easy to see that

$$\begin{aligned} &\int_0^1 \left[\sum_{t=0}^{\infty} K_1(t)\chi(t, \alpha)\right]^2 d\alpha = \int_0^1 \left[\sum_{n=1}^{N} a_n\chi(n, \alpha)\right]^2 d\alpha \\ &= \int_0^1 \sum_{n=1}^{N} a_n^2\chi^2(n, \alpha)\, d\alpha = \sum_{n=1}^{N} a_n^2. \end{aligned}$$

Thus

$$\begin{cases} \sum_{t=0}^{\infty} K_1(t)\chi(t, \alpha) \equiv \sum_{n=1}^{N} a_n\chi(n, \alpha) \\ \sum_{n=1}^{N} a_n^2 = 1 \end{cases} \tag{3.11}$$

gives a normalized function of first degree which is orthogonal to zero degree functions. Especially if we take

$$K_1(t) = e_n(t), \tag{3.12}$$

then we get the function $\chi(n, \alpha)$. Since $\{e_n(t)\}$ $(n = 1, 2, \cdots)$ are normal and mutually orthogonal in $L^2(t)$, we obtain a system $\{\chi(n, \alpha)\}$ $(n = 1, 2, \cdots)$, which are normal and orthogonal to constants and to each other in $L^2(\alpha)$. We call them *the first species of functions.*

Now we consider a second degree function in $L^2(\alpha)$ such that

$$\sum_{t_1=1}^{\infty}\sum_{t_2=1}^{\infty} K_2(t_1, t_2)\chi(t_1, \alpha)\chi(t_2, \alpha) + \sum_{t=1}^{\infty} K_1(t)\chi(t, \alpha) + K_0 \tag{3.13}$$
$$= \sum_{m=1}^{M}\sum_{n=1}^{M} a_{mn}\chi(m, \alpha)\chi(n, \alpha) + \sum_{n=1}^{N} a_n\chi(n, \alpha) + K_0.$$

We want (3. 13) to be orthogonal to every constant C, so we have

$$\int_0^1 \left[\sum_{t_1=1}^{\infty}\sum_{t_2=1}^{\infty} K_2(t_1, t_2)\chi(t_1, \alpha)\chi(t_2 \alpha) + \sum_{t=1}^{\infty} K_1(t)\chi(t, \alpha) + K_0\right] C\, d\alpha$$
$$= \int_0^1 \left[\sum_{m=1}^{M}\sum_{n=1}^{M} a_{mn}\chi(m, \alpha)\chi(n, \alpha) + \sum_{n=1}^{N} a_n\chi(n, \alpha) + K_0\right] C\, d\alpha$$
$$= \left[\sum_{m=1}^{M} a_{mm} + K_0\right] C = 0.$$

Thus we have

$$\sum_{m=1}^{M} a_{mm} + K_0 = 0.$$

$$K_0 = -\sum_{m=1}^{M} a_{mm}. \tag{3, 14}$$

We also want to make (3. 13) orthogonal to any first degree function

$$\sum_{t=1}^{\infty} K_1^*(t)\chi(t, \alpha) + K_0^* = \sum_{n=1}^{N} a_n^*\chi(n, \alpha) + K_0^*. \tag{3.15}$$

Since, when (3. 14) is satisfied, the second degree function (3. 13) is orthogonal to any constant K_0^*, it is enough to consider $\sum K_1^*(t)\chi(t, \alpha)$ only. Then by (2. 5) and the fact that

$$\int_0^1 \chi(n, \alpha)\chi(m, \alpha)\chi(l, \alpha)\, d\alpha = 0,$$

we have

$$\int_0^1 \left[\sum_{t_1=1}^{\infty}\sum_{t_2=1}^{\infty} K_2(t_1, t_2)\chi(t_1, \alpha)\chi(t_2, \alpha) + \sum_{t=1}^{\infty} K_1(t)\chi(t, \alpha) - \sum_{m=1}^{M} a_{mm}\right]$$
$$\times \left[\sum_{t-1}^{\infty} K_1^*(t)\chi(t, \alpha)\right] d\alpha$$

$$= \int_0^1 \left[\sum_{n=1}^{N} a_n \chi(n,\alpha)\right]\left[\sum_{n=1}^{N} a_n^* \chi(n,\alpha)\right] d\alpha = \sum_{n=1}^{N} a_n a_n^* = 0.$$

Since $K_1^*(t)$ is arbitrary, we must have $a_n = 0$ for $n = 1, 2, \cdots, N$, that is, $K_1(t) = 0$. Therefore we know the expression which is orthogonal to every zero degree and first degree function is given by

$$\sum_{m=1}^{M}\sum_{n=1}^{M} a_{mn}\chi(m,\alpha)\chi(n,\alpha) - \sum_{m=1}^{M} a_{mm} \qquad (a_{mn} = a_{nm}). \tag{3.16}$$

Thus we have obtained a category of second degree functions which are orthogonal to lower degree functions. We have to normalize them.

$$\int_0^1 \left[\sum_{m=1}^{M}\sum_{n=1}^{M} a_{mn}\chi(m,\alpha)\chi(n,\alpha) - \sum_{m=1}^{M} a_{mm}\right]^2 d\alpha$$

$$= \int_0^1 d\alpha \left[\sum_{m=1}^{M}\sum_{n=1}^{M}\sum_{k=1}^{M}\sum_{l=1}^{M} a_{mn}a_{kl}\chi(m,\alpha)\chi(n,\alpha)\chi(k,\alpha)\chi(l,\alpha) - 2\left(\sum_{m=1}^{M} a_{mm}\right)\left(\sum_{m=1}^{M}\sum_{l=1}^{M} a_{mn}\chi(m,\alpha)\chi(n,\alpha)\right) + \left(\sum_{m=1}^{M} a_{mm}\right)^2\right]$$

$$= \sum_{\substack{m=1 \\ m\neq k}}^{M}\sum_{k=1}^{M} a_{mm}a_{kk} + 2\sum_{m=1}^{M} a_{mm}^2 + \sum_{\substack{m=1 \\ m\neq n}}^{M}\sum_{n=1}^{M} a_{mn}^2 + \sum_{\substack{n=1 \\ n\neq m}}^{M}\sum_{m=1}^{M} a_{nm}^2 - 2\left(\sum_{m=1}^{M} a_{mm}\right)^2 + \left(\sum_{m=1}^{M} a_{mm}\right)^2 \tag{3.17}$$

$$= 2\sum_{\substack{m=1 \\ m\neq n}}^{M}\sum_{n=1}^{M} a_{mn}^2 + \sum_{m=1}^{M} a_{mm}^2 = 1.$$

Here we notice the following relations which will be easily seen from the graphs of $\chi(n,\alpha)$.

$$\begin{aligned}
&\int_0^1 \chi(m,\alpha)\chi(n,\alpha)\chi(k,\alpha)\chi(l,\alpha)d\alpha = 0,\\
&\int_0^1 \chi(m,\alpha)\chi(m,\alpha)\chi(k,\alpha)\chi(l,\alpha)d\alpha = 0,\\
&\int_0^1 \chi(m,\alpha)\chi(m,\alpha)\chi(k,\alpha)\chi(k,\alpha)d\alpha = 1,\\
&\int_0^1 \chi(m,\alpha)\chi(m,\alpha)\chi(m,\alpha)\chi(k,\alpha)d\alpha = 0,
\end{aligned} \tag{3.18}$$

$$\int_0^1 \chi(m,\alpha)\chi(m,\alpha)\chi(m,\alpha)\chi(m,\alpha)d\alpha = 2,$$

(we promise hereafter that different letters m, n, k, l mean different integers respectively).

Putting $a_{mn} = a_{nm} = 1/2$ for specific m,n and $a_{pq} = 0$ for other indices, we get $\chi(m,\alpha)\ \chi(n,\alpha)$. Also putting $a_{mm} = 1$ for a fixed m and $a_{pq} = 0$ for other indices, we get $\chi^2(m,\alpha) - 1$. We call the family

$$\{\chi(m,\alpha)\chi(n,\alpha),\ \chi^2(m,\alpha) - 1\} \quad (m \neq n;\ m = 1, 2, \cdots;\ n = 1, 2, \cdots) \tag{3.19}$$

the second species of functions. It is easily seen that the functions of this species are mutually orthogonal.

Now we are in position to consider the third degree functions,

$$\sum_{l=1}^{L}\sum_{m=1}^{L}\sum_{n=1}^{L} a_{lmn}\chi(l,\alpha)\chi(m,\alpha)\chi(n,\alpha) + \sum_{m=1}^{M}\sum_{n=1}^{M} a_{mn}\chi(m,\alpha)\chi(n,\alpha) + \sum_{n=1}^{M} a_n\chi(n,\alpha) + K_0, \tag{3.20}$$

where $a_{lmn} = a_{mnl} = a_{nlm} = a_{lnm} = a_{nml} = a_{mln}$ and $a_{mn} = a_{nm}$.

Elementary computations show that (3.20) is orthogonal to any functions of 0-th, 1st and 2nd degree when the following conditions are satisfied:

$$\sum_{m=1}^{M} a_{mm} + K_0 = 0, \tag{3.21}$$

$$3\sum_{l=1}^{L} a_{llk} - a_{kkk} + a_k = 0 \qquad \text{for every } k, \tag{3.22}$$

$$a_{mn} = 0 \qquad \text{for all } m,n. \tag{3.23}$$

Thus we have a category of third degree functions having the following form:

$$\sum_{l=1}^{L}\sum_{m=1}^{L}\sum_{n=1}^{L} a_{lmn}\chi(l,\alpha)\chi(m,\alpha)\chi(n,\alpha) - \sum_{n=1}^{L}\left(3\sum_{l=1}^{L} a_{lln} - a_{nnn}\right)\chi(n,\alpha). \tag{3.24}$$

Since (3.24) is orthogonal to every function of first degree, we have

$$\int_0^1 d\alpha \left[\sum\sum\sum a_{lmn}\chi(l,\alpha)\chi(m,\alpha)\chi(n,\alpha) - \sum\left(3\sum a_{lln} - a_{nnn}\right)\chi(n,\alpha)\right]^2$$

$$= \int_0^1 d\alpha \left[\sum\sum\sum a_{lmn}\chi(l,\alpha)\chi(m,\alpha)\chi(n,\alpha) - \sum\left(3\sum a_{lln} - a_{nnn}\right)\chi(n,\alpha)\right]$$

$$\times\left[\sum\sum\sum a_{pqr}\chi(p,\alpha)\chi(q,\alpha)\chi(r,\alpha)\right].$$

Hence by elementary but somewhat complicated computation, the condition of normalization is given as

$$\Big[3!\sum\sum\sum{}^{*} a_{lmn}{}^2 + 3^2\sum\sum\sum{}^{*} a_{llm}a_{mnn}$$

$$+ 2\times 3^2\sum_{l\neq m}\sum a_{llm} + 2\times 3\sum_{l\neq m}\sum a_{lll}a_{lmm} + 4\sum a_{lll}{}^2\Big]$$

$$-\left[3\sum_{p\neq r}\sum a_{ppr}\left(3\sum_{l=1}^{L} a_{llr} - a_{rrr}\right) + 2\sum_{p=1}^{L} a_{ppp}\left(3\sum_{l=1}^{L} a_{llp} - a_{ppp}\right)\right] = 1,$$

where $\sum\sum\sum{}^{*}$ means the summation without $l = m,\ m = n,\ n = l$.

Putting $a_{lmn} = 1/6$ or $a_{lln} = 1/3$ we get third degree functions $\chi(l,\alpha)\times\chi(m,\alpha)\chi(n,\alpha)$ and $\chi^2(l,\alpha)\chi(n,\alpha) - \chi(n,\alpha) = \{\chi^2(l,\alpha) - 1\}\chi(n,\alpha)$. We call the family of third degree functions

(3.26) $$\{\chi(l,\alpha)\chi(m,\alpha)\chi(n,\alpha)\,;\,[\chi^2(l,\alpha) - 1]\chi(n,\alpha)\}$$

the third specices of functions. They are normalized and orthogonal to each other and to lower degree functions.

We should like to continue this processes for fourth degree functions and so on. In those cases the computations will become more and more complicated and so we confine ourselves to state that we get the family

(3.27) $$\begin{gathered}\{\chi(k,\alpha)\chi(l,\alpha)\chi(m,\alpha)\chi(n,\alpha);\\ [\chi^2(l,\alpha) - 1]\chi(m,\alpha)\chi(n,\alpha);\\ [\chi^2(l,\alpha) - 1][\chi^2(m,\alpha) - 1]\}\end{gathered}$$

as *the fourth species of functions.* Conversely we show that the total family of functions of the n-th species constitute a complete orthonormal system in $L^2(\alpha)$. That is to say, we get

THEOREM 1. *Put*

(3.28) $$\varphi_{n,0}(\alpha) = \chi(n,\alpha),\ \varphi_{n,1}(\alpha) = \chi^2(n,\alpha) - 1 \qquad (n = 1, 2,\cdots)$$

and $\varphi_{0,1}(\alpha) = 1$. *Then the set of finite products of these functions*

(3.29) $$\{\varphi_{l,\cdot}(\alpha)\varphi_{m,\cdot}(\alpha)\cdots\varphi_{n,\cdot}(\alpha)\} \qquad l > m > \cdots > n \geqq 0$$

forms a complete orthonormal system in $L^2(\alpha)$. (*See* Fig. 1).

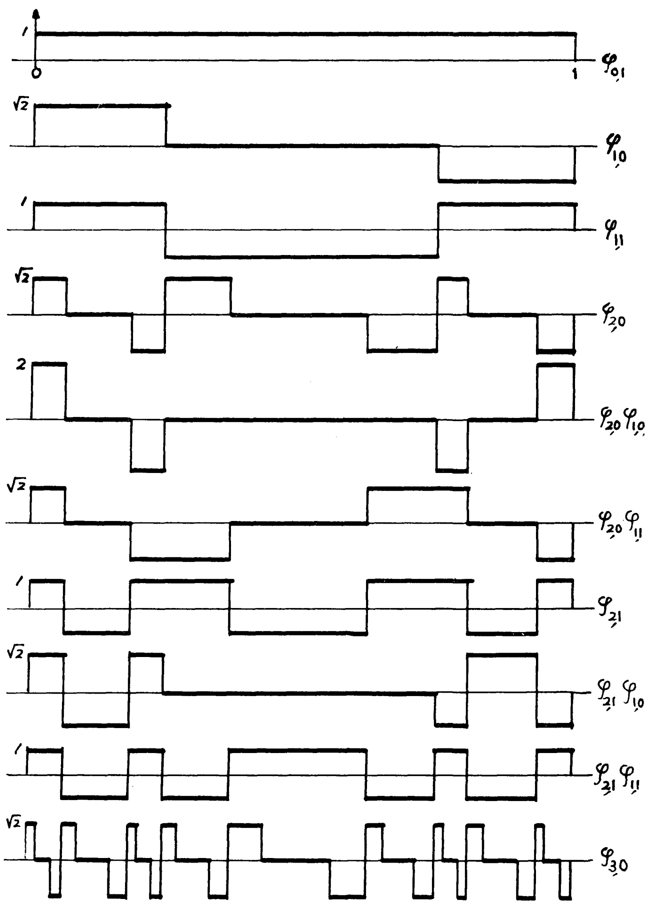
PROOF. Let

$$\varphi_{l,\cdot}(\alpha)\varphi_{m,\cdot}(\alpha)\cdots\varphi_{n,\cdot}(\alpha) \tag{3.30}$$

and

$$\varphi_{p,\cdot}(\alpha)\varphi_{q,\cdot}(\alpha)\cdots\varphi_{r,\cdot}(\alpha) \tag{3.31}$$

be any two functions of the system. They are mutually orthogonal. We may suppose to be $l \geqq p$. In case of $l > p$



(Fig. 1)

$$\int_0^1 \varphi_{l,\cdot}(\alpha)\varphi_{m,\cdot}(\alpha)\cdots\varphi_{n,\cdot}(\alpha)\cdot\varphi_{p,\cdot}(\alpha)\varphi_{q,\cdot}(\alpha)\cdots\varphi_{r,\cdot}(\alpha)\,d\alpha = 0 \tag{3.32}$$

because $\varphi_{m,\cdot}(\alpha), \cdots, \varphi_{n,\cdot}(\alpha), \varphi_{p,\cdot}(\alpha), \cdots, \varphi_{r,\cdot}(\alpha)$ are constant on every interval $I(l-1, k)$ and

$$\int_{I(l-1,k)} \varphi_{l,\cdot}(\alpha)\,d\alpha = 0 \qquad (1 \leqq k \leqq 3^{l-1}). \tag{3.33}$$

If $l = p$, we may divide into four cases: 1) $\varphi_{l,\cdot}(\alpha) \equiv \varphi_{l,1}(\alpha)$, $\varphi_{p,\cdot}(\alpha) \equiv \varphi_{p1}(\alpha)$; 2) $\varphi_{l,\cdot}(\alpha) \equiv \varphi_{l,1}(\alpha)$, $\varphi_{p,\cdot}(\varphi) \equiv \varphi_{p,0}(\alpha)$; 3) $\varphi_{l,\cdot}(\alpha) \equiv \varphi_{l,0}(\alpha)$, $\varphi_{p,\cdot}(\alpha) \equiv \varphi_{p,1}(\alpha)$; 4) $\varphi_{l,\cdot}(\alpha) \equiv \varphi_{l,0}(\alpha)$, $\varphi_{p,\cdot}(\alpha) \equiv \varphi_{p,0}(\alpha)$. In case 1), $\varphi_{l,1} = \varphi_{p,1}$ and we may omit the terms $\varphi_{l,\cdot}$ and $\varphi_{p,\cdot}$ in (3. 32) since $\varphi_{l,1}^2(\alpha) \equiv 1$. Then it is enough to compare the next terms of (3. 30) and (3. 31), and we may proceed the arguments that we are now discussing. In cases 2) and 3), $\varphi_{l,\cdot} \equiv \varphi_{l,1}$, $\varphi_{p,\cdot} \equiv \varphi_{p,0}$ or $\varphi_{l,\cdot} \equiv \varphi_{l,0}$, $\varphi_{p,\cdot} \equiv \varphi_{p,1}$ and (3.32) is also evident since $\varphi_{l,0}(\alpha) \cdot \varphi_{l,1}(\alpha) = \varphi_{l,0}(\varphi)$. In case 4), $\varphi_{l,0} = \varphi_{p,0}$ and $\varphi_{m,\cdot}(\alpha) \neq \varphi_{q,\cdot}(\alpha)$. It is enough to consider the case $m \neq q$. We may suppose $m > q$. Then

$$\int_0^1 \varphi_{l,0}^2(\alpha)\varphi_{m,\cdot}(\alpha)\,d\alpha = \sum_{k=1}^{3^{m-1}} \int_{I(m-1,k)} \varphi_{l,0}^2(\alpha)\varphi_{m,\cdot}(\alpha)\,d\alpha = 0 \tag{3.34}$$

and the other functions of integrand in (3. 32) are constants on every intervals $I(m-1, k)$. Thus (3. 32) is proved.

The function (3. 29) is normalized, because $\varphi_{k,1}^2(\alpha) = 1$ for every k and $\varphi_{m_1,0}^2 \cdot\cdot \cdot \varphi_{m_\nu,0}^2(\alpha) = 2^\nu$ at a point set of measure $\left(\frac{1}{2}\right)^\nu$ and $= 0$ elsewhere.

We show the completeness of this system. For this purpose it is sufficient to show that any characteristic function on an interval with dyadic rational end points can be approximated by a sum of the functions of this system. First we show

$$\begin{aligned} f_1(\alpha) &= 1 \text{ for } \alpha \in [0, 1/4), &&= 0 \text{ elsewhere,} \\ g_1(\alpha) &= 1 \text{ for } \alpha \in [3/4, 0), &&= 0 \text{ elsewhere,} \\ k_1(\alpha) &= 1 \text{ for } \alpha \in [1/4, 3/4), &&= 0 \text{ elsewhere} \end{aligned} \tag{3.35}$$

are expressed by sums of functions of this system. That is,

$$\begin{aligned} f_1(\alpha) &= \frac{1}{2^2}\varphi_{0,1}(\alpha) + \frac{1}{2\sqrt{2}}\varphi_{1,0}(\alpha) + \frac{1}{2^2}\varphi_{1,1}(\alpha), \\ g_1(\alpha) &= \frac{1}{2^2}\varphi_{0,1}(\alpha) - \frac{1}{2\sqrt{2}}\varphi_{1,0}(\alpha) + \frac{1}{2^2}\varphi_{1,1}(\alpha), \\ k_1(\alpha) &= \frac{1}{2}\varphi_{0,1}(\alpha) - \frac{1}{2}\varphi_{1,1}(\alpha). \end{aligned}$$

Similarly we put

$$\begin{aligned} f_n(\alpha) &= \frac{1}{2^2}\varphi_{0,1}(\alpha) + \frac{1}{2\sqrt{2}}\varphi_{n,0}(\alpha) + \frac{1}{2^2}\varphi_{n,1}(\alpha), \\ g_n(\alpha) &= \frac{1}{2^2}\varphi_{0,1}(\alpha) - \frac{1}{2\sqrt{2}}\varphi_{n,0}(\alpha) + \frac{1}{2^2}\varphi_{n,1}(\alpha), \\ k_n(\alpha) &= \frac{1}{2}\varphi_{0,1}(\alpha) - \frac{1}{2}\varphi_{n,1}(\alpha) \end{aligned} \tag{3.36}$$

then we have

$$f_n(\alpha) = \max \frac{1}{\sqrt{2}} [\varphi_{n,0}(\alpha), 0]$$

$$g_n(\alpha) = \max \frac{1}{\sqrt{2}} [-\varphi_{n,0}(\alpha), 0]$$

and

$$k_n(\alpha) = 1 - f_n(\alpha) - g_n(\alpha).$$

Then the characteristic function

$$h(\alpha) = \begin{cases} 1, & \alpha \in \left(\frac{2^n - 1}{2^{n+1}}, \frac{2^{n+1} - 1}{2^{n+2}}\right) \\ 0, & \text{elsewhere} \end{cases}$$

can be represented as

$$h(\alpha) = k_1(\alpha)k_2(\alpha)\cdots k_n(\alpha)f_{n+1}(\alpha). \tag{3.37}$$

Consequently

$$\sum_{n=1}^{\infty} k_1(\alpha)k_2(\alpha)\cdots k_n(\alpha)f_{n+1}(\alpha) \tag{3.38}$$

$$= \sum_{n=1}^{\infty} \frac{1}{2^{n+1}} (1 - \varphi_{1,1}(\alpha))(1 - \varphi_{2,1}(\alpha))\cdots(1 - \varphi_{n,1}(\alpha))$$

$$\times\left(\frac{1}{2} + \frac{1}{\sqrt{2}} \varphi_{n+1,0}(\alpha) + \frac{1}{2} \varphi_{n+1,1}(\alpha)\right)$$

represents a characteristic function of $\left[\frac{1}{4}, \frac{2}{4}\right)$. Thus we have succeded in expressing the characteristic functions of $\left[0, \frac{1}{4}\right)$, $\left[\frac{1}{4}, \frac{2}{4}\right)$, $\left[\frac{2}{4}, \frac{3}{4}\right)$ and $\left[\frac{3}{4}, 1\right)$ by the desired forms.

This method tells us that if we change the combination of terms of (3. 38) we can obtain the characteristic function of an arbitrary interval with dyadic rational end points. For example, if we replace $k_1(\alpha)$ by $f_1(\alpha)$ in (3. 38) we get the characteristic function of $\left[\frac{1}{16}, \frac{2}{16}\right)$.

4. **Fundamental inequality.** At first we rearrange the complete orthonormal system by the number of partitions of α-space, not by the degree of functions. In Theorem 1, we put

$$\varphi_{m,0}(\alpha) = \chi(m, \alpha)$$

$$\varphi_{m,1}(\alpha) = \chi^2(m, \alpha) - 1 \qquad (m = 1, 2, 3, \cdots).$$

Here we expand each positive integer n triadically:

$$n = 3^{n_1}2^{m_1} + 3^{n_2}2^{m_2} + \cdots + 3^{n_\nu}2^{m_\nu} \tag{4. 1}$$

where $n_1 > n_2 > \cdots > n_\nu \geqq 0$, $m_1, m_2, \cdots, m = 0$ or 1. Then we put

$$\psi_0(\alpha) \equiv 1 \tag{4. 2}$$
$$\psi_n(\alpha) \equiv \varphi_{n_1+1,m_1}(\alpha) \cdot \varphi_{n_2+1,m_2}(\alpha) \cdots \varphi_{n_\nu+1,m_\nu}(\alpha).$$

That is,

$$\begin{array}{lll} \psi_1(\alpha) \equiv \varphi_{1,0}(\alpha), & \psi_2(\alpha) \equiv \varphi_{1,1}(\alpha), & \psi_3(\alpha) \equiv \varphi_{2,0}(\alpha), \\ \psi_4(\alpha) \equiv \varphi_{2,0}(\alpha)\varphi_{1,0}(\alpha), & \psi_5(\alpha) \equiv \varphi_{2,0}(\alpha)\varphi_{1,1}(\alpha), & \psi_6(\alpha) \equiv \varphi_{2,1}(\alpha) \\ \psi_7(\alpha) \equiv \varphi_{2,1}(\alpha)\varphi_{1,0}(\alpha), & \psi_8(\alpha) \equiv \varphi_{2,1}(\alpha)\varphi_{1,1}(\alpha), & \psi_9(\alpha) \equiv \varphi_{3,0}(\alpha) \end{array} \tag{4. 3}$$

etc. As is easily seen, the system $\{\psi_m(\alpha)\}$ is nothing but the rearrangement of the complete ortho-normal system obtained in Theorem 1. By this numbering, the functions $\psi_m(\alpha)$ $(0 \leqq m \leqq 3^n - 1)$ are all constant on the common intervals $I(n, \nu)$ $(\nu = 1, 2, \cdots, 3^n)$.

We show this value by $\psi_m(I(n, \nu))$. We know that the intervals on which $\psi_m(\alpha)$ takes constants $\psi_m(I(n, \nu))$ become narrower as m increases. We shall give here *an alternative proof of Theorem* 1, that is, *the completess of* $\{\psi_m(\alpha)\}$. Suppose that $f(\alpha) \in L^2(\alpha)$ and

$$\int_0^1 f(\alpha)\psi_m(\alpha)\, d\alpha = 0 \qquad \text{for} \qquad 0 \leqq m \leqq 3^n - 1. \tag{4. 4}$$

Then we have

$$\int_0^1 f(\alpha)\psi_m(\alpha) d\alpha = \sum_{\nu=1}^{3^n} \left[\int_{I(n,\nu)} f(\alpha)\, d\alpha \cdot \psi_m(I(n, \nu)) \right] = 0. \tag{4. 5}$$

Let us consider the determinant of 3^n-th degree

$$|\psi_m(I(n, \nu))| \qquad (0 \leqq m \leqq 3^n - 1,\ 1 \leqq \nu \leqq 3^n). \tag{4. 6}$$

This determinant (4. 6) does not vanish, because by the orthogonality condition

$$\int_0^1 \psi_m(\alpha)\psi_n(\alpha)\, d\alpha = \delta_{mn},$$

we know that

$$|\psi_m(I(n,\nu))|^2$$

is non-zero. Thus 3^n vectors in 3^n-dimensional space

$$(\psi_m(I(n,1)),\ \psi_m(I(n,2)),\cdots,\psi_m(I(n,3^n))) \qquad 0\leqq m\leqq 3^n-1$$

are linearly independent and we have by (4. 5)

$$\int_{I(n,\nu)} f(\alpha)d\alpha = 0 \qquad (\nu = 1,2,\cdots,3^n).$$

The set of terminating points of all the intervals $I(n,\nu)$ $(n = 1,2,\cdots)$ forms a dense set of points in α-space. Now let us suppose (4. 4) holds for every n, then the continuous function defined by

$$F(\alpha) = \int_0^\alpha f(t)\,dt$$

must be a constant. It follows that $f(\alpha)$ is equivalent to zero. This proves the completeness of the system.

Let $f(\alpha)\in L(\alpha)$ and write

$$f(\alpha)\sim\sum_{n=0}^{\infty} c_n\psi_n(\alpha), \tag{4. 7}$$

where c_n is given by

$$c_n = \int_0^1 f(\alpha)\ \psi_n(\alpha)\,\alpha. \tag{4. 8}$$

The n-th partial sum of (4. 7) is

$$S_n(\alpha) = \sum_{k=0}^{n-1} c_k\psi_k(\alpha). \tag{4. 9}$$

The 3^n-th partial sum $S_{3^n}(\alpha)$ is then given as

$$\begin{aligned} S_{3^n}(\alpha) &= \sum_{k=0}^{3^n-1}\left[\int_0^1 f(\beta)\psi_k(\beta)d\beta\right]\psi_k(\alpha) \\ &= \int_0^1 f(\beta)\left[\sum_{k=0}^{3^n-1}\psi_k(\beta)\psi_k(\alpha)\right]d\beta = \int_0^1 f(\beta)D_{3^n}(\alpha,\beta)\,d\beta, \end{aligned} \tag{4.10}$$

where

$$D_{3^n}(\alpha,\beta)\equiv\sum_{k=0}^{3^n-1}\psi_k(\beta)\psi_k(\alpha).$$

Since $\psi_m(\alpha)$ is defined as a finite product of $\varphi_{n,0}(\alpha)$ or $\varphi_{n,1}(\alpha)$, we know immediately

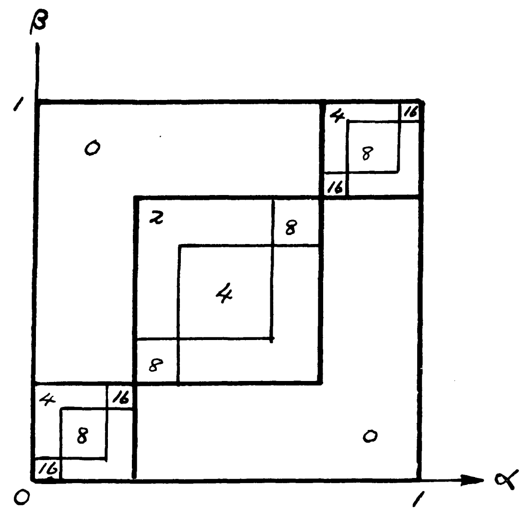
(4.11) $$D_{3^n}(\alpha,\beta)=\prod_{r=1}^{n}[1+\varphi_{r,0}(\alpha)\varphi_{r,0}(\beta)+\varphi_{r,1}(\alpha)\varphi_{r,1}(\beta)].$$

Thus we have

(4.12) $$D_{3^n}(\alpha,\beta)=\begin{cases}\dfrac{1}{|I(n,k)|} & \text{for } \alpha\in I(n,k) \text{ and } \beta\in I(n,k)\\ 0 & \text{elsewhere,}\end{cases}$$

where $|I(n,k)|$ shows the length of the interval $I(n,k)$. (See Fig. 2).

(Fig. 2)



$D_3(\alpha,\beta)$ and $D_{3^2}(\alpha,\beta)$

By (4. 10) and (4. 12) we get

THEOREM 2. (Kaczmarz) $S_{3^n}(\alpha)\to f(\alpha)$ *a. e. as* $n\to\infty$.

PROOF. By (4. 10) and (4. 12), we have

(4.13) $$S_{3^n}(\alpha)=\frac{1}{|I(n,k)|}\int_{I(n,k)}f(\beta)d\beta$$

whenever $\alpha\in I(n,k)$. Since $|I(n,k)|\to 0$ as $n\to\infty$, we have the theorem.

For $f(\alpha)\in L^2(\alpha)$, we define the coefficient c_m by

$$c_m=\int_0^1 f(\alpha)\psi_m(\alpha)d\alpha$$

then by the completeness of $\{\psi_m(\alpha)\}$ we obtain Parseval's equality

$$\int_0^1 |f(\alpha)|^2 d\alpha = \sum_{n=0}^{\infty} c_m^2. \tag{4.14}$$

Here we assume $c_0 = 0$, then

$$\int_0^1 |f(\alpha)|^2 d\alpha = \sum_{m=1}^{\infty} c_m^2 = \sum_{n=0}^{\infty} \sum_{m=3^n}^{3^{n+1}-1} c_m^2 = \sum_{n=0}^{\infty} \int_0^1 \Delta_n^2(\alpha) d\alpha = \int_0^1 \left\{ \sum_{n=0}^{\infty} \Delta_n^2(\alpha) \right\} d\alpha$$

where

$$\Delta_n(\alpha) = \sum_{m=3^n}^{3^{n+1}-1} c_m \psi_m(\alpha). \tag{4.15}$$

We generalize this equality for $f(\alpha) \in L^k(\alpha)$ $(1 < k < \infty)$. This result was originally obtained for the Walsh functions by Paley [5].

THEOREM 3. (Paley). *Let* $f(\alpha)$ *have the* $\{\psi_m(\alpha)\}$*-expansion*

$$f(\alpha) \sim \sum_{m=0}^{\infty} c_m \psi_m(\alpha).$$

Let $\Delta_n(\alpha)$ *denote the partial sum*:

$$\Delta_n(\alpha) = \sum_{m=3^n}^{3^{n+1}-1} c_m \psi_m(\alpha) \qquad (n = 0, 1, 2, \cdots).$$

For simplicity let $c_0 = 0$. *Then, for* $1 < k < \infty$,

$$B_k \int_0^1 \left\{ \sum_{n=0}^{\infty} \Delta_n^2(\alpha) \right\}^{k/2} d\alpha \leqq \int_0^1 |f(\alpha)|^k d\alpha \leqq B_k \int_0^1 \left\{ \sum_{n=0}^{\infty} \Delta_n^2(\alpha) \right\}^{k/2} d\alpha^{*)} \tag{4.16}$$

whenever either member exists.

We divide the proof into several steps.

LEMMA 1. *Let* $n(\alpha)$ *be an integer which varies arbitrarily with* α. *Then*

$$\int_0^1 |S_{3^{n(\alpha)}}(\alpha)|^k d\alpha \leqq B_k \int_0^1 |f(\alpha)|^k d\alpha \qquad (k > 1);$$

$$\int_0^1 |S_{3^{n(\alpha)}}(\alpha)| d\alpha \leqq B \int_0^1 |f(\alpha)| \log^+ |f(\alpha)| d\alpha + B; \tag{4.18}$$

*) We denote B_k with an index k a constant which depends only on k and need not be the same in different contexts.

$$|S_{3^{n(\alpha)}}(\alpha)| \leqq \text{ess. sup. } |f(\alpha)|,$$

provided that the right hand side exists.

PROOF. These are results deduced from Hardy-Littlewood's maximal theorem [8; vol. I, p. 29.] and (4. 13).

LEMMA 2. *Let* $m_1 > \max(m_2, m_3, \cdots, m_q)$, *then*

$$\int_0^1 \Delta_{m_1}(\alpha)\Delta_{m_2}(\alpha)\cdots\Delta_{m_q}(\alpha)\, d\alpha = 0. \tag{4.19}$$

PROOF. We expand each of $\Delta_{m_2}(\alpha)$, $\Delta_{m_3}(\alpha), \cdots, \Delta_{m_q}(\alpha)$ into a series of $\{\psi_m(\alpha)\}$ by (4. 15). By (4. 2) each $\psi_m(\alpha)$ is a finite product of $\varphi_{n,0}(\alpha)$ or $\varphi_{n,1}(\alpha)$, but since

$$m_1 > \max(m_2, m_3, \cdots, m_q)$$

none of terms $\psi_m(\alpha)$ in these series contains $\varphi_{l,0}(\alpha)$ or $\varphi_{l,1}(\alpha)$ $(l \geqq m_1)$. Hence each of $\Delta_{m_2}(\alpha), \cdots, \Delta_{m_q}(\alpha)$ is constant on the interval $I(m_1 - 1, k)$. On the other hand, by(3. 33)

$$\int_{I(m_1-1,k)} \varphi_{m_1,\cdot}(\alpha)\, d\alpha = 0.$$

Thus we have the lemma.

LEMMA 3. *For* $q \geqq 2$,

$$\left(\sum_{n=0}^{\infty} \int_0^1 |\Delta_n(\alpha)|^q d\alpha\right)^{1/q} \leqq B_q \left(\int_0^1 |f(\alpha)|^q d\alpha\right)^{1/q}. \tag{4.20}$$

PROOF. In virtue of Parseval's equality we have (4.20) for $q = 2$. It holds for $q = \infty$ also, because

$$\Delta_n(\alpha) = \int_0^1 f(\beta)[\varphi_{n+1,0}(\alpha)\varphi_{n+1,0}(\beta) + \varphi_{n+1,1}(\alpha)\varphi_{n+1,1}(\beta)]$$

$$\times \prod_{r=1}^{n} [1 + \varphi_{r,0}(\alpha)\varphi_{r,0}(\beta) + \varphi_{r,1}(\alpha)\varphi_{r,1}(\beta)]\, d\beta$$

$$= \int_0^1 f(\beta)\left\{\prod_{r=1}^{n+1} - \prod_{r=1}^{n}\right\} [1 + \varphi_{r,0}(\alpha)\varphi_{r,0}(\beta) + \varphi_{r,1}(\alpha)\varphi_{r,1}(\beta)]\, d\beta$$

and (4. 12) shows

$$\sup|\Delta_n(\alpha)| \leqq \text{ess. sup } 2|f(\alpha)|. \tag{4.21}$$

To prove (4. 20) for general $q \geqq 2$, we have only to use the well known convexity theorem of M. Riesz [8, vol. II. p. 95].

LEMMA 4. *The conclusion of Theorem* 3 *is true when* $k=2\nu$ *is an even intger.*

PROOF. Put

$$\delta_n(\alpha)=\sum_{m=0}^{n-1}\Delta_m(\alpha).$$

Then we have

$$\int_0^1\{\delta_{n+1}^k(\alpha)-\delta_n^k(\alpha)\}d\alpha$$
$$=\int_0^1\left\{k\Delta_n(\alpha)\delta_n^{k-1}(\alpha)+\frac{k(k-1)}{2}\Delta_n^2(\alpha)\delta_n^{k-2}(\alpha)+\cdots+\Delta_n^k(\alpha)\right\}d\alpha$$
$$=\int_0^1\left\{\frac{k(k-1)}{2}\Delta_n^2(\alpha)\delta_n^{k-2}(\alpha)+\cdots+\Delta_n^k(\alpha)\right\}d\alpha$$

by Lemma 2. By Hölder's inequality and an inequality

$$x^ty^{1-t}\leqq x+y \qquad (x\geqq 0,\ y\geqq 0,\ 1>t>0),$$

we have, for $3\leqq\mu\leqq k-1$,

$$\left|\int_0^1\Delta_n^\mu(\alpha)\delta_n^{k-\mu}(\alpha)d\alpha\right|\leqq\left[\int_0^1\Delta_n^2(\alpha)\delta_n^{k-2}(\alpha)d\alpha\right]^{(k-\mu)/(k-2)}$$
$$\cdot\left[\int_0^1\Delta_n^k(\alpha)\,d\alpha\right]^{(\mu-2)/(k-2)}$$
$$\leqq\int_0^1\Delta_n^2(\alpha)\delta_n^{k-2}(\alpha)\,d\alpha+\int_0^1\Delta_n^k(\alpha)\,d\alpha.$$

Hence

$$\left|\int_0^1\{\delta_{n+1}^k(\alpha)-\delta_n^k(\alpha)\}\,d\alpha\right|\leqq B_k\left[\int_0^1\Delta_n^2(\alpha)\delta_n^{k-2}(\alpha)\,d\alpha+\int_0^1\Delta_n^k(\alpha)\,d\alpha\right].$$

Summing up from $n=0$ to $n=N$, by the first inequality of Lemma 1 and Hölder's inequality we have

$$\int_0^1\delta_{N+1}^k(\alpha)\,d\alpha\leqq B_k\int_0^1\left[\sum_{n=0}^N\Delta_n^2(\alpha)\right]\max_{0\leqq n\leqq N}[\delta_n^{k-2}(\alpha)]\,d\alpha+B_k\int_0^1\sum_{n=0}^N\Delta_n^k(\alpha)\,d\alpha$$
$$\leqq B_k\left[\int_0^1\left\{\sum_{n=0}^N\Delta_n^2(\alpha)\right\}^{k/2}d\alpha\right]^{2/k}\left[\int_0^1\max\{\delta_n^k(\alpha)\}\,d\alpha\right]^{(k-2)/k}$$
$$+B_k\int_0^1\sum_{n=0}^N\Delta_n^k(\alpha)d\alpha$$

$$\leqq B_k \left[\int_0^1 \left\{\sum_{k=0}^{N} \Delta_n^2(\alpha)\right\}^{k/2} d\alpha\right]^{2/k} \left[\int_0^1 \delta_{N+1}^k(\alpha)\, d\alpha\right]^{(k-2)/k} + B_k \int_0^1 \left\{\sum_{n=0}^{N} \Delta_n^2(\alpha)\right\}^{k/2} d\alpha.$$

Therefore we may deduce that

$$\int_0^1 \delta_{N+1}^k(\alpha) d\alpha \leqq B_k \int_0^1 \left\{\sum_{n=0}^{\infty} \Delta_n^2(\alpha)\right\}^{k/2} d\alpha.$$

The same argument will show that, if $N < M$,

$$\int_0^1 \left\{\delta_M(\alpha) - \delta_N(\alpha)\right\}^k d\alpha \leqq B_k \int_0^1 \left\{\sum_{N+1}^{M} \Delta_n^2(\alpha)\right\}^{k/2} d\alpha,$$

and thus $\delta_N(\alpha)$ tends strongly to $f(\alpha)$ as $N \to \infty$, because the limit has the same coefficients $\{c_n\}$ defined by (4. 8) with $f(\alpha)$.
Thus we have

$$\int_0^1 f^k(\alpha) d\alpha \leqq B_k \int_0^1 \left\{\sum_{n=0}^{\infty} \Delta_n^2(\alpha)\right\}^{k/2} d\alpha.$$

The proof of the opposite inequality is quite the same as that of Paley's. We shall show it for the sake of completeness. Consider the integral

$$\text{(4.22)} \qquad \int_0^1 \Delta_{n_1}^2(\alpha)\Delta_{n_2}^2(\alpha)\cdots\Delta_{n_{\nu-1}}^2(\alpha)\delta_N^2(\alpha)d\alpha$$

where $\nu = k/2$ and $N-1 \geqq n_1 > n_2 > \cdots > n_{\nu-1}$.

$$\delta_N^2(\alpha) = \left[\sum_{m=0}^{N-1} \Delta_m(\alpha)\right]^2 = \left[\ \delta_n(\alpha) + \sum_{m=n}^{N-1} \Delta_m(\alpha)\right]^2$$

$$= \delta_n^2(\alpha) + 2\sum_{m=n}^{N-1} \Delta_m(\alpha)\delta_n(\alpha) + \sum_{m=n}^{N-1} \Delta_m^2(\alpha) + 2\sum\sum_{m \neq h} \Delta_m(\alpha)\Delta_h(\alpha).$$

Hence by Lemma 2,

$$0 \leqq (4.\,22) = \int_0^1 \Delta_{n_1}^2(\alpha)\Delta_{n_2}^2(\alpha)\cdots\Delta_{n_{\nu-1}}^2(\alpha)\delta_n^2(\alpha)d\alpha$$

$$+\sum_{m=n}^{N-1}\int_0^1 \Delta_{n_1}^2(\alpha)\Delta_{n_2}^2(\alpha)\cdots\Delta_{n_{\nu-1}}^2(\alpha)\Delta_m^2(\alpha)\,d\alpha.$$

It follows that

$$\sum_{m=n}^{N-1}\int_0^1 \Delta_{n_1}^2(\alpha)\Delta_{n_2}^2(\alpha)\cdots\Delta_{n_{\nu-1}}^2(\alpha)\Delta_m^2(\alpha)\,d\alpha$$
$$\leqq \int_0^1 \Delta_{n_1}^2(\alpha)\Delta_{r_2}^2(\alpha)\cdots\Delta_{n_{\nu-1}}^2(\alpha)\delta_N^2(\alpha)\,d\alpha.$$

Summing over all possible combinations of the numbers $n_1, n_2, \cdots, n_{\nu-1}$ for which $\max(n_1, n_2, \cdots, n_{\nu-1}) \leqq N-1$, we get, the sum being taken over the combinations $(n_1, n_2, \cdots, n_{\nu-1}, m)$ for which $\max(n_1, n_2, \cdots, n_{\nu-1}, m) \leqq N-1$,

$$\sum_{n_1}\cdots\sum_{n\nu-1}\sum_{m}\int_0^1 \Delta_{n_1}^2(\alpha)\cdots\Delta_{n_{\nu-1}}^2(\alpha)\Delta_m^2(\alpha)\,d\alpha$$

(4.23) $$\leqq \sum\cdots\sum\sum\int_0^1 \Delta_{n_1}^2(\alpha)\cdots\Delta_{n\nu-1}^2(\alpha)\Delta_m^2(\alpha)\,d\alpha$$

$$\leqq \int_0^1 \delta_N^2(\alpha)\left[\sum_{j=0}^{N-1}\Delta_j^2(\alpha)\right]^{(k-2)/2} d\alpha.$$

By Lemma 3,

$$\sum_{n=0}^{N-1}\left\{\int_0^1 \Delta_n^k(\alpha)\,d\alpha\right\} \leqq B_k\int_0^1 \delta_N^k(\alpha)\,d\alpha.$$

Put S the summation on the left-hand side of (4. 23), then a little consideration shows that

$$\int_0^1\left[\sum_{n=0}^{N-1}\Delta_n^2(\alpha)\right]^{k/2} d\alpha \leqq B_k\left[S+\sum_{n=0}^{N-1}\int_0^1 \Delta_n^k(\alpha)\,d\alpha\right]$$
$$\leqq B_k\left[\int_0^1 \delta_N^2(\alpha)\left\{\sum_{n=0}^{N-1}\Delta_n^2(\alpha)\right\}^{(k-2)/2} d\alpha + \int_0^1 \delta_N^k(\alpha)d\alpha\right]$$
$$\leqq B_k\left\{\left[\int_0^1 \delta_N^k(\alpha)\,d\alpha\right]^{2/k}\left[\int_0^1\left\{\sum_{n=0}^{N-1}\Delta_n^2(\alpha)\right\}^{k/2} d\alpha\right]^{(k-2)/k} + \int_0^1 \delta_N^k(\alpha)\,d\alpha\right\},$$

and thus

$$\int_0^1\left[\sum_{n=0}^{N-1}\Delta_n^2(\alpha)\right]^{k/2} d\alpha \leqq B_k\int^1 \delta_N^k(\alpha)\,d\alpha \leqq B_k\int_0^1 f^k(\alpha)\,d\alpha,$$

from which the desired result follows.

PROOF OF THE THEOREM. The assertion (4. 16) is equivalent to following statement which is convenient for interpolation; let $\varepsilon_0, \varepsilon_1, \cdots$ be a set of arbitrary unit factors. Let

$$\Delta^*(\alpha) = \sum_{n=0}^{\infty} \varepsilon_n \Delta_n(\alpha). \tag{4.24}$$

Then

$$B_k \int_0^1 |\Delta^*(\alpha)|^k d\alpha \leqq \int_0^1 |f(\alpha)|^k d\alpha \leqq B_k \int_0^1 |\Delta^*(\alpha)|^k d\alpha \qquad (1 < k < \infty) \tag{4.25}$$

whenever either side exists.

That (4. 25) follows from (4. 16) is immediate because of $\varepsilon_n^2 = 1$ for $n = 0, 1, 2, \cdots$. For the opposite result is a consequence of Kintchine's inequality [3; p. 131].

Now let

$$\delta_N^*(\alpha) = \sum_{n=0}^{N-1} \varepsilon_n \Delta_n(\alpha).$$

Then, if k is an even integer, we have

$$\left[\int_0^1 |\delta_N^*(\alpha)|^k d\alpha\right]^{1/k} \leqq B_k \left[\int_0^1 |\delta_N(\alpha)|^k d\alpha\right]^{1/k} \leqq B_k \left[\int_0^1 |f(\alpha)|^k d\alpha\right]^{1/k}. \tag{4.26}$$

We may use the Riesz's convexity theorem to interpolate between two consequtive even integer. Hence we have (4. 26) for all $k \geqq 2$. Thus $\delta_N^*(\alpha)$ tends strongly, with index k, to a limit function $\Delta^*(\alpha)$, whose $\{\psi_m(\alpha)\}$-Fourier series is obtained by expanding $\sum \varepsilon_n \Delta_n(\alpha)$, and

$$\int_0^1 |\Delta^*(\alpha)|^k d\alpha \leqq B_k \int_0^1 |f(\alpha)|^k d\alpha.$$

Since $f(\alpha)$ is obtained from $\Delta^*(\alpha)$ in the same way as $\Delta^*(\alpha)$ is obtained from $f(\alpha)$, we have also

$$\int_0^1 |f(\alpha)|^k d\alpha \leqq B_k \int_0^1 |\Delta^*(\alpha)|^k d\alpha.$$

Thus we proved the Theorem for $k \geqq 2$.

The case $1 < p \leqq 2$ is reduced to the case $p \geqq 2$ by the conjugacy argument. Let k' be the conjugate exponent of k, i.e.

$$\frac{1}{k} + \frac{1}{k'} = 1.$$

Then for $f(\alpha) \in L^k(\alpha)$ and $g(\alpha) \in L^{k'}(\alpha)$, by Hölder's inequality and (4. 26)

$$\int_0^1 \Delta_N^*(\alpha) g(\alpha) d\alpha = \int_0^1 \Delta_N^{**}(\alpha) f(\alpha) d\alpha$$

$$\leqq \left[\int_0^1 |\Delta_N^{**}(\alpha)|^{k'} d\alpha\right]^{1/k'} \cdot \left[\int_0^1 |f(\alpha)|^k d\alpha\right]^{1/k} \tag{4.27}$$

$$\leqq B_k \left[\int_0^1 |g(\alpha)|^{k'} d\alpha\right]^{1/k'} \cdot \left[\int_0^1 |f(\alpha)|^k d\alpha\right]^{1/k}$$

where $\Delta^{**}(\alpha)$ is a function formed from $g(\alpha)$ in the same way as $\Delta^*(\alpha)$ is obtained from $f(\alpha)$. (4. 27) is satisfied for every $g \in L^{k'}$. Thus we have

$$\int_0^1 |\Delta_N^*(\alpha)|^k d\alpha \leqq B_k \int_0^1 |f(\alpha)|^k d\alpha$$

and then arguing as before we have

$$\int_0^1 |\Delta^*(\alpha)|^k d\alpha \leqq B_k \int_0^1 |f(\alpha)|^k d\alpha \qquad (1 < k \leqq 2).$$

5. A modification. In the preceding sections our discussions are founded on a time series having the values $\sqrt{2}$, 0 and $-\sqrt{2}$. In this section we shall consider more general time series having three different values.

Let $\chi(t, \alpha)$ be a time series having the values $\sqrt{\frac{1}{2c}}$, 0 and $-\sqrt{\frac{1}{2c}}$ with probabilities c (c is a fixed number such that $0 < c \leqq 1/2$), $1 - 2c$ and c respectively.

Repeating the same argument as in paragraphs 2 and 3 we have

$$\varphi_{1,0}(\alpha) = \begin{cases} \sqrt{\frac{1}{2c}}, & \alpha \in [0, c) \\ 0, & \alpha \in [c, 1 - c) \\ -\sqrt{\frac{1}{2c}}, & \alpha \in [1 - c, 1), \end{cases} \tag{5. 1}$$

$$\varphi_{1,1}(\alpha) = \sqrt{\frac{2c}{1 - 2c}} [\varphi_{1,0}^2(\alpha) - 1] \tag{5. 2}$$

$$= \begin{cases} \sqrt{\dfrac{1-2c}{2c}}, & \alpha \in [0,c) \cup [1-c,1) \\ -\sqrt{\dfrac{2c}{1-2c}}, & \alpha \in [c,1-c) \end{cases}$$

and

$$(5.\ 3) \qquad \varphi_{2,0}(\alpha) = \begin{cases} \varphi_{1,0}\left(\dfrac{1}{c}\alpha\right), & \alpha \in [0,c) \\ \varphi_{1,0}\left(\dfrac{1}{(1-2c)}(\alpha-c)\right), & \alpha \in [c,1-c) \\ \varphi_{1,0}\left(\dfrac{1}{c}(\alpha-1+c)\right), & \alpha \in [1-c,1) \end{cases}$$

and

$$(5.\ 4) \qquad \varphi_{2,1}(\alpha) = \sqrt{\frac{2c}{1-2c}}\,[\varphi_{2,0}^2(\alpha)-1]$$

and so on.

Put

$$\psi_0(\alpha) \equiv 1$$

and

$$\psi_n(\alpha) = \varphi_{n_1+1,m_1}(\alpha)\ \varphi_{n_2+1,m_2}(\alpha)\cdots\varphi_{n_\nu+1,m_\nu}(\alpha)$$

corresponding to the triadic expansion of the positive integer n,

$$n = 3^{n_1}2^{m_1} + 3^{n_2}2^{m_2} + \cdots + 3^{n_\nu}2^{m_\nu}$$

where

$$n_1 > n_2 > \cdots > n_\nu \geqq 0$$
$$m_1, m_2, \cdots, m_\nu = 0 \text{ or } 1.$$

Then the systems $\{\psi_n(\alpha)\}$ are shown to be a complete orthonormal system. The proof is quite similar to the before.

Putting $c=\frac{1}{2}$ we have the Rademacher system $\{\varphi_{n,0}(\alpha)\}$ and Walsh system $\{\psi_n(\alpha)\}$. The system $\{\varphi_{n,1}(\alpha)\}$ degenerates to zero in this case. Our discussions in the preceding sections are just the case $c=\frac{1}{4}$. In this case every function $\psi_n(\alpha)$ have at most three values but in the other cases $\psi_n(\alpha)$ generally takes more than three values. The case taking $c=\frac{1}{3}$ may be especially noteworthy because every interval $I(n,\nu)$ for each ν has the equal length 3^{-n} regularly.

REFERENCES

[1] H. E. CHRESTENSON, A class of generalized Walsh functions, Pacific Journ. of Math., 5(1955), 17-31.
[2] N. J. FINE, On Walsh functions, Trans. Amer. Math. Soc., 65(1949), 372-414.
[3] S. KACZMARZ-H. STEINHAUS, Theorie der Orthogonalreihen, Monografje Matematyczne, Warsaw, (1935).
[4] P. Lévy, Sur une généralisation des fonctions orthogonales de M. Rademacher, Comm. Math. Helv., 16(1944), 146-152.
[5] R. E. A. C. PALEY, A remarkable series of orthogonal functions (I), Proc. London Math. Soc., 34(1932) 241-264.
[6] C. WATARI, On generalized Walsh Fourier series, Tôhoku Math. Journ., 10(1958), 211-241.
[7] N. WIENER, Nonlinear problems in random theory, Technology Press Research Monographs, (1958).
[8] A. ZYGMUND, Trigonometric series, I. II. Cambridge Univ. Press, (1959).

THE FACULTY OF ENGINEERING, IBARAKI UNIVERSITY, HITACHI, JAPAN