OPERATION AND CO-OPERATION IN THE GENERAL CATEGORIES

KISUKE TSUCHIDA

(Received August 2, 1963)

1. Introduction. Let $\mathfrak{T}$ be a category consisting of the based topological spaces and based continuous maps. Eckmann and Hilton [2] defined an operation $\pi(B, F) \times \pi(B, \Omega Y) \to \pi(B, F)$ for a fibre map $P: X \to Y$ with fibre F in $\mathfrak{T}$, where ΩY denotes the loop space in Y and B any space. Dually they defined a co-operation $\rho^*: \pi(F, B) \times \pi(\Sigma Y, B) \to \pi(F, B)$ for a cofibre map $i: Y \to X$ with cofibre F in $\mathfrak{T}$, where ΣY denotes the suspension space of Y.

In this paper we try to define the corresponding one in the framework of the general categories.

Using a non-trivial semi-simplicial standard construction in the sense of Godement [4], Huber [5] developed a semi-simplicial homotpy theory in the framework of general categories.

Huber pointed out that the standard construction induces a fibration $K(Y): CY \to Y$, (See §3 for the definition) and the fibre ΩY of $K(Y)$ has the properties approximately corresponding to those of a loop space, and the dual standard construction induces a cofibration $K(X): X \to CX$ and the cofibre ΣX of $K(X)$ has the formal properties of a suspension. Thus, we start from above consider-ation and try to study our problem.

In §3, by Huber's semi-simplicial homotopy, we shall define Kan homotopy groups in the general category, an operation $\rho_*: \pi_n(X, Y_3) \times \pi_n(X, \Omega Y_1) \to \pi_n(X, Y_3)$ for a fibration sequence $Y_3 \to Y_2 \to Y_1$, and, in §5, dually a cooperation $\rho^*: \pi_n(X_3, Y) \times \pi_n(\Sigma X_i, Y) \to \pi_n(X_3, Y)$ for a cofibration sequence $X_1 \to X_2 \to X_3$.

Since Kan homotopy groups under the consideration are naturally isomorphic to Eckmann-Hilton homotopy groups which are higher by one dimension than the former [5], our operation and co-operation in the special case include the similar operation in [2] only in the group case. It is known that any map $v: X \to Y$ is factorized in the form $X \xrightarrow{u} E_v \xrightarrow{p} Y$, where u is a homotopy equi-valence and p a fibre map with fibre F_v, and F_v may also be interpreted as the fibre space over X induced by the map v.

In §4 we shall introduce a notion of the trace of morphisms corresponding the above F_v and obtain the similar results in [2]. From theorems 4.3 and 4.5 in §4, there exists a fibration sequence $Y_1 \to F_v \to Y_2$ for a morphism $v; Y_2 \to Y_1$ and so, in the category $\mathfrak{T}$, F_v may be interpreted as the fibre space over Y_2 induced by the map $v: Y_2 \to Y_1$. In the framework of general categries, theorems 4.8 and 4.11 play the corresponding roles in theorems 3.11 and 5.15 in [2] respectively. Last section is devoted to the dual discussions.

2. Preliminaries. A category $\mathfrak{K}$ consists of a non-empty class $\mathfrak{K}$ of objects X, Y, $\cdots$ together with sets $\mathrm{Hom}(X, Y)$ of morphisms $f: X \to Y$ $(X, Y \in \mathfrak{K})$, and of an associative composition of morphisms $\circ$: $\mathrm{Hom}(X, Y) \times \mathrm{Hom}(Y, Z) \to \mathrm{Hom}(X, Z)$, $(f \cdot g) \to g \circ f$, which has both-sided identity $1_X \in (X, X)$.

An object $o \in \mathfrak{K}$ is called a zero object if, for all X, the sets $\mathrm{Hom}(X, 0)$ and $\mathrm{Hom}(o, X)$ consists of exactly one element 0. If $\mathfrak{K}$ has a zero object, then in each set $\mathrm{Hom}(X, Y)$ we have a distinguished morphism 0, called the zero morphism. A morphism $f: X \to Y$ is called an equivalence if there is a morphism $g: \mathrm{Y} \to X$ such that $g \circ f = 1$ and $f \circ g = 1$. A morphism f is called an epimorphism if for any Z and any $v_i: Y \to Z$, $i = 1, 2$, the relation $v_1 \circ f = v_2 \circ f$ implies $v_1 = v_2$. A morphism f is called a monomorphism if for any W and any $w_i: W \to X$, $i = 1, 2$, the relation $f \circ w_1 = f \circ w_2$ implies $w_1 = w_2$.

Notice that if f is an equivalence then f is both epimorphism and monomorphism, but the converse is in general false.

Let $\mathfrak{K}'$, $\mathfrak{K}''$, $\mathfrak{M}'$ and $\mathfrak{M}''$ be arbitrary categories. Let $F, G: \mathfrak{K}' \to \mathfrak{K}''$ be covariant functors and $\theta: F \to G$ a functor morphism. Let $U: \mathfrak{K}'' \to \mathfrak{M}''$ and $V: \mathfrak{M}' \to \mathfrak{K}'$ be covariant functors. According to (4), (5), we define $U * \theta * V$: $U \circ F \circ V \to U \circ G \circ V$ by

$$(U * \theta * V)(X) = U(\theta(V(X))) \qquad \text{for } X \in \mathfrak{M}'.$$

If U(or V) is the identity functor I, then we abbreviate $U * \theta * V$ to $\theta * V$ (or $U * \theta$). Then the following formulas are valid for any covariant functors and any functor morphisms (cf. [4], [5]);

(1) $$(U \circ V) * \theta = U * (V * \theta)$$

(2) $$\theta * (U \circ V) = (\theta * U) * V$$

(3) $$(U * \theta) * V = U * \theta * V = U * (\theta * V)$$

(4) $$U * (\theta' \circ \theta'') * V = (U * \theta' * V) \circ (U * \theta'' * V)$$

(5) $$(\psi * G) \circ (U * \varphi) = (V * \varphi) \circ (\psi * G)$$

for any two functor morphisms $\varphi: F \to G$, $\psi: U \to V$.

Rule (5) may be remembered with the following commutative diagram:

$$\begin{array}{ccc} U \circ F & \xrightarrow{U * \varphi} & U \circ G \\ {\scriptstyle \psi * F} \downarrow & & \downarrow {\scriptstyle \psi * G} \\ V \circ F & \xrightarrow[V * \varphi]{} & V \circ G \end{array}$$

Throughout this paper we consider the categories with zero objects and zero morphisms and we assume that the functors preserve zero objects and zero morphisms.

Let $\mathfrak{K}$ be arbitrary category. Let $\{C, k, p\}$ be a standard construction in $\mathfrak{K}$, i.e. $C: \mathfrak{K} \to \mathfrak{K}$ is a covariant functor, $k: C \to I$ and $p: C \to C \circ C$ are functor morphisms and the following two axioms are satisfied. (cf [4], [5])

(s$_1$) $$(k*C)\circ p = (C*k)\circ p = \text{identity},$$

(s$_2$) $$(p*C)\circ p = (C*p)\circ p.$$

Each standard construction $\{C, k, p\}$ generates a semi-simplical functor

$$F_* = (F_n, d_n^i, s_n^i), \qquad n \geqq 0$$

i.e. a sequence of functors $F_n: \mathfrak{K} \to \mathfrak{K}$ together with faces and degeneracy morphisms

$$d_n^i: F_n \to F_{n-1},$$
$$s_n^i: F_n \to F_{n+1}, \qquad 0 \leqq i \leqq n$$

F_* is dfined as follows: Let $C^0 = I$ and $C^{n+1} = C \circ C^n$. Then we put

$$F_n = C^{n+1}, \qquad d_n^i = C^i*k*C^{n-i}, \qquad s_n^i = C^i*p*C^{n-i}.$$

The faces and degeneracy morphisms satisfy the usual semi-simplicial commutation rules;

(a) $$d^i d^j = d^{j-1} d^i \qquad i < j$$

(b) $$s^i s^j = s^{j+1} s^i \qquad i \leqq j$$

(c) $$d^i s^j = s^{j-1} d^i \qquad i < j$$

(d) $$d^i s^i = d^{i+1} s^i = \text{identiy}$$

(e) $$d^i s^j = s^j d^{i-1} \qquad i > j + 1.$$

If we apply the semi-simplicial functor F_* to an object $Y \in \mathfrak{K}$, we obtain a semi-simplicial object $F_*(Y) = (F_n(Y), d^i(Y), s^i(Y))$.

Let $\mathfrak{S}$ be the category of sets. If we now take the functor $\mathrm{Hom}(X, \ )$: $\mathfrak{K} \to \mathfrak{S}$ with a fixed first argument X, we obtain a semi-simplicial complex

$$K_*(X, Y) = \mathrm{Hom}(X, F_*(Y)) = (\mathrm{Hom}(X, F_n(Y))\ d^i, s^i).$$

Then we have the Kan homotopy groups (cf [6]) of $K_*(X, Y)$ and we define $\pi_n(X, Y) = \pi_n(K_*(X, Y))$ $(n \geqq 0)$. Here it is assumed that all the $K_*(X, Y)$ satisfy the Kan condition.

Now we define the kernel and the cokernel of a morphism. Let $u: X \to Y$ be a morphism.

A pair (U, j) consisting of an object U and of a monomorphism $j: U \to X$ is called a kernel of u, if

(1) $$u \circ j = 0$$

(2) for each object Z and each morphism: $v: Z \to X$ with $u \circ v = 0$, there exists

one and only one morphism $w: Z \rightarrow U$, so that $j \circ w = v$.

The definition of a cokernel is dually defined.

Throughout this paper, we shall be concerned with a category in which the kernel and the cokernel of morphisms always exist.

Let $K_*(X, Y)$ be the semi-simplicial complex induced by a standard construction $\{C, k, p\}$ in $\mathfrak{K}$.

DEFINITION 2.1 ([5]). A morphism $u: Y_2 \rightarrow Y_1$ is called a fibration if the induced semi-simplicial map

$$u_*: K_*(X, Y_2) \rightarrow K_*(X, Y_1)$$

is a semi-simplicial fibration (cf [6]) for all $X \in \mathfrak{K}$.

If C commutes with kernels (i.e. $C(\text{Ker } u) = \text{Ker } C(u)$), then the kernel Y_3 of u will be called a fibre of u. Then the semi-simplicial fibre of u_* may be identified with $K_*(X, Y_3)$.

PROPOSITION 2.2 (Huber [5]). *Let $\mathfrak{K}$ be a category and let $\{C, k, \rho\}$ be a standard construction in $\mathfrak{K}$, such that the complexes $K_*(X, Y)$ are Kan complexes. Then*

1) *the morphism $k(Y): CY \rightarrow Y$ is a fibration,*

2) $\pi_n(X, CY) = 0 \qquad n \geqq 0.$

If we denote the fibre of $k(Y)$ by ΩY, then we get the followin.

PROPOSITION 2.3 (Huber [5]).

$\partial_{n+1}(k(Y)): \pi_{n+1}(X, Y) \rightarrow \pi_n(X, \Omega Y)$ *is isomorphic onto for* $n > 0$.

3. Operations in Kan homotopy groups. Let $\mathfrak{K}$ be a category and let $K_*(X, Y)$ be the semi-simplicial complex induced by a standard construction $\{C, k, p\}$ in $\mathfrak{K}$. We suppose that all the $K_*(X, Y)$ satisfy Kan condition. Let $u: Y_2 \rightarrow Y_1$ be a fibration and (Y_3, v) its fibre. By proposition 2.2, $k(Y_1): CY_1 \rightarrow Y_1$ is a fibration. We take $(a, b) \in \pi_n(X, Y_3) \times \pi_n(X, \Omega Y_1)$. Let $\varphi: X \rightarrow C^{n+1}Y_3$ be a representative of the homotopy class $a \in \pi_n(X, Y_3)$. By proposition 2.3, there exists only one $c \in \pi_{n+1}(X, Y_1)$ such that $\partial_{n+1}(k(Y_1))c = b$. Let $\tau: X \rightarrow C^{n+2}Y_1$ be a representative of c.

Since $u_*: K_*(X, Y_2) \rightarrow K_*(X, Y_1)$ is a semi-simplicial fibration, there exists a $(n+1)$-simplex $\xi \in K_*(X, Y_2)$ such that $C^{n+2}(u)\xi = \tau$ and $d^i\xi = 0$ for $i \neq n+1$. From the definition of the boundary homomorphism $\partial_{n+1}(u): \pi_{n+1}(X, Y_1) \rightarrow \pi_n(Y, Y_3)$ for the fibration $u: Y_2 \rightarrow Y_1$, $\partial_{n+1}(u)c$ is represented by a morphism $\chi: X \rightarrow C^{n+1}Y_3$ such that $C^{n+1}(v)\chi = d^{n+1}\xi$.

Now, if we take $n+1$ n-simplexes $\sigma_i = 0$ $(0 \leqq i \leqq n-2)$, $\sigma_{n-1} = \varphi$ and $\sigma_{n+1} = \chi$ in $K_*(X, Y_3)$, then we have $d^{j-1}\sigma_i = d^i\sigma_j$ for $i < j$ and $i, j \neq n$. Since

$K_*(X, Y_3)$ is a Kan complex, then there exists a $(n+1)$-simplex $\eta \in K_*(X, Y_3)$ such that $d^{n-1}\eta = \varphi$, $d^{n+1}\ \eta = \chi$ and $d^i\eta = 0$ $(0 \leqq i \leqq n-2)$. From the definition of the product in Kan homotopy group $\pi_n(X, Y_3)$ (cf [6]), the homotopy class of n-simplex $d^n\eta \in K_*(X, Y_3)$ satisfy the relation $\{d^n\eta\} = \{\varphi\}\{\chi\}$ where $\{\ \ \}$ denotes the homotopy class.

We now define an operation $\rho_*: \pi_n(X, Y_3) \times \pi_n(X, \Omega Y_1) \to \pi_n(X, Y_3)$ as

$$\rho_*(a, b) = \{d^n\eta\}, \qquad a \in \pi_n(X, Y_3),\ b \in \pi_n(X, \Omega Y_1).$$

Also the definition of ρ_* may be written in the following abbreviated form:

$$\rho_*(a, b) = a \cdot \partial_{n+1}(u)c,$$

where $\partial_{n+1}(k(Y_1)c = b$ and $\cdot$ denotes the product in Kan homotopy group $\pi_n(X, Y_3)$.

THEOREM 3.1.

(1) $$\rho_*(a, 0) = a \qquad (a \in \pi_n(X, Y_3))$$

(2) $$\rho_*(a, b_1 \cdot b_2) = \rho_*(\rho_*(a, b_1), b_2) \qquad (b_i \in \pi_n(X, \Omega Y_1)\ i = 1, 2).$$

PROOF. Since (1) is evident we only prove (2). We have

$$\rho_*(a, b_1 \cdot b_2) = a \cdot \partial(c_1 \cdot c_2) = a \cdot (\partial c_1 \cdot \partial c_2)$$

$$\rho_*(\rho_*(a, b_1), b_2) = \rho_*(a \cdot \partial c_1, b_2) = (a \cdot \partial c_1) \cdot \partial c_2.$$

Hence (2) follows from the associativity in Kan homotopy group.

4. The trace of morphism. In this section we shall consider a category $\mathfrak{K}$ with direct products. Here we recall the defintion of a direct product in [3].

A direct product of the objects A_1, A_2 is an object $A_1 \times A_2$ and a system of morphisms $p_i: A_1 \times A_2 \to A_i$, $i = 1, 2$, with the property:

(D). For any object X of $\mathfrak{K}$ and any system of morphisms $f_i: X \to A_i$, $i = 1, 2$, there exists a unique morphism $\mathrm{f}: X \to A_1 \times A_2$ with $p_i \cdot f = f_i$.

The morphisms p_i are called the projection of $A_1 \times A_2$, and morphisms f_i are called the component of f; we write $f = \{f_1, f_2\}$, so that

$$p_i\{f_1, f_2\} = f_i.$$

Let $(B \times B,\ p_1',\ p_2')$ be a direct product of the object B's and let morphisms $f_i: A_i \to B$ be given, $i = 1, 2$. The morphism

$$\{f_1p_1, f_2p_2\}: A_1 \times A_2 \to B \times B$$

will be written $f_1 \times f_2$. Then also we have

$$p_1'(f_1 \times f_2) = f_1p_1 \text{ and } p_2'(f_1 \times f_2) = f_2p_2.$$

DEFINITION 4.1. The trace of $f_1 \times f_2: A_1 \times A_2 \to B \times B$ is a pair (Q, ι) consisting of an object Q and monomorphism $\iota: Q \to A_1 \times A_2$ satisfying the conditions:

(i) $$p_1'(f_1 \times f_2)\iota = p_2'(f_1 \times f_2)\iota \qquad \text{(or equivalenty } f_1p_1\iota = f_2p_2\iota)$$

(ii) if D is any object of $\mathfrak{K}$ and $h: D \to A_1 \times A_2$ is a morphism with the property $p_1'(f_1 \times f_2)h = p_2'(f_1 \times f_2)h$ (or equivalently $f_1 p_1 h = f_2 p_2 h$), then h admits a unique factorization $D \to Q \xrightarrow{\iota} A_1 \times A_2$.

For example, if $\mathfrak{K}$ is a category of sets with base points, then the trace of $f_1 \times f_2$ is $Q = \{(a_1, a_2) \in A_1 \times A_2;\ f_1 a_1 = f_2 a_2\}$.

Now we return to the notation in §3. Let $\{C, k, p\}$ be a standard construction in $\mathfrak{K}$ and we assume that functor C is D-functor in the sense of [3; §3], i. e.

$$C(A_1 \times A_2) = C(A_1) \times C(A_2) \text{ and } C(p_i): C(A_1 \times A_2) \to C(A_i),\ i = 1, 2,$$

are projections.

Moreover we assume that if $f: A \to B$ is a monomorphism, then $C(f)$: $C(A) \to C(B)$ is so.

Let $k(Y): CY_1 \to Y_1$ be a morphism in §2 and $v: Y_2 \to Y_1$ any morphism. Then we denote the trace of $k(Y_1) \times v$ by (F_v, j). Let $q_1: CY_1 \times Y_2 \to CY_1$ and $q_2: CY_1 \times Y_2 \to Y_2$ be projections. Then we have the following commutative diagram:

(4. 2)
$$\begin{array}{ccc} Fv & \xrightarrow{q_1 j} & CY_1 \\ {\scriptstyle q_2 j}\downarrow & & \downarrow{\scriptstyle k(Y_1)} \\ Y_2 & \xrightarrow[v]{} & Y_1 \end{array}$$

THEOREM 4. 3. *For any morphism* $v: Y_2 \to Y_1$, $q_2 j: Fv \to Y_2$ *is a fibration.*

PROOF. Let $n(n-1)$-simplexes $\sigma_0, \cdots, \sigma_{k-1}, \sigma_{k+1}, \cdots, \sigma_n \in K_*(X, F_v)$ satisfy $d^{j-1}\sigma_i = d^i\sigma_j$ for $i < j$ and $i, j \neq k$ and let an n-simplex $\tau \in K_*(X, Y)$ satisfy $d^i\tau = C^n(q_2 j)\sigma_i$ for $i \neq k$. If we consider $C^n(q_1 j)\sigma_i$' s $(i \neq k)$ and $C^{n+1}(v)\tau$, then we have

$$d^n(CY_1)C^n(q_1 j)\sigma_j = d^{j-1}(CY_1)C^n(q_1 j)\sigma_i \qquad \text{for } i < j \text{ and } i, j \neq k$$

and
$$C^n(k(Y_1)C^n(q_1 j) = d^i(Y_1)C^{n+1}(v)\tau.$$

By Proposition 2. 2, $k(Y_1): CY_1 \to Y_1$ is a fibration and hence there exists a n-simplex $\sigma \in K_*(X, CY_1)$ such that $d^i(CY_1)\sigma = C^n(q_1 j)\sigma_i$ for $i \neq k$ and $C^{n+1}(k(Y_1))\sigma = C^{n+1}(v)\tau$. Since $C^{n+1}(CY_1 \times Y_2) = C^{n+1}(CY_1) \times C^{n+1}(Y_2)$, there exists a morphism $\xi: X \to C^{n+1}(CY_1 \times Y_2)$ such that $C^{n+1}(q_1)\xi = \sigma$ and $C^{n+1}(q_2)\xi = \tau$. But $C^{n+1}(k(Y_1))C^{n+1}(q_1)\xi = C^{n+1}(v)\ C^{n+1}(q_2)\xi$. Hence, from the property (ii) in Definition 5. 1, ξ may be factorized in the form:

$$X \xrightarrow{\eta} C^{n+1}(Fv) \xrightarrow{C^{n+1}(j)} C^{n+1}(CY_1 \times Y_2).$$

Now, consider an n-simplex $\eta \in (X, Fv)$, we have

$$C^n(q_1)C^n(j)d^i\eta = C^n(q_1)C^n(j)\sigma_i$$

$$C^n(q_2)C^n(j)d^i\eta = C^n(q_2)C^n(j)\sigma_i \qquad \text{for } i \neq k.$$

Since $C^n(q_1)(C^n(q_2))$ is the projection $C^n(Y_1 \times Y_2) \rightarrow C^n(CY_1)(C^n(Y_2))$, by the property of the direct product, we have

$$C^n(j)d^i\eta = C^n(j)\sigma_i \qquad i \neq k.$$

As j is a monomorphism, $C^n(j)$ is so and hence $d^i\eta = \sigma_i$ for $i \neq k$. Also evidently we have $C^{n+1}(q_2 j)\eta = \tau$. Q. E. D.

THEOREM 4.4. *If* $v : Y_2 \rightarrow Y_1$ *is a fibration,* $q_1 j : Fv \rightarrow CY_1$ *is so.*

The proof of Theorem 4.4 is the same as that of Theorem 4.3 and we omit the proof.

In the following we suppose that the kernel of every fibration commutes with C, i. e. every fibration has a fibre in the sense of the definition of §2.

THEOREM 4.5. *The fibre of the fibration* $q_2 j : F_v \rightarrow Y_2$ *is equivalent with* ΩY_1, *where* ΩY_1 *is the fibre of the fibration* $k(Y_1) : CY_1 \rightarrow Y_1$.

PROOF. We denote the fibre of the fibration $q_2 j$ by (K, k). Consider the following diagram:

$$\begin{array}{ccc} K & & \Omega Y_1 \\ k \downarrow & & \downarrow i \\ F_v & \xrightarrow{\quad q_1 j \quad} & CY_1 \\ q_2 j \downarrow & & \downarrow k(Y_1) \\ Y_2 & \xrightarrow{\quad v \quad} & Y_1 \end{array}$$

Since $k(Y_1)q_1 jk = vq_2 jk = 0$, there exists a morphism $m : K \rightarrow \Omega Y_1$ such that $im = q_1 jk$. Let $\iota_1 : CY_1 \rightarrow CY_1 \times Y_2$ be a mophism satisfying $q_1\iota_1 =$ identity and $q_2\iota_1 = 0$. (Such a morphism surely exists from the property of the direct product.)

Consider a morphism $\iota_1 \circ i : \Omega Y_1 \rightarrow CY_1 \times Y_2$, then we have

$$p_1(k(Y_1) \times v)\iota_1 i = p_2(k(Y_1) \times v)\iota_1 i.$$

Hence, by the property of trace F_v, $\iota_1 i$ is factorized in the form:

$$\Omega Y_1 \xrightarrow{w} F_v \xrightarrow{j} CY_1 \times Y_2.$$

$q_2 jw = q_2\iota_1 i = 0$ implies the existence of a morphism $n : \Omega Y_1 \rightarrow K$ such that $kn = w$. Then $imn = q_1 jkn = q_1 jw = q_1\iota_1 i = i$ and $mn =$ identity, for i is a monomophism. Also we have

$$q_1 jknm = q_1\iota_1 im = q_1 jk$$

and

$$q_2jknm = 0 = q_2jk.$$

Hence, by the property of the direct product, we have $jknm = jk$, in which both j and k are monomorphisms. Hence $nm =$ identity. Thus K is equivalent with ΩY_1. Q. E. D.

Henceforth we shall identity K with ΩY_1 and we have the following fibration sequence:

$$\Omega Y_1 \xrightarrow{k} F_v \xrightarrow{q_2j} Y_2.$$

Next let $v: Y_2 \to Y_1$ be a fibration with fibre (Y_3, u). Then the following theorem may be proved analogously as in Theorem 4. 2.

THEOREM 4.6. *If a sequence $Y_3 \to Y_2 \to Y_1$ is a fibration sequence, then a sequence $Y_3 \to F_v \to CY_1$ is so.*

In virtue of Theorems 4. 5 and 4.6, if $v: Y_2 \to Y_1$ is a fibration with fibre (Y_3, u), then we have the following commutative diagram:

$$\begin{array}{ccccc} & & \Omega Y_1 & \xrightarrow{\text{id}} & \Omega Y_1 \\ & & k \downarrow & & \downarrow i \\ Y_3 & \xrightarrow{l} & F_v & \xrightarrow{q_1j} & CY_1 \\ \text{id} \downarrow & & q_2j \downarrow & & \downarrow k(Y) \\ Y_3 & \xrightarrow{u} & Y_2 & \xrightarrow{v} & Y_1 \end{array} \tag{4. 7}$$

THEOREM 4. 8. *If $v: Y_2 \to Y_1$ is a fibration with fibre (Y_3, u). then*

$$l_* : \pi_n(X, Y_3) \to \pi_n(X, F_v)$$

is an isomophism for $n \geqq 0$.

PROOF. The result follows from the Kan homotopy exact sequence of the fibration $q_1j: F_v \to CY_1$ and proposition 2. 2.

Note that Theorem 4.8 is also obtained from the next theorem 4. 9 and the five lemma.

THEOREM 4. 9 *If $v: Y_2 \to Y_1$ is a fibration with fibre (Y_3, u), then the following diagram is commutative;*

$$\begin{array}{ccccccccc}
\longrightarrow \pi_{n+1}(X, Y_1) & \xrightarrow{\partial(v)} & \pi_n(X, Y_3) & \xrightarrow{u_*} & \pi_n(X, Y_2) & \xrightarrow{v_*} & \pi_n(X, Y_1) \longrightarrow \\
\downarrow \partial(k(Y_1)) & \boxed{1} & \downarrow l_* & \boxed{2} & \downarrow \mathrm{id} & \boxed{3} & \downarrow \partial(k(Y_1)) \\
\longrightarrow \pi_n(X, \Omega Y_1) & \xrightarrow[k_*]{} & \pi_n(X, F_v) & \xrightarrow[(q_2 j)_*]{} & \pi_n(X, Y) & \xrightarrow[\partial(q_2 j)]{} & \pi_n(X, \Omega Y_1) \longrightarrow .
\end{array}$$

Here the upper sequence is Kan homotopy exact sequence of the fibration $v: Y_2 \to Y_1$ (cf [5] [6]) *and the lower is that of the fibration* $q_2 j: F_v \to Y_2$.

LEMMA. *Let* $Y_3 \xrightarrow{u} Y_2 \xrightarrow{v} Y_1$ *be a fibration sequence, then the boundary homomorphism* $\partial(v): \pi_{n+1}(X, Y_1) \to \pi_n(X, Y_3)$ *may be defined as follows: let* $a \in \pi_{n+1}(X, Y_1)$ *be represented by a morphism* $\tau: X \to C^{n+2}(Y_1)$ *such that* $d^i\tau = 0$ $(0 \leqq i \leqq n+1)$, *then* $\partial(v)a$ *is represented by a morphism* $\alpha: X \to C^{n+1}(Y_3)$ *determined by the relation*: $C^{n+2}(v)\xi = \tau$, $d\xi = 0$ *for* $i \neq n$ *and* $d^n\xi = C^{n+1}(u)\alpha$, *where* $\xi: X \to C^{n+2}(Y_2)$ *exists since* $v: Y_2 \to Y_1$ *is a fibration,*

PROOF. According to the usual definition of the boundary homomorphism, $\partial(v)a$ is represented by a morphisms $\beta: X \to C^{n+1}(Y_3)$ (cf [6]) determined by the relation; $C^{n+2}(v)\eta = \tau$, $d^i\eta = 0$ for $i \neq n+1$ and $d^{n+1}\eta = C^{n+1}(u)\beta$, where $\eta: X \to C^{n+2}(Y_2)$ exists since $v: Y_2 \to Y_1$ is a fibration.

In order to prove the lemma we have only to prove the existence of a morphism $\omega: X \to C^{n+2}(Y_2)$ satisfying $C^{n+2}(v)\omega = 0$, $d^n\omega = d^{n+1}\eta$, $d^{n+1}\omega = d^n\xi$ and $d^i\omega = 0$ $i < n$. Take $n+2$ $n+1$-simplexes in $K_*(X, Y)$ $\sigma_i = 0$, $0 \leqq i \leqq n-1$, $\sigma_n = s^{n+1}d^{n+1}\eta$, $\sigma_{n+1} = s^{n+1}d^n\xi$, then $d^i\sigma_j = d^{j-1}\sigma_i$ for $i < j$, $i, j \neq n+2$ and $C^{n+2}(v)\sigma_i = 0$ for $i < n+2$. Since $v: Y_2 \to Y_1$ is a fibration, then there exists $\zeta: X \to C^{n+3}(Y_2)$ such that $C^{n+3}(v)\zeta = 0$ $d^n\zeta = s^{n+1}d^{n+1}\eta$ $d^{n+1}\zeta = s^{n+1}d^n\xi$ and $d^i\zeta = 0$ for $i < n$. If we consider $d^{n+1}\zeta: X \to C^{n+2}(Y_2)$, we have

$$\begin{aligned}
&C^{n+2}(v)d^{n+2}\zeta = 0,\\
&d^n d^{n+2}\zeta = d^{n+1}d^n\zeta = d^{n+1}s^{n+1}d^{n+1}\eta = d^{n+1}\eta,\\
&d^{n+1}d^{n+2}\zeta = d^{n+1}d^{n+1}\zeta = d^{n+1}s^{n+1}d^n\xi = d^n\xi,\\
&d^i d^{n+2}\zeta = d^{n+1}d^i\zeta = 0 \ (i < n).
\end{aligned}$$

Hence $d^{n+2}\zeta: X \to C^{n+2}(Y_2)$ is a required morphism.

THE PROOF OF THEOREM 4.9.

i) Commutativity of $\boxed{1}$. Let $\tau: X \to C^{n+2}(Y_1)$ be a representative of $a \in \pi_{n+1}(X, Y_1)$.

Since $k(Y_1): CY_1 \to Y_1$ is a fibration, there exists a morphism $\eta: X \to C^{n+2}(CY_1)$ such that $C^{n+2}(k(Y_1))\eta = \tau$ and $d^i\eta = 0$ for $i \leqq n$. Then we have a morphism $\psi: X \to C^{n+1}(\Omega Y_1)$ satisfying $C^{n+1}(i)\psi = d^{n+1}\eta$.

Since $q_1 j: F_v \to CY_1$ is a fibration and $C^{n+1}(q_1 j)C^{n+1}(k)\psi = C^{n+1}(i)\psi = d^{n+1}\eta$, there exists a morphism $\alpha : X \to C^{n+2}(F_v)$ such that $C^{n+2}(q_1 j)\alpha = \eta$, $d^i\alpha = 0$ for $i < n$, and $d^{n+1}\alpha = C^{n+1}(k)\psi$. Then we have $C^{n+2}(v)C^{n+2}(q_2 j)\alpha = C^{n+2}(k(Y_1)) C^{n+2}(q_1 j)\alpha = C^{n+2}(k(Y_1))\eta = \tau$ and $d^i C^{n+2}(q_2 j)\alpha = 0$ for $i \neq n$.

Since $C^{n+1}(v)C^{n+1}(q_2 j)^n d\alpha = 0$ and $v: Y_2 \to Y_1$ is a fibration, by the lemma, the morphism $\zeta : X \to C^{n+1}(Y_3)$ determined by $C^{n+1}(q_2 j)d^n\alpha = C^{n+1}(u)\zeta$ may be considered as a representative of $\partial(v)a$.

Since
$$C^{n+1}(q_2 j)C^{n+1}(l)\zeta = C^{n+1}(u)\zeta = C^{n+1}(q_2 j)d^n\alpha,$$
$$C^{n+1}(q_1 j)C^{n+1}(l)\zeta = 0 \quad \text{and}$$
$$C^{n+1}(q_1 j)d^n\alpha = d^n\eta = 0,$$

by the property of the direct product, we have $C^{n+1}(j)C^{n+1}(l)\zeta = C^{n+1}(j)d^n\alpha$. But j is a monomophism and hence $d^n\alpha = C^{n+1}(l)\zeta$.

Accordingly $\alpha : X \to C^{n+2}(F_v)$ satisfies the following relations:

$$d^i\alpha = 0 \text{ for } i < n, \; d^n\alpha = C^{n+1}(l)\zeta \text{ and } d^{n+1}\alpha = C^{n+1}(k)\psi.$$

As $C^{n+1}(l)\zeta$ and $C^{n+1}(k)\psi$ are the representatives of $l_*\partial(v)a$ and $k_*\partial(k(Y_1))a$ respectively, it follows that $l_*\partial(v)a = k_*\partial(k(Y_1))a$.

Thus the commutativity of [1] is established.

ii) Commutativity of [2] immediately follows from $q_2 jl = u$ in (4. 7).

iii) Commutativity of [3]. Let $\tau : X \to C^{n+1}(Y_2)$ be a representative of $a \in \pi_n(X, Y_2)$. Then by the definition $\partial(q_2 j)a$ is represented by a morphism $\chi : X \to C^n(\Omega Y_1)$ determined by the following relations:

$$C^{n+1}(q_2 j) = \tau, \; d^i\xi = 0 \; i < n \text{ and } C^n(k)\psi = d^n\xi,$$

where $\xi : X \to C^{n+1}(F_v)$ exists since $q_2 j$ is a fibration.

On the other hand $\partial(k(Y_1))\, v_* a$ is represented by a morphism $\psi : X \to C^n(\Omega Y_1)$ determined by $C^n(i)\psi = d^n C^{n+1}(q_1 j)\xi$, since $C^{n+1}(k(Y_1))C^{n+1}(q_1 j)\xi = C^{n+1}(v)C^{n+1}(q_2 j)\xi = C^{n+1}(v)\tau$ and $d^i C^{n+1}(q_1 j)\xi = 0$ for $i < n$.

It is easily verified that $C^n(i)\psi = C^n(i)\chi$. As i is a monomorphism, we have $\psi = \chi$. This proves the commutativity of [3].

Now we shall define an operation $\rho_{v_*} : \pi_n(X, F_v) \times \pi_n(X, \Omega Y_1) \to \pi_n(X, F_v)$ for any morphism $v : Y_2 \to Y_1$.

Take $(a, b) \in \pi_n(X, F_v) \times \pi_n(X, \Omega Y_1)$ and let $\sigma : X \to C^{n+1}(F_v)$ and $\tau : X \to C^{n+1}(\Omega Y_1)$ be the representatives of a and b respectively. Then from the Kan condition there exists a morphism $\rho : X \to C^{n+2}(F_v)$ such that $d^{n-1}\rho = \sigma$, $d^{n+1}\rho = C^{n+1}(k)\tau$.

Then we define $\rho_{v_*}(a, b)$ by $\{d^n\rho\}$, that is to say, $\rho_{v_*}(a, b) = a \cdot k_*(b)$.

REMARK. Let $v : Y_2 \to Y_1$ be a fibration. Consider the diagram

$$\begin{array}{ccc} \pi_n(X, Y_3) \times \pi_n(X, \Omega Y_1) & \xrightarrow{\rho_*} & \pi_n(X, Y_3) \\ l_* \times 1 \downarrow & & \downarrow l_* \\ \pi_n(X, F_v) \times \pi_n(X, \Omega Y_1) & \xrightarrow{\rho_{v_*}} & \pi_n(X, F_v). \end{array}$$

According to Theorem 4.8, l_* is an isomorphism for $n \geqq 0$, and for $a \in \pi_n(X, Y_3), b \in \pi_n(X, \Omega Y), l_*\rho_*(a, b) = l_*(a \cdot \partial(v)\partial^{1-}(k(Y_1))b) = l_*a \cdot l_*\partial(v)\partial^{-1}(k(Y_1))$ b; according to Theorem 4.9, $l_*\partial(v)\partial^{-1}(k(Y_1)) = k_*$. Hence $l_*\rho_*(a, b) = l_*a \cdot k_*(b) = \rho_{v_*}(l_*a \cdot b) = \rho_{v_*} \cdot (l_* \times 1)(a, b)$. Thus the above diagram is commutative. Therefore we may conclude that if $v: Y_2 \to Y_1$ is a fibration, ρ_{v_*} is equivalent to ρ_* defined in §2.

THEOREM 4.10.

(i) *For* $b_1, b_2 \in \pi_n(X, \Omega Y_1)$, $\rho_{v_*}(k_*b_1, b_2) = k_*(b_1 \cdot b_2)$.

(ii) *If* a_1, $a_2 \in \pi_n(X, F_v)$, *then* $a_1 = \rho_{v_*}(a_2, b)$ *for some* $b \in \pi_n(X, \Omega Y_1)$ *if and only if* $(q_2j)_*\, a_1 = (q_2j)_*a_2$.

PROOF. (i) and the necessity of (ii) are obvious. We only prove the sufficiency of (ii). Let $\sigma_i : X \to C^{n+1}(F_v)$ be a representative of $a_i \in \pi_n(X, F_v)$ $(i = 1, 2)$. From $(q_2j)_*a_1 = (q_2j)_*a_2$, there exists a morphism $\xi : X \to C^{n+2}(Y_2)$ such that

$$d^i\xi = 0 \qquad (i < n),\ d^n\xi = C^{n+1}(q_2j)\sigma_1,\ d^{n+1}\xi = C^{n+1}(q_2j)\sigma_2.$$

Also since $q_2j : F_v \to Y_2$ is a fibration, there exists a morphism $\sigma : X \to C^{n+2}(F_v)$ such that

$$d^i\sigma = 0 \qquad (i < n - 1),\ d^n\sigma = \sigma_1,\ d^{n+}\sigma^1 = \sigma_2 \text{ and } C^{n+2}(q_2j)\sigma = \xi.$$

Then $d^{n+1}\sigma$ represents $a_1 - a_2 \in \pi_n(X, F_v)$ and $C^{n+1}(q_2j)d^{n-1}\sigma = d^{n-1}\xi = 0$. Hence there exists a morphism $\tau : X \to C^{n+1}(\Omega Y_1)$ satisfying $C^{n+1}(k)\tau = d^{n-1}\sigma$. Since $C^{n+1}(k)\, d^i\tau = d^id^{n-1}\ \sigma = d^{n-2}d^i\sigma = 0$ for $i < n - 1$ and k is a morphism, $d^i\tau = 0$ for $i < n - 1$. Similary we may deduce that $d^{n-1}\ \tau = 0$ and $d^n\tau = 0$.

Now let b be the homotopy class of τ. Then the relations $d^i\sigma = 0$ $(i < n - 1)$, $d^{n-1}\sigma = C^{n+1}(k)\tau$, $d^n\sigma = \sigma_1$, $d^{n+1}\sigma = \sigma_2$ imply $a_1 = \rho_{v^*}(a_2, b)$.

Next we consider a monomorphism $u: Y_3 \to Y_2$ in the fibration sequence

$$Y_3 \xrightarrow{u} Y_2 \xrightarrow{v} Y_1.$$

Denote the trace of $k(Y_2) \times u : CY_2 \times Y_3 \to Y_2 \times Y_2$ by (F_u, h). Let $r_1 : CY_2 \times Y_3 \to CY_2$ and $r_2 : CY_2 \times Y_3 \to Y_3$ be projections, and let $s_i : Y_2 \times Y_2 \to Y_2$ $(i = 1, 2)$ be projection on the i-th factor. Then we have the following commutative diagram.

$$(4.11)\qquad \begin{array}{ccccc} \Omega Y_2 & \xrightarrow{\text{id}} & \Omega Y_2 & \xrightarrow{\Omega(v)} & \Omega Y_1 \\ {\scriptstyle k_2}\downarrow & & \downarrow{\scriptstyle i_2} & & \downarrow{\scriptstyle i_1} \\ F_u & \xrightarrow{r_1 h} & CY_2 & \xrightarrow{C(v)} & CY_1 \\ {\scriptstyle r_2 h}\downarrow & & \downarrow{\scriptstyle k(Y_2)} & & \downarrow{\scriptstyle k(Y_1)} \\ Y_3 & \xrightarrow{n} & Y_2 & \xrightarrow{v} & Y_1 \end{array}$$

where the left column corresponds to the middle column in the diageam (4.7), and $\Omega(v):\Omega Y_2 \to \Omega Y_1$ exists since $k(Y_1)C(v)i_2 = vk(Y_2)i_2 = 0$ and $(\Omega(Y_1), i_1)$ is the kernel of $k(Y_1)$.

Now we consider a sequence $F_u \xrightarrow{h} CY_2 \times Y_3 \xrightarrow{C(v)\times u} CY_1 \times Y_2$.

Since
$$k(Y_1)q_1(C(v)\times u)h = k(Y_1)C(v)r_1h = vk(Y_2)r_1h = vur_2h = 0$$
$$vq_2(C(v)\times u)h = vur_2h = 0,$$

and F_v is the trace of $k(Y_1)\times v$, there exists a morphism $f\colon F_u \to F_v$ satisfying $jf = (C(v)\times u)h$. Then we have

$$q_1jfk_2 = q_1(C(v)\times u)hk_2 = C(v)r_1hk_2 = C(v)i_2,$$
$$q_2jfk_1 = q_2(C(v)\times u)hk_2 = ur_2hk_2 = 0,$$
$$q_1jk_1\Omega(v) = i_1\Omega(v) = C(v)i_2, \qquad \text{where } k_1:\Omega Y_1 \to F_v,$$
$$q_2jk_1\Omega(v) = 0.$$

Therefore by the property of the direct product, $jk_1\Omega(v) = jfk_2$ and hence $k_1\Omega(v) = fk_2$.

Thus we have the following commutative diagram;

$$(4.12)\qquad \begin{array}{ccc} \Omega Y_2 & \xrightarrow{\Omega(v)} & \Omega Y_1 \\ {\scriptstyle k_2}\downarrow & & \downarrow{\scriptstyle k_1} \\ F_u & \xrightarrow{f} & F_v \\ {\scriptstyle r_2 h}\downarrow & & \downarrow{\scriptstyle q_2 j} \\ Y_3 & \xrightarrow{u} & Y \end{array}$$

As $k(Y_1)C(v)r_1h = vur_1h = 0$ in (4.11), there also exists a morphism $t: F_u \to \Omega Y_1$ satisfying $i_1t = C(v)r_1h$. Since $q_1jf = q_1(C(v)\times u)h = C(v)r_1h = i_1t$, $i_1tk_2 = q_1jfk_2 = q_1jk_2\Omega(v) = i_1\Omega(v)$ and so $\Omega(v) = tk_2$.

PROPOSITION 4.13. *The following diagram is commutative.*

$$\begin{array}{ccc} \pi_{n+1}(X, Y_2) & \xrightarrow{v_*} & \pi_{n+1}(X, Y_1) \\ \partial_2 \downarrow & & \downarrow \partial_1 \\ \pi_n(X, \Omega Y_2) & \xrightarrow[(\Omega v)_*]{} & \pi_n(X, \Omega Y_1) \end{array}$$

PROOF. Let $\tau : X \to C^{n+2}(Y_2)$ be a representative of $a \in \pi_{n+1}(X, Y_2)$. Then $\partial_2 a$ is represented by a morphism $\varphi : X \to C^{n+1}(\Omega Y_2)$ determined by the relation $C^{n+2}(k(Y_2))\sigma = \tau$, $d^i\sigma = 0$ $(0 \leqq i \leqq n)$, $C^{n+1}(i_2)\varphi = d^{n+1}\sigma$ where $\sigma : X \to C^{n+2}(CY_2)$ exists since $k(Y_2)$ is a fibration.

One has $C^{n+2}(v)\tau = C^{n+2}(v)C^{n+2}(k(Y_2))\sigma = C^{n+2}(k(Y_1))C^{n+2}(C(v))\sigma$ and $d^iC^{n+2}(v)\tau = 0$ for $i \neq n+1$. Hence $\partial_1 v_* a$ is represented by $\psi : X \to C^{n+1}(\Omega Y_1)$ determined by $C^{n+1}(i_1)\psi = d^{n+1}C^{n+2}(C(v))\sigma$. But $C^{n+1}(i_1)\psi = C^{n+1}(C(v))d^{n+1}\sigma = C^{n+1}(C(v))C^{n+1}(i_2)\varphi = C^{n+1}(i_1)C^{n+1}(\Omega(v))\varphi$. Therefore $\psi = C^{n+1}(\Omega(v))\varphi$, for $C^{n+1}(i_1)$ is a monomorphism. Thus the commutativity in (5. 13) is proved.

THEOREM 4. 14. *If $v : Y_2 \to Y_1$ is a fibration, we have the following commutative diagram :*

$$\begin{array}{ccccccccc} \longrightarrow & \pi_n(X, \Omega Y_2) & \xrightarrow{k_{2*}} & \pi_n(X, F_u) & \xrightarrow{(r_2h)_*} & \pi_n(X, Y_3) & \xrightarrow{\partial(r_2h)} & \pi_{n-1}(X, \Omega Y_2) & \longrightarrow \\ & \partial_2^{-1} \downarrow & \boxed{1} & t_* \downarrow & \boxed{2} & l_* \downarrow & \boxed{3} & \downarrow \partial_2^{-1} & n>0 \\ \longrightarrow & \pi_{n+1}(X, Y_2) & \xrightarrow[(qj)]{} & \pi_n(X, \Omega Y_1) & \xrightarrow[k_{1*}]{} & \pi_n(X, F_v) & \xrightarrow[(q_2j)_*]{} & \pi_n(X, Y) & \longrightarrow \end{array}$$

where the upper sequence is the Kan homotopy exact sequence for the fibration $r_2h : F_u \to Y_3$ and the lower is those for the fibration $q_2j : F_v \to Y_2$.

PROOF. (i) Commutativity of $\boxed{1}$. From the commutativity of $\boxed{3}$ in Theorem 4. 9, $\partial_1 v_* = \partial(q_2 j)$ and by Proposition (4. 13) $\partial_1 v_* = (\Omega v)_* \partial_2$ so that $\partial(q_2 j) = (\Omega v)_* \partial_2$. But $\Omega(v) = tk_2$. Hence $\partial(q_2 j)\, \partial_2^{-1} = t_* k_{2*}$.

(ii) Commutativity of $\boxed{2}$. It is sufficient to prove that $(r_2h)_* = \partial(v)\partial_1^{-1}t_*$.

$$\begin{array}{ccc} \pi_n(X, F_u) & \xrightarrow{(r_2h)_*} & \pi_n(X, Y_3) \\ t_* \downarrow & & \uparrow \partial(v) \\ \pi_n(X, \Omega Y_1) & \xrightarrow{\partial_1^{-1}} & \pi_{n+1}(X, Y) \end{array} \qquad n > 0.$$

Let $a \in \pi_n(X, F_u)$ be represented by a morphism $\tau : X \to C^{n+1}(F_u)$ such that $d^i\tau = 0$ $(0 \leqq i \leqq n)$. Since $\partial_1 : \pi_{n+1}(X, Y_1) \to \pi_n(X, \Omega Y_1)$ is an isomorphism for $n > 0$, there exists only one $c \in \pi_{n+1}(X, Y_1)$ such that $\partial_1 c = t_* a$. Let $c \in \pi_{n+1}(X, Y_1)$ be represented by a morphism $\sigma : X \to C^{n+1}(Y_1)$ such that $d^i\sigma = 0$ $(0 \leqq i \leqq n+1.)$ Then by the definition of ∂_1 there exists a morphism $\eta : X \to C^{n+2}$

(CY_1) such that $C^{n+2}(k(Y_1))\eta = \sigma$, $d^i\eta = 0$ $(0 \leqq i \leqq n)$, $d^{n+1}\eta = C^{n+1}(i_1 t)\tau$.

Therefore $\partial(v)\partial_1^{-1}t_* a$ is represented by a morphism $\xi : X \to C^{n+1}(Y_3)$ determined by the relation $C^{n+2}(v)\chi = \sigma$, $d^i\chi = 0$ $(0 \leqq i \leqq n)$, $d^{n+1}\chi = C^{n+1}(u)\xi$, where $\chi : X \to C^{n+2}(Y_2)$ exists since $v : Y_2 \to Y_1$ is a fibration. On the other hand $(r_2h)_* a$ is represented by $C^{n+1}(r_2h)\tau : X \to Y_3$.

Now consider $C^{n+1}(r_1h)\tau : X \to C^{n+1}(CY_2)$. It follows from $i_1t = C(v)r_1h$ that $C^{n+1}(C(v))C^{n+1}(r_1h)\tau = C^{n+1}(i_1t)\tau = d^{n+1}\eta$. If we regard $C^{n+1}(r_1h)\tau$ as a $(n+1)$-simplex of $K_*(X, Y_2)$, i.e. $C^{n+1}(r_1h)\tau : X \to C^{n+2}(Y_2)$, we have $C^{n+2}(v)C^{n+1}(r_1h)\tau = d^{n+1}\eta$ where η is regarded as a $(n+2)$-simplex of $K_*(X, Y_1)$.

On the other hand $\sigma = C^{n+2}(k(Y_1))\eta = d^{n+2}\eta$ and $d^i\eta = 0$ $(0 \leqq i \leqq n)$. Hence $C^{n+2}(v)C^{n+1}(r_1h)\tau \sim \sigma$ ($C^{n+2}(v)C^{n+1}(r_1h)\tau$ is semi-simplicial homotopic to σ). But $d^{n+1}C^{n+1}(r_1h)\tau = C^{n+1}(k(Y_2))C^{n+1}(r_1h)\tau = C^{n+1}(u)C^{n+1}(r_2h)\tau$ and $d^iC^{n+1}(r_1h)\tau = 0$ $(0 \leqq i \leqq n)$. Hence by the definition of $\partial(v) : \pi_{n+1}(X, Y_1) \to \pi_n(X, Y_3)$, we may regard $C^{n+1}(r_2h)\tau$ as a representative of $\partial(v)c$. Thus $(r_2h)_* = \partial(v)\partial_1^{-1}t_*$.

(iii) Commutativity of $\boxed{3}$. Let $a \in \pi_n(X, Y_3)$ be represented by a morphism $\tau : X \to C^{n+1}(Y_3)$ such that $d^i\tau = 0$ $(0 \leqq i \leqq n)$. By Theorem 4.3, $r_2h : F_u \to Y_3$ is also a fibration and $\partial(r_2h)\, a$ is represented by a morphism $\chi : X \to C^n(\Omega Y_2)$ determined by the relation $C^{n+1}(r_2h)\sigma = \tau$, $d^i\sigma = 0$ $(0 \leqq i \leqq n)$, $C^n(k_2)\chi = d^n\sigma$, where $\sigma : X \to C^{n+1}(F_u)$ exists since r_2h is a fibration.

Since $C^{n+1}(u)\tau = C^{n+1}(u)C^{n+1}(r_2h)\sigma = C^{n+1}(k(Y_2))C^{n+1}(r_1h)\sigma$ and $d^iC^{n+1}(r_1h)\sigma = 0$ $(0 \leqq i \leqq n)$, it follows from the definition of $\partial_2 : \pi_n(X, Y_2) \to \pi_{n-1}(X, \Omega Y_2)$ that a morphism $\psi : X \to C^n(\Omega Y_1)$ determined by $d^nC^{n+1}(r_1h)\sigma = C^n(i_2)\psi$ represents an element of $\pi_{n-1}(X, \Omega Y_2)$. Then $C^n(i_2)\psi = d^nC^{n+1}(r_1h)\sigma = C^n(r_1h)\, d^n\sigma = C^n(r_1h)C^n(k_2)\chi = C^n(i_2)\chi$, where $(r_1h)k_2 = i_2$ follows from (4. 11). As $C^n(i_2)$ is monomorphic this implies $\psi = \chi$ and $\partial_2 u_* = \partial(r_2h)$ follows.

On the other hand $q_2jl = u$ and hence $\partial_2(q_2j)_*l_* = \partial(r_2h)$.

From Theorems 4.8 and 4.14 and the five lemma we obtain the following theorem:

THEOREM 4.15. *If* $Y_3 \xrightarrow{u} Y_2 \xrightarrow{v} Y_1$ *is a fibration sequence, then*

$$t_* : \pi_n(X, F_u) \to \pi_n(X, \Omega Y_1)$$

is an isomorphism for $n > 1$.

Here we note that Theorem 4.15 corresponds to Proposition 4.7 in [2].

5. The dual statements. Finally we shall consider the dual statement. A dual standard construction in $\mathfrak{K}$ is a triple $\{C, k, p\}$ consisting of a covariant functor $C : \mathfrak{K} \to \mathfrak{K}$, and of functor morphisms $k : I \to C$ and $p : CC \to C$, such that the axioms

(SC 1′) $\quad p \circ (k * C) = p * (C * k) = \text{identity}$,

(SC 2′) $\quad p \circ (p * C) = p \circ (C * p)$

are satisfied.

The functor $F^* = (F^n, d^i, S^i)$ belonging to a dual standard construction has a dual semi-simplicial structure, i. e. the faces and degeneracy morphisms

$$d_n^i : F^{n-1} \longrightarrow F^n$$
$$s_n^i : F^{n+1} \longrightarrow F^n$$

go into the opposite direction and satisfy the relation dual to (a)-(e) in §2. Let $K_*(X, Y)$ be the semi-simplicial complex induced by a dual standard construction $\{C, k, p\}$, i. e.

$$K_*(X, Y) = \mathrm{Hom}(F^*(X), Y) = (\mathrm{Hom}(F^n(X), Y), d^i, s^i).$$

Then we have the Kan homotopy groups $\pi_n(X, Y) = \pi_n(K_*(X, Y)$. Here we shall assume that all the $K_*(X, Y)$ satisfy the Kan condition.

DEFINITION 2. 1′([5]). A morphism $u : X_1 \to X_2$ is called a cofibration if the induced semi-simplicial map

$$u^* : K_*(X_2, Y) \to K_*(X_1, Y)$$

is a semi-simplicial fibration (cf. [6]).

If C commutes with cokernels (i. e. $C(\mathrm{Coker}\ u) = \mathrm{Coker}\ C(u)$, then the cokernel X_3 of u will be called cofibre of u.

PROPSITION. 2. 2′ (Huber [5]).

1) *The morphism* $k(X) : X \to CX$ *is a cofibration.*

2) $$\pi_n(CX, Y) = 0 \qquad \textit{for } n \geqq 0.$$

If we denote ΣX the cokernel of $k(X)$ and C commutes ΣX, then

PROPOSITION. 2. 3′ (Huber [5])

$$\partial(k(X)) : \pi_{n+1}(X, Y) \to \pi_n(\Sigma X, Y) \textit{ is an isomorphism for } n > 0.$$

In the following we shall assume the existences of cofibres.

Let $u : X_1 \to X_2$ be a cofibration in $\mathfrak{K}$ and let (X_3, v) its cofibre. By Proposition 2. 2′, $k(X_1) : X_1 \to CX_1$ is a cofibration. Then as dual in §3 we may define a co-operation

$$\rho^* : \pi_n(X_3, Y) \times \pi_n(\Sigma X_1, Y) \to \pi_n(X_3, Y).$$

THEOREM 3. 1′.

1) $\rho^*(a, 0) = a$, *where* $a \in \pi_n(X_3, Y)$ *and* $0 \in \pi_n(\Sigma X_1, Y)$ *denotes the unit element.*

2) $\rho^*(a, b_1 \cdot b_2) = \rho^*(\rho^*(a, b_1), b_2)$, *where* $a \in \pi_n(X_3, Y)$, $b_i \in \pi_n(\Sigma X_i, Y)$ $(i = 1, 2)$.

Let $\mathfrak{K}$ be the category of topological spaces with base points, $\mathrm{Hom}(X, Y)$ being the set of base points preserving continuous maps $X \to Y$, with the natural rule of composition. Then a dual standard construction $\{C, k, p\}$ in $\mathfrak{K}$ was defined in [5] as the cone construction:

$$CX = X \times I/X \times \{0\} \cup \{*\} \times I,$$

I denoting the real interval $0 \leqq t \leqq 1$, with the base point 0.

Again let $\mathfrak{K}$ be a general category and suppose that $\mathfrak{K}$ has inverse products. Here we recall the definition of an inverse product in [3].

An inverse product of the object A_1, A_2 is an object $A_1 * A_2$ and a system of morphisms $q_j : A_i \to A_1 * A_2$ $(j = 1, 2)$ with the property

(I) For any object X of $\mathfrak{K}$ and any system of morphisms $f_j : A_j \to A_1 * A_2$ $(j = 1,2)$, there exists a unique morphism $f : A_1 * A_2 \to X$ with $fq_j = f_j$ $(j = 1, 2.)$

The morphisms q_j are called the injections into $A_1 * A_2$; and the morphisms f_j are called the component of f; we write $f = <f_1, f_2>$, so that

$$<f_1, f_2> q_j = f_j.$$

Let $f_i : B \to A_i$, $i = 1, 2$, be morphisms, then the morphism $< q_1 f_1, q_2 f_2 >:$ $B * B \to A_1 * A_2$ will be written $f_1 * f_2$. Then we have $(f_1 * f_2) q_i' = q_i f_i$ $(i = 1, 2)$, where $q_j' : B \to B * B$, $j = 1, 2$, are injections.

DEFINITION 4.1′. The trace of $f_1 * f_2 : B * B \to A_1 * A_2$ is a pair (Q, ρ) consisting of an object and an epimorphism satisfy the conditions

(i) $\rho(f_1 * f_2) q_1' = \rho(f_1 * f_2) q_2'$ (or equivalently $\rho q_1 f_1 = \rho q_2 f_2$)

(ii) if D is any object of $\mathfrak{K}$ and $h : A_1 * A_2 \to D$ is a morphism with the property $h(f_1 * f_2) q_1' = h(f_1 * f_2) q_2'$ (or equivalently $h q_1 f_1 = h q_2 f_2$), then h admits a unique factorization $A_1 * A_2 \xrightarrow{\rho} Q \to D$.

We assume that the functor C in consideration is I-functor in the sense of [3], i.e. $C(A_1 * A_2) = C(A_1) * C(A_2)$ and $C(q_i) : C(A_i) \to C(A_1 * A_2)$, $i = 1, 2$, are injections. Moreover we assume that if $g : B \to A$ is an epimorphism, then $C(g) : C(B) \to C(A)$ is so.

Let $k(X_1) : X_1 \to CX_1$ be a morphism in Proposition 2.2′ and $v : X_1 \to X_2$ any morphism. Then we denote the trace of $k(X_1) * v$ by (F^v, p). Let $q_1 : CX_1 \to CX_1 * X_2$ and $q_2 : X_2 \to CX_1 * X_2$ be injections, then we have the following commutative diagram;

$$(4.2') \qquad \begin{array}{ccc} X_1 & \xrightarrow{\quad v \quad} & X_2 \\ {\scriptstyle k(X_1)} \downarrow & & \downarrow {\scriptstyle pq_2} \\ CX_1 & \xrightarrow[\quad pq_2 \quad]{} & F^v \end{array}$$

The proofs of the following theorems are quite dual to that of the preceeding ones and we shall omit.

THEOREM 4.3′. *$pq_2 : X_2 \to F^v$ is a cofibration.*

THEOREM 4.4′. *If $v : X_1 \to X_2$ is a cofibration, $pq_1 : CX_1 \to F^v$ is so.*

THEOREM 4.5′. *The cofibre of cofibration $pq_2 : X_2 \to F^v$ is equivalent with ΣX_1, the cofibre of $k(X) : X_1 \to CX_1$.*

THEOREM 4.6′. *If a sequence $X_1 \to X_2 \to X_3$ is a cofibration sequence, then a sequence $CX_1 \to F^v \to X_3$ is so.*

If $v : X_1 \to X_2$ is a cofibration with cofibre X_3, then the following diagram is commutative.

$$
(4.7') \qquad
\begin{array}{ccccc}
X_1 & \xrightarrow{\quad u \quad} & X_2 & \xrightarrow{\quad v \quad} & X_3 \\
{\scriptstyle k(X_1)}\downarrow & & \downarrow{\scriptstyle pq_2} & & \downarrow{\scriptstyle \mathrm{id}} \\
CX_1 & \xrightarrow{\quad pq_1 \quad} & F^v & \xrightarrow{\quad l \quad} & X_3 \\
{\scriptstyle i_1}\downarrow & & \downarrow{\scriptstyle k_1} & & \\
\Sigma X_1 & \xrightarrow[\quad \mathrm{id} \quad]{} & \Sigma X_1 & &
\end{array}
$$

THEOREM 4.8′. *If a morphism $v : X_1 \to X_2$ is a cofibration with cofibre X_3, then $l_* : \pi_n(X_3, Y) \to \pi_n(F^v, Y)$ is an isomorphism for $n \geqq 0$.*

THEOREM 4.9′. *If $v : X_1 \to X_2$ is a cofibration with cofibre X_3, then the following diagram is commutative*:

$$
\begin{array}{ccccccccc}
\longrightarrow & \pi_{n+1}(X_1, Y) & \xrightarrow{\partial(v)} & \pi_n(X_3, Y) & \xrightarrow{u^*} & \pi_n(X_2, Y) & \xrightarrow{v^*} & \pi_n(X_1, Y) & \longrightarrow \\
& {\scriptstyle \partial(k(X_1))}\downarrow & & {\scriptstyle l^*}\downarrow & & {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle \partial(k(X_1))} & \\
\longrightarrow & \pi_n(\Sigma X_1, Y) & \xrightarrow[k^*]{} & \pi_n(F^v, Y) & \xrightarrow[(pq_2)]{} & \pi_n(X_2, Y) & \xrightarrow[\partial(pq_2)]{} & \pi_{n-1}(\Sigma X_1, Y) &
\end{array}
$$

where the upper sequence is the Kan homotopy exact sequence of a cofibration $v : X_1 \to X_2$ and the lower is that of a cofibration $pq_2 : X_2 \to F^v$.

For any morphism $v : X_1 \to X_2$ co-operation $\rho_v^* : \pi_n(F^v, Y) \times \pi_n(\Sigma X_1, Y) \to \pi_n(F^v, Y)$ is defined by

$$\rho_v^*(a, b) = a \cdot k^*(b), \qquad a \in \pi_n(F^v, Y),\ b \in \pi_n(\Sigma X_1, Y).$$

If $v : X_1 \to X_2$ is a cofibration, we may show that ρ_v^* is equivalent with ρ^*.

THEOREM 4.10′.

(i) *For $b_1, b_2 \in \pi_n(\Sigma X_1, Y)$, $\rho_v^*(k^*(b_1), b_2) = k^*(b_1 \cdot b_2)$*

(ii) *If $a_1, a_2 \in \pi_n(F^v, Y)$, then $a = \rho_v^*(a_2, b)$ for some $b \in \pi_n(\Sigma X_1, Y)$ if and* *any if* $(pq_2)^*a_1 = (pq_2)^*a_2$.

Let $X_1 \xrightarrow{v} X_2 \xrightarrow{u} X_3$ be a cofibration sequence. We denote the trace of $k(X_2)*u : X_2*X_2 \longrightarrow CX_2*X_3$ by (F^u, h). Let $r_1,\ r_2 : CX_2, X_3 \longrightarrow CX_2*X_3$ be injections. Then (4.11) and (4. 12) may be dualized as follows.

$$
(4.11') \qquad \begin{array}{ccccc}
X_1 & \xrightarrow{v} & X_2 & \xrightarrow{u} & X_3 \\
{\scriptstyle k(X_1)}\downarrow & & {\scriptstyle k(X_2)}\downarrow & & \downarrow{\scriptstyle hr_2} \\
CX_1 & \xrightarrow{C(v)} & CX_2 & \xrightarrow{hr_1} & F^u \\
{\scriptstyle i_1}\downarrow & & {\scriptstyle i_2}\downarrow & & \downarrow{\scriptstyle k_2} \\
\Sigma X_1 & \xrightarrow{\Sigma(v)} & \Sigma X_2 & \xrightarrow{\mathrm{id}} & \Sigma X_2
\end{array}
$$

$$
(4.12') \qquad \begin{array}{ccc}
X_2 & \xrightarrow{u} & X_3 \\
{\scriptstyle pq_2}\downarrow & & \downarrow{\scriptstyle hr_2} \\
F^v & \xrightarrow{f} & F^u \\
{\scriptstyle k_1}\downarrow & & \downarrow{\scriptstyle k_2} \\
\Sigma X_1 & \xrightarrow{\Sigma(v)} & \Sigma X_2
\end{array}
$$

Then there exists a morphism $t : \Sigma X_1 \to F^u$ such that $ti_1 = hr_1 \cdot C(v)$. Also $\Sigma(v) = k_2 t$.

PROPOSITION 4. 13$'$. *The following diagram is commutative.*

$$
\begin{array}{ccc}
\pi_{n+1}(X_2, Y) & \xrightarrow{v^*} & \pi_{n+1}(X_1, Y) \\
{\scriptstyle \partial_2}\downarrow & & \downarrow{\scriptstyle \partial_1} \\
\pi_n(\Sigma X_2, Y) & \xrightarrow{\Sigma(v)^*} & \pi_n(\Sigma X_1, Y)
\end{array}
$$

THEOREM 4. 14.$'$ *If* $v : X_1 \to X_2$ *is a cofibration with cofibre* X_3, *then we have the following commutative diagram.*

$$
\begin{array}{ccccccccc}
\longrightarrow & \pi_n(\Sigma X_2, Y) & \xrightarrow{k_2^*} & \pi_n(F^u, Y) & \xrightarrow{(hr_2)^*} & \pi_n(X_3, Y) & \xrightarrow{\partial(hr_2)} & \pi_{n-1}(\Sigma X_2, Y) \\
& {\scriptstyle \partial_2^{-1}}\downarrow & & {\scriptstyle t^*}\downarrow & & {\scriptstyle l^*}\downarrow & & \downarrow{\scriptstyle \partial_2^{-1}} \\
\longrightarrow & \pi_{n+1}(X_2, Y) & \xrightarrow[\partial(pq_2)]{} & \pi_n(\Sigma X_1, Y) & \xrightarrow[k_1^*]{} & \pi_n(F^v, Y) & \xrightarrow[(pq_2)^*]{} & \pi_n(X_2, Y)
\end{array}
$$

where the upper sequence is the Kan homotopy exact sequence of a cofibration $hr_2: X_3 \to F^u$ *and the lower is that of a cofibration* $pq_2: X_2 \to F^v$.

THEOREM 4.15'. *If* $X_1 \xrightarrow{v} X_2 \xrightarrow{u} X_3$ *is a cofibration sequence then*

$$t^*: \pi_n(F^u, Y) \to \pi_n(\Sigma X_1, Y)$$

is an isomorphism for $n > 1$.

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HIROSAKI UNIVERSITY.