STRONG SUMMABILITY OF WALSH FOURIER SERIES

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1. Introduction. Let $f(x)$ be an integrable and periodic function with period 1. Let $\{\psi_n(x)\}$ $(n=0, 1, 2, \cdots)$ be the orthogonal system of Walsh (We refer to [4] for definition of the system), and

$$(1) \qquad \sum_{n=0}^{\infty} a_n \psi_n(x), \qquad a_n = \int_0^1 f(x)\psi_n(x)\,dx.$$

be the Walsh Fourier series of $f(x)$. We denote the partial sum of (1) by

$$s_n(x) = \sum_{\nu=0}^{n-1} a_\nu \psi_\nu(x)$$

and the strong Cesàro mean of (1) by

$$R_n^\delta(x) = \frac{1}{A_n^\delta}\sum_{\nu=0}^{n} A_{n-\nu}^{\delta-1}|s_\nu(x)|,$$

where

$$A_n^\delta = \binom{n+\delta}{n} \cong \frac{n^\alpha}{\Gamma(\alpha+1)}.$$

Paley [4] stated the following theorem without proof.

THEOREM A. *If $f(x)$ belongs to the class $L^p(p>1)$ and $\delta>1/p$, then**)

$$\int_0^1 (\max_{0\leqq n<\infty} |R_n^\delta(x)|)^p\,dx \leqq A_p \int_0^1 |f(x)|^p\,dx.$$

And he conjectured that Theorem A would be valid for any $\delta>0$.

In the present note, the author will prove this conjecture in stronger form. In fact, if we set

*) A_p is a constant depending on p only and is not necessarily the same in different occurences.

$$R_n^{\delta,k}(x) = \left(\frac{1}{A_n^\delta}\sum_{\nu=0}^{n} A_{n-\nu}^{\delta-1}|s_\nu(x)|^k\right)^{\frac{1}{k}} \quad (k>0).$$

then we can prove the following theorem.

THEOREM B. *If $f(x)$ belongs to the class $L^p(p>1)$ and $\delta>0$, then*

$$\int_0^1 (\max_{0\leqq n<\infty}|R_n^{\delta,k}(x)|)^p\,dx \leqq B_{p,\delta}\int_0^1 |f(x)|^p\,dx.$$

On the course of the proof, we will also show the following theorem.

THEOREM C. *If $f(x)$ belongs to the class L^p $(p>1)$ and $r\geqq 2$, then*

$$\int_0^1 \left(\sum_{n=1}^{\infty}\frac{|s_n(x)-\sigma_n(x)|^r}{n}\right)^{\frac{p}{r}} dx \leqq C_p\int_0^1 |f(x)|^p\,dx,$$

where $\sigma_n(x)$ is the arithmetic mean of $s_n(x)$.

The trigonometric analogue of this theorem has already given by the author [7]. In the last paragraph, we shall give a remark about the trigonometric Fourier series.

2. A Lemma on the decomposition of a vector-valued function. Mr. Igari [3] gave a decomposition theorem of Hörmander type [2]. We may extend this to a vector-valued function.

Let $u(x)=\{u_1(x), u_2(x), \cdots, u_n(x), \cdots\}$ be an l^r-valued $(r\geqq 1)$ function of $x\in[0,1]$ and measurable in the Bochner sense. If $\|u(x)\|\in L^p$, then we write $f\in L^p(l^r)$.

LEMMA 1. *We set*

$$y_0 = 2\int_0^1 \|u(x)\|\,dx,$$

then, for any $y>y_0$, we can decompose $u(x)$ such as

(1°) $$u(x) = v(x) + w(x), \quad w(x) = \sum_{k=1}^{\infty} w_k(x),$$

(2°) $$\|v(x)\| \leqq 2y, \qquad \textit{almost every } x,$$

(3°) $$\int_0^1 \|v(x)\|\,dx \leqq \int_0^1 \|u(x)\|\,dx,$$

(4°) $$\sum_{k=1}^{\infty}\int_0^1 \|w_k(x)\|\,dx \leqq 2\int_0^1 \|u(x)\|\,dx\,,$$

(5°) *There is a sequence of disjoint intervals* $\{I_k\}$ *such as*

$$\sum_{k=1}^{\infty}|I_k| \leqq \frac{1}{y}\int_0^1 \|u(x)\|\,dx\,, \qquad \textit{support } (w_k)\subset I_k$$

and the end points of I_k *are dyadic-rational,*

(6°) $$\int_0^1 w_k(x)\,dx = \theta, \qquad (k = 1, 2, \cdots),$$

where θ *is the zero element of* l^r.

PROOF. The proof is almost the same to the case of a real valued function. Since $y > y_0$,

$$\frac{1}{2} = \frac{1}{y_0}\int_0^1 \|u(x)\|\,dx \geqq \frac{1}{y}\int_0^1 \|u(x)\|\,dx\,.$$

We divide the interval $[0, 1]$ in two congruent intervals and denote it by $J_{0,i}$ $(i = 1, 2)$, then

$$\frac{1}{|J_{0,i}|}\int_{J_{0,i}}\|u(x)\|\,dx = 2\int_{J_{0,i}}\|u(x)\|\,dx \leqq 2\int_0^1 \|u(x)\|\,dx \leqq y\,.$$

Then we divide $J_{0,i}$ into two equal intervals $J_{0,i}^{(1)}$, $J_{0,i}^{(2)}$, and if there are such intervals that

$$\frac{1}{|J_{0,i}^{(j)}|}\int_{J_{0,i}^{(j)}}\|u(x)\|\,dx \geqq y,$$

then we term them $I_{1,k}$ $(k=1, 2, \cdots)$. The remaining intervals are termed by $J_{1,k}$ $(k=1, 2, \cdots)$, and we divide each of them into two congruent intervals such as $J_{1,k}^{(j)}$ $(j=1, 2, \cdots)$. If there are such interval that

$$\frac{1}{|J_{1,k}^{(j)}|}\int_{J_{1,k}^{(j)}}\|u(x)\|\,dx \geqq y\,,$$

then, we call them $I_{2,k}$ $(k=1, 2, \cdots)$ and the remaining intervals are termed by $J_{2,k}$ $(k=1,2,\cdots)$. We repeat this procedure. Since the number of intervals of $I_{j,k}$ for fixed j is finite, we order these $I_{j,k}$ to a sequence $\{I_k\}$ $(k=1, 2,\cdots)$. Then I_k is evidently disjoint. Since every I_k is one of the type $I_{i,l}$ and is contained in one of $J_{j-1,l}$, which belongs to the preceding division,

$$
\begin{aligned}
(2)\qquad y &\leqq |I_k|^{-1}\int_{I_k}\|u(x)\|\,dx \\
&\leqq |I_{j,l}|^{-1}\,|J_{j-1,l}|\,|J_{j-1,l}|^{-1}\int_{J_{j-1,l}}\|u(x)\|\,dx \\
&\leqq 2y.
\end{aligned}
$$

Let us set

$$
(3)\qquad v(x)=\begin{cases}|I_k|^{-1}\int_{I_k}u(t)\,dt\,, & \text{if}\quad x\in I_k\\ u(x)\,, & \text{if}\quad x\notin \cup I_k\equiv E\,,\end{cases}
$$

$$
w_k(x)=\begin{cases}u(x)-v(x)\,, & x\in I_k\\ \theta\,, & x\notin I_k\end{cases}
$$

and

$$
w(x)=\sum w_k(x)\,.
$$

Then (1°) and (6°) are evident.

If $x\in I_k$, by Minkowski's inequality

$$
\begin{aligned}
\|v(x)\| &= \frac{1}{|I_k|}\left\|\int_{I_k}u(t)\,dt\right\| \\
&= |I_k|^{-1}\Big(\sum_{i=1}^{\infty}\Big|\int_{I_k}u_i(t)\,dt\Big|^r\Big)^{\frac{1}{r}} \qquad (r\geqq 1) \\
&\leqq |I_k|^{-1}\int_{I_k}\Big(\sum_{i=1}^{\infty}|u_i(t)|^r\Big)^{\frac{1}{r}}\,dt \\
&\leqq |I_k|^{-1}\int_{I_k}\|u(t)\|\,dt\,.
\end{aligned}
$$

Hence we have

$$
\int_0^1\|v(x)\|\,dx=\Big(\int_{E^C}+\sum_{k=1}^{\infty}\int_{I_k}\Big)\|v(x)\|\,dx
$$

$$
\leqq\int_{E^C}\|u(x)\|\,dx+\sum_{k=1}^{\infty}\int_{I_k}\|u(x)\|\,dx=\int_0^1\|u(x)\|\,dx,
$$

which is (3°). Since, by (3)

$$\sum_{k=1}^{\infty}\int_0^1 \| w_k(x) \| \, dx \leqq \int_E \{\| u(x) \| + \| v(x) \|\} dx$$
$$\leqq 2\int_0^1 \| u(x) \| \, dx$$

and this is nothing but (4°). If $x \in I_k$, by (2),

$$\| v(x) \| \leqq 2y$$

and if $x \notin \cup I_k$, then there is an interval I containing x and with length smaller than arbitrary given positive number such as

$$\frac{1}{|I|}\int_I \| u(t) \| \, dt \leqq y \, .$$

Hence by the theorem of differentiation, we have

$$\|u(x)\| \leqq y \qquad \text{(a. e.)}.$$

However $u(x)=v(x)$, by (3), and we get

$$\|v(x)\| \leqq 2y \qquad \text{(a. e.)},$$

this is (2°).

By the construction of I_k, we have

$$| I_k | \leqq \frac{1}{y}\int_{I_k} \| u(x) \| \, dx$$

and

$$\sum | I_k | \leqq \frac{1}{y}\int_0^1 \| u(x) \| \, dx \, .$$

Other properties of (5°) are evident. Thus we have proved Lemma completely.

3. Strong summability of Walsh Fourier series. Let $f_n(x) \in L(0,1)$, and its Walsh Fourier series be

$$f_n(x) \sim \sum_{\nu=0}^{\infty} c_\nu^{(n)} \psi_\nu(x)$$

and its partial sums be

$$s_{m(n)}(x,f_n) = \sum_{v=0}^{m(n)-1} c_v^{(n)} \psi_v(x).$$

THEOREM 1. *If* $f_n(x) \in L^r$ $(r>1)$, *and* $p>1$, *then*

$$\int_0^1 \left(\sum_{n=1}^{\infty} | s_{m(n)}(x, f_n)|^r\right)^{\frac{p}{r}} dx \leqq A_{p,r} \int_0^1 \left(\sum_{n=1}^{\infty} |f_n(x)|^r\right)^{\frac{p}{r}} dx.$$

PROOF. When $p=r$, we get this inequality by only addition of known formula.

Next we shall prove that

$$(*) \qquad \mu\left[x \,\middle|\, \left\{\sum_{n=1}^{\infty} |s_{m(n)}(x,f_n)|^r\right\}^{\frac{1}{r}} > y\right] \leqq \frac{1}{y} \int_0^1 \left(\sum_{n=1}^{\infty} |f_n|^r\right)^{\frac{1}{r}} dx,$$

for any $y>0$.

We set

$$\delta_0(x,f_n) = c_0^{(n)}, \quad \delta_{k+1}(x,f_n) = \sum_{v=2^k}^{2^{k+1}-1} c_k^{(n)} \psi_k(x)$$

and

$$g_n(x) = f_n(x) \psi_{m(n)}(x).$$

Then, by the known formula (see Paley [4] or Sunouchi [8])

$$s_{m(n)}(x,f_n) \psi_{m(n)}(x) = \delta_{k_1(n)}(x, g_n) + \delta_{k_2(n)}(x, g_n) + \cdots + \delta_{k\lambda(n)}(x, g_n),$$

where

$$m(n) = 2^{k_1(n)} + 2^{k_2(n)} + \cdots + 2^{k\lambda(n)},$$

$$0 \leqq k_1(n) < k_2(n) < \cdots.$$

We suppose the vector-valued function

$$g(x) \equiv \{g_1(x),\ g_2(x), \cdots,\ g_n(x), \cdots, \}$$

is $u(x)$ in Lemma 1 and decompose it into

$$v(x) = \{v_1(x),\ v_2(x), \cdots,\ v_n(x), \cdots, \}$$

and

$$w(x) = \{w^{(1)}(x),\ w^{(2)}(x), \cdots,\ w^{(k)}(x), \cdots\}.$$

Then

$$s_{m(n)}(x, f_n)\,\psi_{m(n)}(x) = \sum \delta_{k(n)}(x, v_n) + \sum \delta_{k(n)}(x, w^{(n)})$$
$$= V_n(x) + W_n(x), \quad \text{say.}$$

As Watari [9] showed

$$\delta_{k(n)}(x, w^{(n)}) = 0 \quad \begin{pmatrix} k = 1, 2, \cdots, \\ n = 1, 2, \cdots, \end{pmatrix} \quad \text{for} \quad x \notin E = \cup I_n.$$

Hence

$$\left\{x \mid \left(\sum |W_n(x)|^r\right)^{\frac{1}{r}} > y\right\} \subset E$$

and

$$\mu\left\{x \mid \left(\sum |W_n(x)|^r\right)^{\frac{1}{r}} > y\right\} \leqq \|g\|/y \leqq \|f\|/y,$$

by Lemma 1.

On the other hand, concerning $V_n(x)$, it is evident to see,

$$\int_0^1 \left(\sum |V_n|^r\right) dx \leqq \int_0^1 \left(\sum |v_n|^r\right) dx.$$

Thus, from (2°) and (3°) of Lemma, we get

$$\mu\{x \mid \|V\| > y\} \leqq \frac{A}{y^r} \int_0^1 \|v\|^r\, dx$$
$$\leqq \frac{A}{y^r} 2^{r-1} y^{r-1} \int_0^1 \|v\|\, dx$$
$$\leqq \frac{2^r A}{y} \int_0^1 \|u(x)\|\, dx \leqq \frac{B}{y} \|f\|.$$

Thus we have proved (*). Hence, applying generalized interpolation theorem of Marcinkiewicz (See, J. Schwartz [5] or A. Benedek, A. P. Calderón and R. Panzone [1]), we can prove the theorem for $1 < p < r$, and the complete theorem may be gotten by familiar conjugacy argument.

THEOREM 2. *If* $f(x) \in L^p(0, 1)$ $(1 < p < \infty)$, *and* $r \geqq 2$, *then*

$$\int_0^1 \left(\sum_{n=1}^{\infty} \frac{|s_n(x) - \sigma_n(x)|^r}{n}\right)^{\frac{p}{r}} dx \leqq A_p \int_0^1 |f(x)|^p\, dx,$$

where $s_n(x)$ and $\sigma_n(x)$ are the partial sum and the arithmetic mean of Walsh-Fourier series of $f(x)$.

PROOF. Applying Theorem 1, we have

$$\int_0^1 \left(\sum_{n=1}^{\infty} \frac{|s_n(x)-\sigma_n(x)|^r}{n}\right)^{\frac{p}{r}} dx = \int_0^1 \left(\sum_{k=1}^{\infty}\sum_{n=2^{k-1}}^{2^k-1} \frac{|s_n(x)-\sigma_n(x)|^r}{n}\right)^{\frac{p}{r}} dx$$
$$\leqq 2^p \int_0^1 \left(\sum_{k=0}^{\infty} |s_{2^k}(x)-\sigma_{2^k}(x)|^r\right)^{\frac{p}{r}} dx$$
$$\leqq 2^p \int_0^1 \left(\sum_{k=0}^{\infty} |s_{2^k}(x)-\sigma_{2^k}(x)|^2\right)^{\frac{p}{2}} dx ,$$

by Jensen's inequality. From the known reduction (See Sunouchi [8]), we get

$$\int_0^1 \left(\sum_{k=0}^{\infty} |s_{2^k}(x)-\sigma_{2^k}(x)|^2\right)^{\frac{p}{2}} dx$$
$$\leqq A_p \int_0^1 \left(\sum_{k=0}^{\infty} |s_{2^k}(x)-s_{2^{k-1}+1}|^2\right)^{\frac{p}{2}} dx$$
$$\leqq A_p \int_0^1 |f(x)|^p\, dx .$$

Thus the theorem is proved. From Theorem 2, G. Sunouchi and S. Yano [6] deduced the following theorem.

THEOREM 3. *If $f(x)\in L^p(0,1)$ $(1<p<\infty)$, and δ, $k>0$, then*

$$\int_0^1 \left\{\max_{1\leqq n<\infty} \left(\frac{1}{A_n^{\delta}}\sum_{\nu=1}^{n} A_{n-\nu}^{\delta-1}|s_\nu(x)|^k\right)^{\frac{1}{k}}\right\}^p dx \leqq A_{p,\delta}\int_0^1 |f(x)|^p\, dx ,$$

where $s_n(x)$ is the partial sum of Walsh Fourier seeies of $f(x)$.

For the sake of completeness, we reproduce the proof.

PROOF. For a given $\delta>0$, we take s such as

$\delta>1-\dfrac{1}{s}$ $(s>1)$ and set, $\dfrac{1}{r}+\dfrac{1}{s}=1$. Then by Hölder's inequality

$$\frac{1}{A_n^{\delta}}\sum_{\nu=1}^{n} A_{n-\nu}^{\delta-1}|s_\nu(x)-\sigma_\nu(x)|^k$$

$$\leqq \frac{1}{A_n^\delta}\left(\sum_{\nu=1}^{n}\frac{|s_\nu(x)-\sigma_\nu(x)|^{rk}}{\nu}\right)^{\frac{1}{r}}\left(\sum_{\nu=1}^{n}\nu^{\frac{1}{r}}[A_{n-\nu}^{\delta-1}]^s\right)^{\frac{1}{s}}$$

$$\leqq \frac{B}{A_n^\delta}\left(\sum_{\nu=1}^{n}\frac{|s_\nu(x)-\sigma_\nu(x)|^{rk}}{\nu}\right)^{\frac{1}{r}}(n^{\frac{s}{r}}n^{(\delta-1)s+1})^{\frac{1}{s}}$$

$$\leqq \frac{Bn^\delta}{A_n^\delta}\left(\sum_{\nu=1}^{n}\frac{|s_\nu(x)-\sigma_\nu(x)|^{rk}}{\nu}\right)^{\frac{1}{r}}$$

$$\leqq C\left(\sum_{\nu=1}^{n}\frac{|s_\nu(x)-\sigma_\nu(x)|^{rk}}{\nu}\right)^{\frac{1}{r}}.$$

Hence

$$\int_0^1\left\{\max_{1\leqq n<\infty}\left(\frac{1}{A_n^\delta}\sum_{\nu=1}^{n}A_{n-\nu}^{\delta-1}\,|\,s_\nu(x)-\sigma_\nu(x)|^k\right)^{\frac{1}{k}}\right\}^p dx$$

$$\leqq \int_0^{2\pi}\left(\sum_{\nu=1}^{\infty}\frac{|s_\nu(x)-\sigma_\nu(x)|^{rk}}{\nu}\right)^{\frac{p}{rk}} dx.$$

If for a given k, we take s sufficiently near 1, then rk is greater than 2, because $r^{-1}+s^{-1}=1$. So by Theorem 2, we get,

$$\int_0^1\left\{\max_{1\leqq n<\infty}\left(\frac{1}{A_n^\delta}\sum_{\nu=1}^{n}A_{n-\nu}^{\delta-1}\,|\,s_\nu(x)-\sigma_\nu(x)|^k\right)^{\frac{1}{k}}\right\}^p dx$$

$$\leqq A_p\int_0^1 |f(x)|^p\,dx.$$

On the other hand we know a maximal theorem concerning $\sigma_n(x)$, that is

$$\int_0^1\left\{\max_{0\leqq n<\infty}|\sigma_n(x)|\right\}^p dx\leqq B_p\int_0^1|f(x)|^p\,dx.$$

Thus we get the theorem.

4. Trigonometric Fourier series. On the above argument, we couldn't get good theorems for the critical case $p=1$.

If we should follow the above method, the unnecessary logarithmic factors would be added to the right hand side of inequalties. However we can get satisfactory theorems in the trigonometric Fourier series. These are proved by so called complex method. For example, we get

THEOREM 4. *Let* $s_n(e^{i\theta})$ and $\sigma_n(e^{i\theta})$ *be the partial sums and the arithmetic means of Fourier power series of* $\varphi(e^{i\theta})$, *respectively. Then for* $r\geqq 2$

$$\int_0^{2\pi}\left\{\sum_{n=1}^{\infty}\frac{1}{n}\,|s_n(e^{i\theta})-\sigma_n(e^{i\theta})|^r\right\}^{\frac{1}{r}}d\theta\leqq B\left\{\int_0^{2\pi}|\varphi(e^{i\theta})|\log^+|\varphi(e^{i\theta})|\,d\theta\right\}+C,$$

$$\int_0^{2\pi}\left\{\sum_{n=1}^{\infty}\frac{1}{n}\,|s_n(e^{i\theta})-\sigma_n(e^{i\theta})|^r\right\}^{\frac{\mu}{r}}d\theta\leqq A_\mu\left\{\int_0^{2\pi}|\varphi(e^{i\theta})|\,d\theta\right\}^{\mu}\quad(0<\mu<1).$$

J. Schwartz [5] proved the integral analogue of the following Lemma, which is proved by the same method.

LEMMA 2. *Let $f_n(\theta)$ $(n=1,2,\cdots)$ be a sequence of integrable functions and let $\tilde{f}_n(\theta)$ denote the conjugate function of $f_n(\theta)$. Then, for $r>1$,*

$$\int_0^{2\pi}\left(\sum_{n=1}^{\infty}|\tilde{f}_n(\theta)|^r\right)^{\frac{1}{r}}d\theta\leqq B\int_0^{2\pi}\left(\sum_{n=1}^{\infty}|f_n(\theta)|^r\right)^{\frac{1}{r}}\log^+\left(\sum_{n=1}^{\infty}|f_n(\theta)|^r\right)^{\frac{1}{r}}d\theta+C,$$

$$\int_0^{2\pi}\left(\sum_{n=1}^{\infty}|\tilde{f}_n(\theta)|^r\right)^{\frac{\mu}{r}}d\theta\leqq D_\mu\int_0^{2\pi}\left(\sum_{n=1}^{\infty}|f_n(\theta)|^r\right)^{\frac{1}{r}}d\theta\Big\}^{\mu}\qquad(0<\mu<1).$$

From this lemma, the reduction is similar to the case $p>1$. (See Sunouchi [7]).

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