A COMPLEMENT TO "ON THE UNITARY EQUIVALENCE AMONG THE COMPONENTS OF DECOMPOSITIONS OF REPRESENTATIONS OF INVOLUTIVE BANACH ALGEBRAS AND THE ASSOCIATED DIAGONAL ALGEBRAS"

MASAMICHI TAKESAKI

(Received March 24, 1964)

After the publication of my paper, indicated in the title, I obtained the following complementary results of Remark in p.370 and Theorem 3. 4.

THEOREM. *Let $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ be two maximal abelian subalgebras of a von Neumann algebra $\boldsymbol{M}$. Let e_1 and e_2 be non-zero projections of $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ respectively. If there exists a central projection z of $\boldsymbol{M}$ such that $e_1 \leqq z$ and $e_2 \leqq (I - z)$, then $\boldsymbol{A}_1 e_1$ and $\boldsymbol{A}_2 e_2$ are unrelated. Hence if $\boldsymbol{A}_1 e_1$ and $\boldsymbol{A}_2 e_2$ are similar, then e_1 and e_2 have the same central carrier.*

PROOF. We choose a weakly dense uniformly separable C^*-subalgebra $\mathfrak{A}$ of $\boldsymbol{M}'$ which contains z. Let $\boldsymbol{A}_1 = L^\infty(\Gamma_1, \mu_1)$ and $\boldsymbol{A}_2 = L^\infty(\Gamma_2, \mu_2)$. Let E_1 and E_2 be the Borel subsets of Γ_1 and Γ_2 associated with e_1 and e_2 respectively. If we decompose the underlying Hilbert space $\mathfrak{H}$ of $\boldsymbol{M}$ and the operator x of $\mathfrak{A}$ with respect to $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ as follows;

$$\mathfrak{H} = \int_{\Gamma_1}^{\oplus} \mathfrak{H}^1(\gamma_1)\, d\mu_1(\gamma_1), \qquad \mathfrak{H} = \int_{\Gamma_2}^{\oplus} \mathfrak{H}^2(\gamma_2) d\mu_2(\gamma_2),$$

and

$$x = \int_{\Gamma_1}^{\oplus} x^1(\gamma_1) d\mu_1(\gamma_1), \qquad x = \int_{\Gamma_2}^{\oplus} x^2(\gamma_2)\, d\mu_2(\gamma_2).$$

Then we have $z^1(\gamma_1) = I$ for every $\gamma_1 \in E_1$ and $z^2(\gamma_2) = 0$ for every $\gamma_2 \in E_2$ by elimination of null sets from E_1 and E_2. Hence $\mathfrak{R}^{\boldsymbol{M}, \mathfrak{A}, \Phi^1, \Phi^2}_{\boldsymbol{A}_1, \boldsymbol{A}_2}(\gamma_1, \gamma_2)$ does not hold for every $(\gamma_1, \gamma_2) \in E_1 \times E_2$, where Φ^i is the family of the representation of $\mathfrak{A}$ defined by $x \to x^i(\gamma_i)$ $(i = 1, 2)$. The second assertion is a direct consequence of the first.

As an interpretation of the above theorem, we get the following

COROLLARY. *Let φ_1 and φ_2 be two representations of an involutive Banach algebra $\mathfrak{B}$ over Hilbert spaces $\mathfrak{H}_1$ and $\mathfrak{H}_2$. Decompose φ_1 and φ_2 into direct integrals of irreducible representations over some standard measure* *spaces* (Γ_1, μ_1) *and* (Γ_2, μ_2) *as follows*;

$$\varphi_1 = \int_{\Gamma_1}^{\oplus} \varphi_1(\gamma_1) d\mu_1(\gamma_1) \qquad \text{and} \qquad \varphi_2 = \int_{\Gamma_2}^{\oplus} \varphi_2(\gamma_2) d\mu_2(\gamma_2).$$

If φ_1 *and* φ_2 *are disjoint, then there exist null sets* $N_1 \subset \Gamma_1$ *and* $N_2 \subset \Gamma_2$ *such that* $\varphi_1(\gamma_1) \not\simeq \varphi_2(\gamma_2)$ *for every* $(\gamma_1, \gamma_2) \in \complement N_1 \times \complement N_2$. *Besides, if for each non-negligible subset* $E_i \subset \Gamma_i$ *the set* F_j *of all* γ_j*'s of satisfying the relation* $\varphi_1(\gamma_1) \simeq \varphi_2(\gamma_2)$ *for some* $\gamma_i \in E_i (i \neq j,\ i,j = 1, 2)$ *is not negligible, then* φ_1 *and* φ_2 *are quasi-equivalent.*

According to the above corollary, we can avoid the rather pathological phenomena described in Remark of p.370 in [1] and in Theorem 2 of [2].

REFERENCES

[1] M. TAKESAKI, On the unitary equivalence among the components of decompositions of representations of involutive Banach algebras and the associated diagonal algebras, Tôhoku Math. Journ., 15(1963), 365-393.

[2] M. TAKESAKI, On some representations of C^*-algebras, Tôhoku Math. Journ., 15 (1963), 79-95.

MATHEMATICAL INSTITUTE, TÔHOKU UNIVERSITY