SUMMABILITY METHODS OF BOREL TYPE AND TAUBERIAN SERIES

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1. Introduction. Let $t_p = \sum_{k=0}^{n} c_{pk}s_k$ denote a linear transformation of a sequence $s_n = \sum_{k=0}^{n} u_k$ where $\{u_k\}$ is a real or complex sequence. When a sequence $\{u_n\}$ satisfies Tauberian condition of the form $\lambda_n u_n = O(1)$[1], it is sometimes possible to estimate $\limsup |t_p - s_n|$ even when $\{s_n\}$ and $\{t_p\}$ are divergent. Such estimation was initiated by H. Hadwiger [5]. R. P. Agnew [1], [2], [3] and [4] gave such estimations for Borel, Abel and integral transforms.

In a recent paper, A.Meir [7] defined summability methods of Borel type $B(a, q)$ which contained Borel, Valiron, Euler, Taylor and S_α transformation and showed the following fact:

If $t_p = \sum_{k=0}^{\infty} c_{pk}s_k$ belongs to $B(a, q)$,

$$\limsup_{\alpha\to\infty} |\sqrt{n}\, u_n| = L < +\infty \tag{1. 1}$$

and $n = n(\alpha)$, $p = p(\alpha)$ are positive increasing functions tending to $+\infty$ as $\alpha\to\infty$ such that

$$\limsup_{\alpha\to\infty} |n - q|/\sqrt{q} = M < +\infty, \tag{1. 2}$$

then

$$\limsup_{\alpha\to\infty} |t_p - s_n| \leqq A\cdot L, \tag{1. 3}$$

where A is a finite constant depending only on M.

In the present paper, the author will consider the case

$$\limsup_{\alpha\to\infty} |n - q|/\sqrt{q} = +\infty$$

1) We have $\lambda_n = \sqrt{n}$ for Borel transforms and $\lambda_n = n$ for Abel transforms.

and show that with the same constant A, (1. 3) is also true for the series satisfying the more general Tauberian condition of Schmidt type when $\{t_p\}$ belongs to $B(a,q)$.

In section 4, we shall consider a problem on limit points of $\{t_p\}$ and $\{s_n\}$. We shall show by a counter example that the statement on limit points in [7] is not generally valid and shall give a substitute theorem on this problem.

Finally I wish to express my gratitude to Professor G. Sunouchi for his kind suggestions.

2. Summability Methods of Borel Type. After A. Meir let us say that the linear transformation $t_p = \sum_{k=0}^{\infty} c_{pk}s_k$ belongs to $B(a,q)$, if the matrix $[c_{pk}]$ satisfies the following conditions: p is a discrete or continuous parameter, a is a positive constant and $q = q(p)$ is a positive increasing function such that for every fixed δ, $\frac{1}{2} < \delta < \frac{2}{3}$

$$(2.1)\quad c_{pk} = \left(\frac{a}{\pi q}\right)^{1/2} \exp\left(-\frac{a(k-q)^2}{q}\right)\left(1 + O\left(\frac{|k-q|+1}{q}\right) + O\left(\frac{|k-q|^3}{q^2}\right)\right)$$

as $p \to \infty$ uniformly in k for $|k-q| \leqq q^{\delta}$,

$$(2.2)\qquad \sum_{|k-q|>q^{\delta}} kc_{pk} = O(\exp(-q^{\eta}))$$

where η is some positive number independent of p, and

$$(2.3)\qquad c_{pk} \geqq 0.$$

It is known that the family $B(a, q)$ with appropriate a and q contains such transformations as Borel, Valiron, S_{α}, Euler and Taylor, see [6] and [7].

THEOREM 2. 1. *Suppose that a sequence* $\{s_n\}$ $(s_n = \sum_{k=0}^{n} u_k)$ *satisfies*

$$(1.1)\qquad \limsup |\sqrt{n}\, u_n| = L < +\infty$$

and that $\{t_p\}$ *belongs to* $B(a, q)$. *Let* $n = n(\alpha)$ *and* $p = p(\alpha)$ *be integer-valued increasing functions of a parameter* α *such that*

$$\lim_{\alpha\to\infty} n(\alpha) = +\infty, \quad \textit{and} \quad \lim_{\alpha\to\infty} p(\alpha) = +\infty.$$

i) *If*

$$\limsup_{\alpha\to\infty} |n-q|/\sqrt{q} = M < +\infty, \tag{2. 4}$$

then we have

$$\limsup_{\alpha\to\infty} |t_p - s_n| \leqq A\cdot L, \tag{2. 5}$$

where
$$A = A_M = (a\pi)^{-\frac{1}{2}}\left(e^{-aM^2+2aM}\int_0^M e^{-ax^2}dx\right)$$
$$= \left(\frac{a}{\pi}\right)^{\frac{1}{2}}\int_{-\infty}^{+\infty} |x-M|\, e^{-ax^2}dx.$$

Moreover, the constant A_M *is the best possible in the sense that there exists a real sequence* $\{s_n\}$ *such that* $\limsup |\sqrt{n}\ u_n| = L < +\infty$ *and the members of* (2. 5) *are equal.*

ii) *If*
$$\lim_{\alpha\to\infty} q(p(\alpha)) = +\infty$$

and

$$\limsup_{\alpha\to\infty} |n-q|/\sqrt{q} = +\infty, \tag{2. 6}$$

then A in the formula (2. 5) *is infinite in the sense that there exists a real sequence* $\{s_n\}$ *such that* $\limsup |\sqrt{n}\ u_n| = L < +\infty$ *and* $\limsup |t_p - s_n| = +\infty$.

For the proof of this theorem we require the following lemmas.

LEMMA 2. 1. *If* $\{a_k(\alpha)\}$ *is a sequence of real functions defined for* $\alpha > 0$, *such that*

$$\limsup_{\alpha\to\infty} \sum_{k=1}^{\infty} |a_k(\alpha)| = M, \tag{2. 7}$$

where M is finite or infinite and

$$\lim_{\alpha\to\infty} a_k(\alpha) = 0 \qquad \textit{for } k = 1, 2, 3, \cdots, \tag{2. 8}$$

then each bounded real or complex sequence $\{x_n\}$ *has a transformation* $y(\alpha)=\sum_{k=1}^{\infty} a_k(\alpha)x_k$ *such that*

$$\limsup_{\alpha\to\infty} |y(\alpha)| \leqq M \limsup_{n\to\infty} |x_n|. \tag{2. 9}$$

Moreover there is a real sequence $\{x_n\}$ *such that* $0<\limsup_{n\to\infty} |x_n|<+\infty$ *and the members of* (2.9) *are equal.*

For the proof of this lemma, see R. P. Agnew [2].

LEMMA 2.2. *If the matrix* $[c_{pk}]$ *belongs to* $B(a,q)$, *then*

$$\sum_{k=0}^{\infty} c_{pk} = 1 + o(q^{-\frac{1}{2}}) \qquad \text{as } p\to\infty. \tag{2. 10}$$

The proof follows from (2.1), (2.2) and (2.3) by simple calculations.

LEMMA 2.3. *If we put* $\sum_{k=m+1}^{n} k^{-\frac{1}{2}} = \int_m^n x^{-\frac{1}{2}}\,dx - \varepsilon_{m,n}$,

where $0\leqq m<n$, *then we have*

$$0<\varepsilon_{m,n}<m^{-\frac{1}{2}} \qquad \text{when } m>1,$$

and

$$0<\varepsilon_{m,n}<2 \qquad \text{when } m=0.$$

3. Proof of Theorem 2.1. Since the first part of this theorem has been proved by A. Meir [7], we shall prove the second part.

By using Lemma 2.2 and setting $\sqrt{k}\,u_k = x_k$, we get

$$\begin{aligned} t_p - s_n &= \sum_{k=0}^{\infty} c_{pk}\, s_k - s_n \\ &= -u_0\Bigl(1-\sum_{j=0}^{\infty} c_{pj}\Bigr) - \sum_{k=1}^{n} u_k\Bigl(1-\sum_{j=k}^{\infty} c_{pj}\Bigr) + \sum_{k=n+1}^{\infty} u_k \sum_{j=k}^{\infty} c_{pj} \\ &= o(1) - \sum_{k=1}^{n} x_k k^{-\frac{1}{2}}\Bigl(1-\sum_{j=k}^{\infty} c_{pj}\Bigr) + \sum_{k=n+1}^{\infty} x_k k^{-\frac{1}{2}} \sum_{j=k}^{\infty} c_{pj}, \end{aligned}$$

because the series are absolutely convergent.

If we set

$$(3.\ 1)\qquad a_k(\alpha)=\begin{cases} -k^{-\frac{1}{2}}\left(1-\sum_{j=k}^{\infty}c_{pj}\right) & \text{for } 1\leqq k\leqq n,\\ k^{-\frac{1}{2}}\sum_{j=k}^{\infty}c_{pj} & \text{for } n<k,\end{cases}$$

then we get

$$(3.\ 2)\qquad t_p - s_n = o(1)+\sum_{k=1}^{\infty} a_k(\alpha)x_k.$$

Since (2.6) holds, there is no loss of generality in setting $\lim_{\alpha\to\infty} |n-q|/\sqrt{q} = \lim_{\alpha\to\infty} |w| = +\infty$, where $w=(n-q)/\sqrt{q}$.

1°) The case where $w/\sqrt{q}=O(1)$.
Using Lemma 2.2 and 2.3, we have

$$\begin{aligned}\sum_{k=1}^{\infty}|a_k(\alpha)| &\geqq \sum_{k=0}^{n} k^{-\frac{1}{2}}\sum_{j=0}^{k-1}c_{pj}+\sum_{k=n+1}^{\infty} k^{-\frac{1}{2}}\sum_{j=k}^{\infty}c_{pj}-|o(q^{-\frac{1}{2}})n^{1/2}|\\ &= o(1)+\sum_{k=0}^{n-1}c_{pk}\sum_{j=k+1}^{n} j^{-\frac{1}{2}}+\sum_{k=n+1}^{\infty}c_{pk}\sum_{j=n+1}^{k} j^{-\frac{1}{2}}\\ &= O(1)+2\sum_{k=1}^{\infty}|\sqrt{k}-\sqrt{n}|c_{pk}.\end{aligned}$$

Now we shall set

$$(3.\ 4)\qquad F(\alpha)=2\sum_{k=1}^{\infty}|\sqrt{k}-\sqrt{n}|c_{pk}.$$

In the case $n>q$, w is positive and therefore we get

$$n=q+w\sqrt{q}\leqq q+wq^{\delta}$$

$$\sqrt{k}+\sqrt{n}\leqq 2\sqrt{q+wq^{\delta}}\qquad \text{for } \max(1,\ q-wq^{\delta})\leqq k\leqq q+wq^{\delta},$$

and

$$|k-n|=n-k\geqq w\sqrt{q}\qquad \text{for}\quad \max(1,\ q-wq^{\delta})\leqq k\leqq q.$$

Then from (3.4) we have

$$F(\alpha) \geqq 2 \sum_{q-wq^\delta \leqq k \leqq q+wq^\delta} |\sqrt{k}-\sqrt{n}\,| c_{pk}$$

$$\geqq \frac{1}{\sqrt{q+wq^\delta}} \sum_{q-wq^\delta \leqq k \leqq q} |k-n| c_{pk}$$

$$\geqq \frac{w\sqrt{q}}{\sqrt{q+wq^\delta}} \sum_{q-wq^\delta \leqq k \leqq q} c_{pk}.$$

If we take α large enough, then we get from (2.2) and Lemma 2.2

$$F(\alpha) \geqq \frac{1}{3} \frac{w\sqrt{q}}{\sqrt{q+wq^\delta}}.$$

In the case $n < q$, we get the followings similarly for sufficiently large α

$$F(\alpha) \geqq \frac{|w|\sqrt{q}}{\sqrt{q+|w|q^\delta}} \sum_{q \leqq k \leqq q+wq^\delta} c_{pk} \geqq \frac{1}{3} \frac{|w|\sqrt{q}}{\sqrt{q+|w|q^\delta}}.$$

Consequently we have for sufficiently large α

$$F(\alpha) \geqq \frac{1}{3} \frac{|w|\sqrt{q}}{\sqrt{q+|w|q^\delta}} \tag{3.5}$$

and then

$$\limsup_{\alpha\to\infty} F(\alpha) \geqq \limsup_{\alpha\to\infty} \frac{1}{3} \frac{|w|\sqrt{q}}{\sqrt{q+|w|q^\delta}} = +\infty. \tag{3.6}$$

Since for each fixed k, we have easily

$$\lim_{\alpha\to\infty} a_k(\alpha) = 0, \tag{3.7}$$

then from Lemma 2.1, (3.3) and (3.4) we get

$$\mathrm{A} = \limsup_{\alpha\to\infty} \sum_{k=1}^{\infty} |a_k(\alpha)| = +\infty. \tag{3.8}$$

2°) The case where $\limsup_{\alpha\to\infty} |w|/\sqrt{q} = +\infty$.

Since we have

$$\frac{1}{\sqrt{q}}\sum_{k=1}^{n} k^{-\frac{1}{2}} \leqq \frac{1}{\sqrt{q}}\left(1+\int_1^n x^{-\frac{1}{2}}\,dx\right) \leqq \sqrt{\frac{n}{q}} = \left(\frac{q+|w|\sqrt{q}}{q}\right)^{1/2},$$

it follows from Lemma 2.2 and 2.3 that

$$\begin{aligned}
(3.9)\qquad \sum_{k=1}^{\infty} |a_k(\alpha)| &= \sum_{k=1}^{n}\left| - k^{-\frac{1}{2}}\sum_{j=0}^{k-1} c_{pj} + o\left(\frac{1}{\sqrt{q}}\right)n^{-\frac{1}{2}}\right| + \sum_{k=n+1}^{\infty} k^{-\frac{1}{2}}\sum_{j=k}^{\infty} c_{pj} \\
&\geqq \sum_{k=1}^{n} k^{-\frac{1}{2}}\sum_{j=0}^{k-1} c_{pj} + \sum_{k=n+1}^{\infty} k^{-\frac{1}{2}}\sum_{j=k}^{\infty} c_{pj} - \frac{1}{3}\sqrt{\frac{n}{q}} \\
&= O(1) + F(\alpha) - \frac{1}{3}\left(\frac{q+|w|\sqrt{q}}{q}\right)^{1/2}.
\end{aligned}$$

Hence we get from (3.5) and Lemma 2.2 for sufficiently large α

$$\begin{aligned}
(3.10)\qquad F(\alpha) - \frac{1}{3}\left(\frac{q+|w|\sqrt{q}}{q}\right)^{1/2} &\geqq \frac{1}{3}\left\{\frac{|w|\sqrt{q}}{\sqrt{q+|w|q^{\delta}}} - \left(\frac{q+|w|\sqrt{q}}{q}\right)^{1/2}\right\} \\
&= \frac{1}{3}\,\frac{|w|\sqrt{q}}{\sqrt{q+|w|q^{\delta}}}\left(1-\left(1+\frac{q}{|w|}\right)^{1/2}\frac{(|w|^{-1}+q^{\delta-1})^{1/2}}{\sqrt{q}}\right) \\
&\geqq \frac{1}{6}\,\frac{|w|\sqrt{q}}{\sqrt{q+|w|q^{\delta}}}.
\end{aligned}$$

Now (3.7) also holds in this case and then we get from (3.9), and (3.10)

$$\mathrm{A} = \limsup_{\alpha\to\infty}\sum_{k=1}^{\infty} |a_k(\alpha)| = +\infty.$$

Thus Theorem 2.1 is completely proved.

4. Tauberian constant and Limit points. The constant A_M mentioned above increases with M, and A_M attains to its minimum value $A_0 = (a\pi)^{-\frac{1}{2}}$ when and only when $M = 0$, that is $\lim (n-q)/\sqrt{q} = 0$.

We shall define that this constant $A_0 = (a\pi)^{-\frac{1}{2}}$ is Tauberian constant of summability method $B(a, q)$.

Now we can derive the following two theorems from Theorem 2.1. The same results on Borel transformation have been proved by R. P. Agnew [4].

THEOREM 4.1. *Let* $t_p = \sum_{k=0}^{\infty} c_{pk}s_k$ *belong to* $B(a, q)$ *and let* $q(p)$ *tend to* *infinity as* $\alpha\to\infty$. *A sequence* $\{\alpha_i\}$ *for which* α_i *tends to infinity is such that,*

$$\limsup_{\alpha_i\to\infty} |t_p - s_n| \leqq (a\pi)^{-\frac{1}{2}} \limsup_{n\to\infty} |\sqrt{n}\, u_n|, \tag{4. 1}$$

whenever $\sum u_n$ *satisfies Tauberian condition* (1. 1) *in which L is positive, if and only if*

$$\lim_{\alpha_i\to\infty} (n-q)/\sqrt{q} = 0. \tag{4. 2}$$

There is no sequence $\{\alpha_i\}$ *such that* α_i *tends to infinity and*

$$\limsup_{\alpha_i\to\infty} |t_p - s_n| < (a\pi)^{-\frac{1}{2}} \limsup_{n\to\infty} |\sqrt{n}\, u_n|, \tag{4. 3}$$

whenever $\sum u_n$ *satisfies Tauberian condition* (1. 1) *in which L is positive.*

THEOREM 4. 2. *Let* $t_p = \sum_{k=0}^{\infty} c_{pk}s_k$ *and* $q(p)$ *satisfy the same conditions as in Theorem* 4. 1.

A function $n(\alpha)$ *which is integer-valued for* $\alpha > 0$ *and tends to infinity as* $\alpha\to\infty$ *is such that,*

$$\limsup_{\alpha\to\infty} |t_p - s_n| \leqq (a\pi)^{-\frac{1}{2}} \limsup_{n\to\infty} |\sqrt{n}\, u_n|, \tag{4. 4}$$

whenever $\sum u_n$ *satisfies Tauberian condition* (1. 1) *in which L is positive, if and only if*

$$\lim_{\alpha\to\infty} (n-q)/\sqrt{q} = 0. \tag{4. 5}$$

There is no function $n(\alpha)$ *such that* $n(\alpha)$ *tends to infinity and*

$$\limsup_{\alpha\to\infty} |t_p - s_n| < (a\pi)^{-\frac{1}{2}} \limsup_{n\to\infty} |\sqrt{n}\, u_n|, \tag{4. 6}$$

whenever $\sum u_n$ *satisfies Tauberian condition* (1. 1) *in which L is positive.*

Now we take the theorem concerning limit points of $\{t_p\}$ and $\{s_n\}$. It is mentioned in [7] without proof that if (1. 1) is satisfied and Z and

Z_B denote the set of limit points of $\{s_n\}$ and $\{t_p\}$ respectively, then for each $s \in Z$ there exists at least one $t \in Z_B$ such that

$$|t - s| \leqq (a\pi)^{-\frac{1}{2}} \cdot L.$$

However this statement is not generally valid without some appropriate condition on $q(p)$. This fact is shown by the following example.

EXAMPLE. Define a sequence $\{u_n\}$ by

$$u_0 = 1 \tag{4. 7}$$

$$u_n = \begin{cases} -\nu^{-2} & (\nu^4 - \nu^3 < n \leqq \nu^4,\ \nu = 1, 2, 3, \cdots) \\ \nu^{-2} & (\nu^4 < n \leqq \nu^4 + \nu^3 + \nu^2,\ \nu = 1, 2, \cdots) \\ 0 & (\text{for other } n). \end{cases}$$

Here we can easily see

$$L = \limsup_{n\to\infty} |\sqrt{n}\, u_n| = 1 \tag{4. 8}$$

and

$$s_{\nu^4} = 0, \qquad \nu = 1, 2, 3, \cdots. \tag{4. 9}$$

Hence $s = 0$ is a point of Z.

Let a summability matrix $[c_{pk}]$ belong to $B(a, q)$, where $q = q(p)$ is a strictly increasing continuous function of p and $q(p)$ tends to infinity as $p \to \infty$.

Now we set for $\nu = 1, 2, 3, \cdots$

$$q_\nu = q(p_\nu) = \nu^4 + 2(\nu^3 + \nu^2) + \frac{\nu}{2} \tag{4.10}$$

and construct a new summability matrix $[c_{p_\nu, k}]$ from $[c_{pk}]$. Since $[c_{pk}]$ belongs to $B(a, q)$ the new matrix $[c_{p_\nu, k}]$ also belongs to $B(a, q_\nu)$. We divide the summation of $t_{p_\nu} = \sum_{k=0}^{\infty} c_{p_\nu k} s_k$ into two parts and set t'_{p_ν}, t''_{p_ν} as follows:

$$(4.11)\qquad \begin{aligned} t_{p_\nu} &= \sum_{k=0}^{\infty} c_{p_\nu k} s_k \\ &= \Big(\sum_{|k-q_\nu|\leqq q_\nu^\delta} + \sum_{|k-q_\nu|<q_\nu^\delta}\Big) c_{p_\nu k} s_k \\ &= t'_{p_\nu} + t''_{p_\nu}. \end{aligned}$$

If we take ν large enough, then we get

$$(4.12)\qquad s_k = \nu + 1 \quad \text{for } |k - q_\nu| \leqq q_\nu^\delta$$

and applying Lemma 2.2, formulas (4.9), (2.1), (2.2) and (2.3) to (4.11), we get

$$(4.13)\qquad \begin{aligned} t'_{p_\nu} &= (\nu + 1) \sum_{|k-q_\nu|\leqq q_\nu^\delta} c_{pk} \\ &\geqq (\nu + 1)\left\{\frac{1}{2} + o(1/\sqrt{q_\nu}) - O(e^{-q_\nu^\eta})\right\} \end{aligned}$$

and

$$(4.14)\qquad \begin{aligned} t''_{p_\nu} &= \sum_{|k-q_\nu|>q_\nu^\delta} c_{pk} s_k \\ &\leqq \sum_{|k-q_\nu|>q_\nu^\delta} c_{p_\nu k} = O(e^{-q_\nu^\eta}) = o(1), \end{aligned}$$

as $\nu \to \infty$.

Then we get from (4.11), (4.13) and (4.14) $\liminf_{\nu\to\infty} t_{p_\nu} = +\infty$ and thus we have shown that there is no point $t \in Z_B$ such that $\limsup |t - s| \leqq (a\pi)^{-\frac{1}{2}}$ for $s = 0 \in Z$.

A.Meir have proved in [7], using the sequence $\{u_n\}$ defined by (4.7), that the least constant A satisfying the condition that for every sequence $\{s_n\}$ which satisfies Tauberian condition (1.1) and each $s \in Z$, there should exist at least one $t \in Z_B$ such that $|t - s| \leqq A \cdot L$, is $A_0 = (a\pi)^{-\frac{1}{2}}$

If we assume that $q = q(p)$ is continuous and tends to infinity as $p \to \infty$, then using Theorem 2.1 the statement mentioned above is valid and the following theorem shows that this statement is true to some generalized condition on $q(p)$.

THEOREM 4.3. *Let a summability matrix belong to* $B(a, q)$ *where* $q = q(p)$ *is a increasing function of* p *and tends to infinity as* $p \to \infty$, *and let a sequence* $\{s_n\}$ *satisfy the Tauberian condition* (1.1). $J(p)$ *denotes* $q(p+0) - q(p-0)$.

If we suppose $J(p) = o(\sqrt{q(p)})$[2], *then for each* $s \in Z$ *there exists at least one* $t \in Z_B$ *such that*

$$|t - s| \leqq (a\pi)^{-\frac{1}{2}} \cdot L.$$

Using Theorem 2.1 we can easily prove this theorem.

5. Schmidt Condition. It is well known and is easy to prove that if a series $\sum u_n$ satisfies Tauberian condition (1.1), then its partial sums $s_n = \sum_{k=0}^{n} u_k$ satisfy the more general Tauberian condition of Schmidt type

(5.1) $\limsup_{p\to\infty} \max_{|q-p| \leqq \lambda\sqrt{p}} |s_q - s_p| \leqq L \cdot \lambda$, where λ is positive.

Letting $n(\alpha)$, $p(\alpha)$ be defined as in section 1, we shall determine the least constant A' which depends upon the functions $n(\alpha)$ and $q(p)$ such that

$$\limsup_{\alpha\to\infty} |t_p - s_n| \leqq A'L, \tag{5.2}$$

where the sequence $\{s_n\}$ satisfies Schmidt condition (5.1).

Now we shall prove the following theorem.

THEOREM 5.1. *Suppose that the sequence* $\{s_n\}$ $(s_n = \sum_{k=0}^{n} u_k)$ *satisties Schmidt condition* (5.1) *and* $\{t_p\}$ *belongs to* $B(q, q)$, *where* $q(p)$ *tends to infinity as* $p \to \infty$ *and* $n(\alpha)$, $p(\alpha)$ *are the functions of parameter* α *as in Theorem* 2.1.

Then the least constant A' *for which* (5.2) *holds is equal to the constant* A *in Theorem* 2.1.

Introducing the following two lemmas, we can prove this theorem with the same method as in [4] by R. P. Agnew.

LEMMA 5.1. *Suppose that the sequence* $\{s_n\}$ *satisfies Schmidt condition* (5.1), $n(\alpha)/q(\alpha) \to 1$ *as* $\alpha \to \infty$, *and*

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2) If $p = p_i$ $(i = 1, 2, \cdots)$, then the condition is replaced by $q(p_i) - q(p_{i-1}) = o(\sqrt{q(p_{i-1})})$.

$$f(\alpha)=\max_{|k-n|\leqq X(\alpha)\sqrt{q}}|s_k-s_n|, \tag{5. 3}$$

where $X(\alpha)$ *is bounded. Then for each* $\varepsilon>0$ *there exists a number* α_0 *such that*

$$f(\alpha)< L\cdot X(\alpha)+\varepsilon, \qquad \alpha>\alpha_0. \tag{5. 4}$$

For the proof, see R. P. Agnew [4].

LEMMA 5.2. *If*

$$f(\alpha)=\sum_{k=1}^{n} g_k(\alpha)h_k(\alpha) \quad , \tag{5. 5}$$

where $g_k(\alpha)$ *and* $h_k(\alpha)$ *are nonnegative and bounded and* $\lim_{\alpha\to\infty} g_k(\alpha)=G_k$ *for each* k, *then*

$$\limsup_{\alpha\to\infty} f(\alpha)=\limsup_{\alpha\to\infty}\sum_{k=1}^{n} G_k h_k(\alpha). \tag{5. 6}$$

PROOF OF THEOREM 5.1. Since each series satisfying Tauberian condition (1.1) also satisfies Schmidt condition (5.1), it is evident from the definition of A and A' in (2.5) and (5.2) that $A'\geqq A$. Then we can prove $A'=A$, provided that we show $A'\leqq A$. In the case where $\limsup |n-q|/\sqrt{q}=\limsup|w|=+\infty$ we have $A=+\infty$ from (ii) of Theorem 2.1 and the inequality $A'\leqq A$ is evidently satisfied. Next, we consider the case where $\limsup|w|=M<+\infty$.

Since the sequence $\{s_n\}$ satisfies Schmidt condition (5.1), we can easily obtain $s_n=O(\sqrt{n})$. From this fact and Lemma 2.2 and since $n/q\to 1$ as $\alpha\to\infty$, we have

$$\begin{aligned} t_p-s_n &= \sum_{k=0}^{\infty} c_{pk}s_k - s_n\left(\sum_{k=0}^{\infty} c_{pk}+o(1/\sqrt{q})\right) \\ &= \sum_{k=0}^{\infty} c_{pk}(s_k-s_n)+o(1) \\ &= H(\alpha)+o(1), \end{aligned} \tag{5. 7}$$

where

$$H(\alpha)=\sum_{k=0}^{\infty} c_{pk}(s_k - s_n). \tag{5.8}$$

Now we shall estimate $|H(\alpha)|$, dividing summation of (5.8) into three parts and we set $H_1(\alpha)$, $H_2(\alpha)$ $H_3(\alpha)$ as follows:

$$\begin{aligned} |H(\alpha)| &\leqq \sum_{k=0}^{\infty} |s_k - s_n| c_{pk} \\ &= \left(\sum_{|k-q|\leqq T\sqrt{q}} + \sum_{T\sqrt{q}<|k-q|q^{\delta}} + \sum_{|k-q|>q^{\delta}}\right) |s_k - s_n| c_{pk} \\ &= H_1(\alpha)+ H_2(\alpha)+ H_3(\alpha), \end{aligned} \tag{5.9}$$

where T is a fixed constant large enough.

At first we consider $H_1(\alpha)$. We have

$$\begin{aligned} H_1(\alpha) &= \sum_{|k-q|\leqq T\sqrt{q}} |s_k - s_n| c_{pk} \\ &= \sum_{r=-N}^{N-1} \sum_{k\in E(r,\alpha)} |s_k - s_n| c_{pk} \\ &\leqq \sum_{r=-N}^{N-1} f_r(\alpha) \sum_{k\in E(r,\alpha)} c_{pk}, \end{aligned} \tag{5.10}$$

where $E(r, \alpha)$ is the set of nonnegative integer k for which

$$q + r\frac{T}{N}\sqrt{q} \leqq k \leqq q + (r+1)\frac{T}{N}\sqrt{q} \tag{5.11}$$

and

$$\begin{aligned} f_r(\alpha) &= \max_{k\in E(r,\alpha)} |s_k - s_n| \\ &= \max_{\left(r\frac{T}{N}-w\right)\sqrt{q}\leqq k-n\leqq\left((r+1)\frac{T}{N}-w\right)\sqrt{q}} |s_k - s_n|. \end{aligned} \tag{5.12}$$

Appliying Lemma 5.1 to $f_r(\alpha)$ in (5.10) we find that there is a number α_0 such that for each integer r satisfying $-N \leqq r \leqq N-1$ and for each $\varepsilon > 0$,

$$f_r(\alpha) \leqq L\cdot X(r,\alpha) + \varepsilon, \qquad \alpha > \alpha_0, \tag{5.13}$$

where

$$X(r, \alpha)= \max\left(|r\frac{T}{N} - w|,\ (r+1)\frac{T}{N} - w|\right). \tag{5.14}$$

Then we have

$$\begin{aligned} H_1(\alpha) &\leqq \sum_{r=-N}^{N-1} (L\cdot X(r, \alpha)+ \sum_{k\in E(r,\alpha)} c_{pk} \\ &\leqq \varepsilon + o(1)+ L\sum_{r=-N}^{N-1} X(r, \alpha) \sum_{k\in E(r,\alpha)} c_{pk}. \end{aligned} \tag{5.15}$$

Since the summability matrix $[c_{pk}]$ belongs to $B(a, q)$ we get for sufficiently large α

$$\begin{aligned} \sum_{k\in E(r,\alpha)} c_{pk} &= \sum_{rT/N\leqq(k-q)/\sqrt{q}\leqq(r+1)T/N} c_{pk} \\ &= \sqrt{\frac{a}{\pi q}} \sum_{rT/N\leqq(k-q)/\sqrt{q}\leqq(r+1)T/N} \exp\left(\frac{(-a(k-q)^2}{q}\right)\left(1+O\left(\frac{|k-q|+1}{\sqrt{q}}\right)+O\left(\frac{|k-q|^3}{q^2}\right)\right) \\ &= \sqrt{\frac{a}{\pi}} \int_{rT/N}^{(r+1)T/N} \exp(-ax^2)\left(1+O\left(\frac{|x|+1}{\sqrt{q}}\right)+O\left(\frac{|x|^3}{\sqrt{q}}\right)\right)dx + o(1) \\ &= \sqrt{\frac{a}{\pi}} \int_{rT/N}^{(r+1)T/N} \exp(-ax^2)dx + o(1). \end{aligned} \tag{5.16}$$

Applying Lemma 5. 2 to (5. 15) we have

$$\begin{aligned} \limsup_{\alpha\to\infty} H_1(\alpha) &= L\cdot\limsup_{\alpha\to\infty} \sum_{r=-N}^{N-1} X(r, \alpha)\sqrt{\frac{a}{\pi}} \int_{rT/N}^{(r+1)T/N} \exp(-ax^2)dx \\ &= L\cdot\sqrt{\frac{a}{\pi}} \limsup_{\alpha\to\infty} H_1'(\alpha), \end{aligned} \tag{5.17}$$

where

$$H_1'(\alpha)= \sum_{r=-N}^{N-1} X(r, \alpha) \int_{rT/N}^{(r+1)T/N} \exp(-ax^2)dx. \tag{5.18}$$

By the definition of $X(r, \alpha)$, (5. 14), when $rT/N\leqq x_r \leqq(r+1)T/N$ and when N is large enough, from (5.18) we have the following inequality

$$(5.19)\qquad H_1'(\alpha)\leqq\sum_{r=-N}^{N-1}\left(\frac{T}{N}+|x_r-w|\int_{rT/N}^{(r+1)T/N}\exp(-ax^2)dx\right)$$
$$=\frac{T}{N}\int_{-T}^{T}\exp(-ax^2)dx+\int_{-T}^{T}|x-w|\exp(-ax^2)\,dx+\varepsilon.$$

If we take T large enough and for this T we take N large enough such as

$$(5.20)\qquad \int_{|x|>T}|x-w|^{-ax^2}\,dx<\varepsilon \text{ and } \frac{T}{N}\int_{-T}^{T}e^{-ax^2}\,dx<\varepsilon,$$

then we get from (5.17), (5.19) and (5.20)

$$(5.21)\qquad \limsup_{\alpha\to\infty} H_1(\alpha)\leqq 3\,\varepsilon+L\sqrt{\frac{a}{\pi}}\int_{-\infty}^{+\infty}|x-M|\,e^{-ax^2}dx.$$

In the next place we shall estimate $H_2(\alpha)$ defined by (5. 9). Since the sequence $\{s_n\}$ satisfies Schmidt condition (5. 1) and from the fact that $|k-n|>\sqrt{n}$ for $T\sqrt{q}<|k-n|\leqq q^\delta$, we obtain

$$(5.22)\qquad |s_k-s_n|\leqq c|\sqrt{k}-\sqrt{n}\,| \quad\text{for}\quad T\sqrt{q}<|k-n|\leqq q^\delta$$

where c is a constant independent of k and n.

Then we have

$$(5.23)\qquad H_2(\alpha)=\sum_{T\sqrt{q}<|k-q|\leqq q^\delta}|s_k-s_n|\;c_{pk}\leqq c\sum_{T\sqrt{q}<|k-q|\leqq q^\delta}|\sqrt{k}-\sqrt{n}|\,c_{pk}$$
$$\leqq\frac{c}{\sqrt{n}}\sum_{T\sqrt{q}<|k-q|\leqq q^\delta}|k-n|\,c_{pk}$$
$$\leqq\frac{c}{\sqrt{n}}\sum_{T\sqrt{q}<|k-q|\leqq q^\delta}(|k-q|+|w|\sqrt{q}\,)c_{pk}$$
$$=\frac{c}{\sqrt{n}}\sum_{T\sqrt{q}<|k-q|\leqq q^\delta}|k-q|\,c_{pk}+\frac{c|w|}{\sqrt{n}}\sum_{T\sqrt{q}<|k-q|\leqq q^\delta}\sqrt{q}\;c_{pk}$$
$$=H_{21}(\alpha)+H_{22}(\alpha),$$

where

$$H_{21}(\alpha)=\frac{c}{\sqrt{n}}\sum_{T\sqrt{q}<|k-q|\leqq q^\delta}|k-q|\,c_{pk},$$

and
$$H_{22}(\alpha)=\frac{c|w|}{\sqrt{n}}\sum_{T\sqrt{q}<|k-q|\leqq q^{\delta}}\sqrt{q}\,c_{pk}.$$

Since the summability matrix $[c_{pk}]$ belongs to $B(a,q)$, we have for sufficiently large α

$$H_{21}(\alpha)=\frac{c}{\sqrt{n}}\sum_{T\sqrt{q}<|k-q|\leqq q^{\delta}}|k-q|\sqrt{\frac{a}{\pi q}}\exp\left(\frac{-a(k-q)^2}{q}\right)\left(1+O\left(\frac{|k-q|+1}{q}\right)\right)$$
$$\leqq\frac{2c}{\sqrt{n}}\sqrt{\frac{qa}{\pi}}\int_T^{\infty}xe^{-ax^2}\,dx+O\left(\frac{1}{\sqrt{n}}\int_T^{\infty}x^2e^{-ax^2}\,dx\right)$$
$$+O\left(\frac{1}{\sqrt{n}}\int_T^{\infty}x^4e^{-ax^2}\,dx\right)+o(1).$$

If we take T large enough, we get

$$H_{21}(\alpha)\leqq\varepsilon+o(1). \tag{5.24}$$

At the same time we obtain for sufficiently large T

$$H_{22}(\alpha)=\frac{c|w|}{\sqrt{n}}\sum_{T\sqrt{q}<|k-q|\leqq q^{\delta}}\sqrt{q}\,c_{pk} \tag{5.25}$$
$$=O\left(\sum_{T<|k-q|\sqrt{q}}c_{pk}\right)<\varepsilon.$$

From (5.23), (5.24) and (5.25) we consequently obtain

$$H_2(\alpha)\leqq 2\varepsilon+o(1). \tag{5.26}$$

Finally we consider $H_3(\alpha)$ defined by (5.9). Since $n/q\to 1$ as $\alpha\to\infty$ and

$$|\sqrt{k}-\sqrt{n}|\leqq n \qquad \text{for } k<q-q^{\delta},$$
$$|\sqrt{k}-\sqrt{n}|\leqq k \qquad \text{for } q+q^{\delta}<k,$$

and from (5.22) and Lemma 2.2, we have

$$H_3(\alpha)=\sum_{|k-q|>q^{\delta}}|s_k-s_n|c_{pk} \tag{5.27}$$

$$\leqq c \sum_{|k-q|>q^{\delta}} |\sqrt{k} - \sqrt{n}|c_{pk}$$

$$\leqq c\left(\sum_{0 \leqq k < q - q^{\delta}} + \sum_{q+q^{\delta}<k}\right) |\sqrt{k} - \sqrt{n}|c_{pk}$$

$$\leqq c(n+1) \sum_{|k-q|>q^{\delta}} kc_{pk}$$

$$= O(q\exp(-q^{\eta}) + \exp(-q^{\eta})) = o(1).$$

From (5.9), (5.21), (5.26) and (5.27) we obtain

$$\limsup_{\alpha\to\infty} |H(\alpha)| \leqq \limsup_{\alpha\to\infty} H_1(\alpha) + \limsup_{\alpha\to\infty} H_2(\alpha) + \limsup_{\alpha\to\infty} H_3(\alpha)$$
$$\leqq 5\varepsilon + L\sqrt{\frac{a}{\pi}} \int_{-\infty}^{+\infty} |x - M| e^{-ax^2} dx. \tag{5.28}$$

Since (5.28) holds for each $\varepsilon > 0$, it implies that

$$\limsup_{\alpha\to\infty} |H(\alpha)| \leqq L\sqrt{\frac{a}{\pi}} \int_{-\infty}^{+\infty} |x - M| e^{-ax^2}\, dx$$

and therefore

$$A' \leqq \sqrt{\frac{a}{\pi}} \int_{-\infty}^{+\infty} |x - M| e^{-ax^2} dx = A.$$

Thus we get $A' = A$ and prove Theorem 5.1, completely.

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