ON THE REALIZABILITY OF WHITEHEAD PRODUCTS

KOICHI IWATA

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1. Introduction. Let π_2, π_3, and π_4 be abelian groups and let $T_{2,2} : \pi_2 \otimes \pi_2 \rightarrow \pi_3$, $T_{2,3} : \pi_2 \otimes \pi_3 \rightarrow \pi_4$ be bilinear homomorphisms. In this paper, we shall give a necessary and sufficient condition under which the given homomorphisms $T_{2,2}$ and $T_{2,3}$ are realizable simultaneously as the Whitehead product operations in spaces of the type $K(\pi_2,2\ ;\ \pi_3,3\ ;\ k^{(4)};\ \pi_4,4\ ;\ k^{(5)}\ ;\ \cdots)$ in the case π_2 is free. This problem was handled by H. Miyazaki [5], but his solution is not complete as was pointed out by P. J. Hilton in [3].

Our present method depends on the theory of cohomology operations by F. P. Peterson [7], [8]. In §2, we state some properties of the Eilenberg-MacLane complex $K_N(\pi,2)$ and its homology, for later use. In §3 we show a cohomology relation, giving a connection between Postnikov invariants and Whitehead products in a space. In the last section the realizability theorem is stated and proved.

2. The complex $K_N(\pi, 2)$ and its homology. Let π be an abelian group. Following Eilenberg-MacLane [2], we recall some properties of the R-complex $K_N(\pi, 2)$ with multiplication Δ.

For each $u \in \pi$ and each integer $t \geqq 0$, there corresponds a $2t$-cycle $\kappa_{2t}(u, 2)$ of $K_N(\pi, 2)$ which satisfies the equations

$$\kappa_0(u, 2) = 1,$$

$$\kappa_{2s}(u, 2)\Delta\kappa_{2t}(u, 2) = \binom{s+t}{s} \kappa_{2(s+t)}(u, 2)$$

and the following homologies

$$\kappa_{2t}(ru, 2) \sim r^t\kappa_{2t}(u, 2),$$

$$\kappa_{2t}(u + v, 2) \sim \sum_{i+j=t} \kappa_{2i}(u, 2)\Delta\kappa_{2j}(v, 2).$$

By [2], II, Theorem 21.1, under the mapping

$$\kappa_{2t}(u, 2) \rightarrow \text{homology class of } \kappa_{2t}(u, 2)$$

$H_{2t}(\pi, 2)$, for $t \leqq 3$, is isomorphic to the commutative graded ring (with multiplication Δ) which is generated by $\kappa_{2s}(u, 2)$ for each $u \in \pi$ and integer $s \leqq t$, subject to the above equations and homologies.

In low dimensions, the FD-structure of $\kappa_{2t}(u, 2)$ is as follows.

$$F_0\kappa_2(u,2) = F_1\kappa_2(u, 2) = F_2\kappa_2(u, 2) = 1_1 \text{ (unit one cell)},$$

$$\kappa_4(u, 2) = (D_3D_2\kappa_2(u, 2))(D_1D_0\kappa_2(u,2)) - (D_3D_1\kappa_2(u, 2))(D_2D_0\kappa_2(u, 2)) + (D_2D_1\kappa_2(u, 2))(D_3D_0\kappa_2(u, 2)),$$

$$\kappa_6(u, 2) = \sum{}' (-1)^{\varepsilon(\mu)}(D_{\nu_4}D_{\nu_3}D_{\nu_2}D_{\nu_1}\kappa_2(u, 2))(D_{\mu_2}D_{\mu_1}\kappa_4(u, 2)),$$

$$\kappa_2(u, 2)\Delta\kappa_4(v, 2) = \sum (-1)^{\varepsilon(\mu)}(D_{\nu_4}D_{\nu_3}D_{\nu_2}D_{\nu_1}\kappa_2(u, 2))\,(D_{\mu_2}D_{\mu_1}\kappa_4(v, 2)),$$

where $\varepsilon(\mu) = \sum_{i=1}^{p} (\mu_i - (i-1))$ and the sums $\sum$ (resp. $\sum'$) being taken over all (2.4) shuffles (μ, ν) (resp. all (2.4) shuffles (μ, ν) such that $\mu_1 = 0$).

In the following sections, we often use the symbol $\kappa_{2t}(u, 2)$ to denote the homology class of $\kappa_{2t}(u, 2)$, if no confusion is expected.

3. A cohomology relation. Let π_2, π_3 be abelian groups and $k^{(4)} \in H^4(\pi_2, 2; \pi_3)$. Then there exists a space E of the type $K(\pi_2, 2; \pi_3, 3; k^{(4)})$. E is considered as the total space of a principal fibre space in the sense of [6], whose base space B is of the type $K(\pi_2, 2)$ and fibre F is of the type $K(\pi_3, 3)$. We identify

$$\pi_2 = \pi_2(E) = p_{\#}^{-1}\, \pi_2(B), \qquad \pi_3 = \pi_3(E) = i_{\#}\pi_3(F),$$

where $p: E \to B$ is the projection and $i: F \to E$ is the inclusion.

Under the identification $H^4(\pi_2, 2; \pi_3) = \operatorname{Hom}(H_4(\pi_2, 2); \pi_3)$, we define a map $\eta: \pi_2 \to \pi_3$ by

$$\eta(u) = k^{(4)}\kappa_4(u, 2), \qquad u \in \pi_2.$$

Then for any $u, v \in \pi_2(E)$, the Whitehead product $[u, v] \in \pi_3(E)$ is

$$[u, v] = \eta(u + v) - \eta(u) - \eta(v).$$

Let π_4 be an abelian group and assume that a bilinear homomorphism $T_{2,3}: \pi_2 \otimes \pi_3 \to \pi_4$ and a homomorphism $E\eta: \pi_3/2\pi_3 \to \pi_4$ are given. By [2], III, Theorem 17.4, $E\eta$ determines the class $\theta \in H^6(\pi_3, 4; \pi_4)$. Moreover, let

$\psi \in H^2(\pi_2, 2; \pi_2)$ be the fundamental class. Then we have

PROPOSITION. *In case* Ext $(H_5(\pi_2, 2); \pi_4)= 0$, *the relation*

$$\theta(k^{(4)}) + \psi \cup k^{(4)} = 0,$$

where the cup product $\cup$ *is taken relative to* $T_{2,3}$, *holds in* $H^6(\pi_2, 2; \pi_4)$ *if and only if there exist the following relations*:

$$T_{2,3}(u \otimes \eta(u)) = 0,$$
$$T_{2,3}(u \otimes \eta(v)) + T_{2,3}(v \otimes [u,v]) = E\eta[u, v],$$

for all $u, v \in \pi_2$.

PROOF. By the hypotheses, we can identify $H^6(\pi_2, 2 ; \pi_4) = \mathrm{Hom}(H_6(\pi_2, 2); \pi_4)$. By the definition of $\cup$-product, we have

$$\begin{aligned}\psi \cup k^{(4)}\kappa_6(u, 2) &= T_{2,3}(\psi(F_3F_4F_5F_6\kappa_6(u, 2)) \otimes k^{(4)}(F_0F_1\kappa_6(u, 2))) \\ &= T_{2,3}(\psi\kappa_2(u, 2) \otimes k^{(4)}\kappa_4(u, 2)) \\ &= T_{2,3}(u \otimes \eta(u)),\end{aligned}$$
$$\psi \cup k^{(4)}(\kappa_2(u, 2)\Delta\kappa_4(v, 2)) = T_{2,3}(u \otimes \eta(v)) + T_{2,3}(v \otimes [u, v]).$$

On the other hand, by [2], III, Theorm 17.4,

$$\theta(k^{(4)}) = k^{(4)} \cup_2 k^{(4)},$$

where the $\cup_2$-product is taken relative to the pairing $\phi : \pi_3 \otimes \pi_3 \to \pi_4$ such that

$$\phi(\alpha,\alpha) = E\eta(a),$$
$$\phi(\alpha, \beta) + \phi(\beta, \alpha) = 0,$$
$$2\phi(\alpha, \beta) = 0,$$

for any $\alpha, \beta \in \pi_3$. By the definition of $\cup_2$-product ([9]), we get

$$\begin{aligned}k^{(4)} \cup_2 k^{(4)}(\kappa_6(u, 2)) = {}& \phi\ (k^{(4)}(F_5F_6\kappa_6(u,2)), k^{(4)}(F_2F_3\kappa_6(u, 2))) \\ &+ \phi(k^{(4)}(F_5F_6\kappa_6(u, 2)),\ k^{(4)}(F_0F_3\kappa_6(u, 2))) + \phi(k^{(4)}(F_5F_6\kappa_6(v, 2)), k^{(4)}(F_0F_1\kappa_6(u,2))) \\ &+ \phi(k^{(4)}(F_1F_6\kappa_6(u, 2)),\ k^{(4)}(F_3F_4\kappa_6(u, 2))) + \phi(k^{(4)}(F_2F_6\kappa_6(u, 2)), k^{(4)}\ (F_0F_4\kappa_6(u, 2))) \\ &+ \phi(k^{(4)}(F_3F_6\kappa_6(u, 2)),\ k^{(4)}(F_0F_1\kappa_6(u, 2))) + \phi(k^{(4)}(F_1F_2\kappa_6(u, 2)), k^{(4)}(F_4F_5\kappa_6(u, 2))) \\ &+ \phi(k^{(4)}(F_2F_3\kappa_6(u, 2)),\ k^{(4)}(F_0F_5\kappa_6((u, 2))) + \phi\ (k^{(4)}(F_3F_4\kappa_6(u, 2)), k^{(4)}(F_0F_1\kappa_6(u,2))) \\ &= \phi([u, u],\ [u,u]) = E\eta(2\eta(u)) = 0,\end{aligned}$$

and similarly

$$k^{(4)}\cup_2 k^{(4)}(\kappa_2(u,2)\Delta\kappa_4(v,2))=\phi([u,v],\ [u,v])=E\eta[u,v].$$

The conclusion follows immediately.

4. The main theorem. Let $\eta\in\pi_3(S^2)$ be the Hopf class and $E\eta\in\pi_4(S^3)$ be its suspension. In any space X, it is well known that the following relations hold:

$$[u,v]=(u+v)\circ\eta-u\circ\eta-v\circ\eta,$$
$$(-u)\circ\eta=u\circ\eta,$$
$$[u,u\circ\eta]=0,$$
$$[u,v\circ\eta]+[v,[u,v]]=[u,v]\circ E\eta,$$

for $u,v\in\pi_2(X)$. Using the result of the previous section, we shall prove the following realizability theorem.

THEOREM. *Let π_2, π_3 and π_4 be abelian groups such that* Ext $(H_5(\pi_2,2);\pi_4)=0$. *For given bilinear homomorphisms $T_{2,2}:\pi_2\otimes\pi_2\to\pi_3$ and $T_{2,3}:\pi_2\otimes\pi_3\to\pi_4$, there exists a space of the type $K(\pi_2,2;\pi_3,3;k^{(4)};\ \pi_4,4;k^{(5)};\cdots)$ in which $T_{2,2}$, $T_{2,3}$ are realized simultaneously as the Whitehead product operations if and only if the following conditions hold*:

(i) *There exists a map $\eta:\pi_2\to\pi_3$ such that,*

$$T_{2,2}(u\otimes v)=\eta(u+v)-\eta(u)-\eta(v),$$
$$\eta(-u)=\eta(u),$$

for $u,v\in\pi_2$.

(ii) *There exists a homomorphism $E\eta:\pi_3/2\pi_3\to\pi_4$, such that*

$$T_{2,3}(u\otimes\eta(u))=0,$$
$$T_{2,3}(u\otimes\eta(v))+T_{2,3}(v\otimes T_{2,2}(u\otimes v))=E\eta(T_{2,2}(u\otimes v)),$$

for $u,v\in\pi_2$.

REMARK. By [2], II, Theorem 22.1, there exists the epimorphism

$$\mathrm{Tor}\ (\pi_2,\pi_2)+\Gamma_4({}_2\pi_2)\to H_5(\pi_2,2).$$

Therefore, when π_2 has no element of finite order or is cyclic of finite order prime to 2, $H_5(\pi_2,2)$ vanishes and the condition Ext $(H_5(\pi_2,2);\pi_4)=0$ is satisfied.

PROOF. The necessity is stated above. Therefore, to prove the theorem, it is sufficient to show the existence of a space realizing $T_{2,2}$, $T_{2,3}$. If the condition (i) holds, then by [4] Theorem 1, there exists a space E of the type $K(\pi_2,2;\pi_3,3;k^{(4)})$ in which $T_{2,2}$ is realized. Now, suppose that the condition (ii) holds. As in §3, consider the principal fibre space (E, p, B) with fibre F. By the proposition of §3, for $\theta \in H^6(\pi_3, 3; \pi_4)$ associated with $E\eta$, the relation $(\theta + \psi \cup) k^{(4)} = 0$ holds. Since $p^* k^{(4)} = 0$, we can define ([7] p. 297)

$$(\theta + \psi \cup)_p k^{(4)} \in H^5(E, \pi_4)/p^* H^5(B, \pi_4) + ({}^1\theta + p^*\psi \cup) H^3(E, \pi_3),$$

where ${}^1\theta$ denotes the suspension of θ. Let k_1 be a representative of $(\theta + \psi \cup)_p k^{(4)}$ and let $\mu : F \times E \to E$ be the operation of F n E. Then, by [8] Theorem 1,

$$\mu^* k_1 = 1 \otimes k_1 + i^* k_1 \otimes 1 + x' \otimes p^* \psi,$$

where ${}^1\theta(x') = {}^1\theta(\iota) = i^* k_1$ (ι denotes of the fundametal class of $H^3(F, \pi_3)$).

As ${}^1\theta$ is additive, we have ${}^1\theta(\iota - x') = 0$. Hence $\iota - x' = 2x''$ for some $x'' \in H^3(F, \pi_3)$. In the spectral sequence associated with (E, p, B), $\psi \otimes (\iota - x')$ defines a class of $E_2^{2,3} \approx E_4^{2,3}$, and $d_4\{\psi \otimes (\iota - x')\} = \{\psi \cup 2\bar{\bar{x}}'' k^{(4)}\} = \{2\bar{x}''(\psi \cup k^{(4)})\} = \{\bar{\bar{x}}'' 2\theta(k^{(4)})\} = \{0\} \in E_4^{6,0}$, where $\bar{x}''$, $\bar{\bar{x}}''$ are endomorphisms of $H^4(B, \pi_3)$, $\psi \cup H^4(B, \pi_3)$ induced by x'' respectively. Thus $\psi \otimes (\iota - x')$ defines a class of $E_\infty^{2,3}$ which is represented by some element $k_0 \in H^5(E, \pi_4)$, and (see e. g. [1])

$$\mu^* k_0 = (\iota - x') \otimes p^* \psi + 1 \otimes k_0,$$
$$i^* k_0 = 0.$$

Let $k^{(5)} = k_0 + k_1$. Then we have

$$\mu^* k^{(5)} = \mu^* k_0 + \mu^* k_1 = 1 \otimes k^{(5)} + i^* k^{(5)} \otimes 1 + \iota \otimes p^* \psi,$$
$$i^* k^{(5)} = {}^1\theta(\iota).$$

Let $f : E \to K(\pi_4, 5)$ be a map representing the homotopy class determined by $k^{(5)}$. f induces a principal fibre space (X, q, E) with fibre Y of the type $K(\pi_4, 4)$. We identify $\pi_2(X) = q_\#^{-1} \pi_2(E)$, $\pi_3(X) = q_\#^{-1} \pi_3(E)$ and $\pi_4(X) = j_\# \pi_4(Y) = \pi_4$, where $j : Y \to X$ is the inclusion. Then by [8] Theorem 2, for $u \in \pi_2(X) = \pi_2$, $\alpha \in \pi_3(X) = \pi_3$,

$$[u, \alpha] = T_{2,3}(u \otimes \alpha),$$

thus $T_{2,3}$ is realized in X. As it is supposed that $T_{2,2}$ is realized in E, under the above identification, $T_{2,3}$ is realized in X. The theorem is proved.

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TÔHOKU UNIVERSITY.