A REMARK ON TRANSFORMATIONS OF A K-CONTACT MANIFOLD

SHÛKICHI TANNO

(Received February 1, 1964)

1. K-contact manifold is a differentiable manifold with a contact metric structure (ϕ, ξ, η, g) such that ξ is a Killing vector field. Especially a normal contact manifold is a K-contact manifold. In this report, the completeness of g is assumed. Keeping the notations and terminologies as in [1] and [2], we prove theorems A, B and C. Of course, as usual, manifolds are supposed to be connected.

THEOREM A. *If a complete K-contact manifold M is not an R-product bundle over an almost Kähler manifold* $B = M/\xi$. *The identity component* Φ^0 *of the Lie group* Φ *composed of all transformations leaving* ϕ *invariant coincides with the identity component* $\mathbf{A}^0$ *of the automorphism group* $\mathbf{A}$.

To prove this it suffices to prove the following

LEMMA. *If a complete K-contact manifold M is not homeomorphic to Euclidean space of the same dimension* $2n+1$ $(\geqq 3)$ *as M. Then there does not exist any infinitesimal transformation X which leaves* ϕ *invariant and satisfies* $\mathfrak{L}(X)\eta = \eta$.

In fact, considering the contraposition of the Theorem, if Φ^0 does not coincide with $\mathbf{A}^0$, we have an infinitesimal transformation X such that $\mathfrak{L}(X)\phi = 0$ and $\mathfrak{L}(X)\eta = \eta$. Here by virtue of the Lemma M is homeomorphic to Euclidean space, particularly M is simply connected. Therefore M is an R-product bundle over an almost Kähler manifold M/ξ [2].

Now we demonstrate the Lemma in its contraposition. As M is complete X generates a global 1-parameter group of transformations $\gamma_t = \exp tX$ $(-\infty < t < \infty)$ such that $\gamma_t^*\eta = e^t\eta$ [1-II]. We see that γ_t has at least one fixed point. Moreover $\gamma_t (t \neq 0)$ cannot have two fixed points, as is seen from the fact that for a curve l of finite length its image $\gamma_t(l)$ has longer or shorter length than that of l according to positive or negative sign of t. Since the fixed point of γ_t does not depend on t, we have the following

(i). There exists a point p in M which is a unique fixed point of γ_t for all t. Thus $X = 0$ at p and $X \neq 0$ in $M - p$.

(ii). For any point $x(\neq p)$ in M, the part $(\gamma_t(x): 0 \leqq t < \infty)$ of the trajectory is infinite, while that of $(\gamma_t(x): -\infty < t \leqq 0)$ is finite.

For this we have only to recall the identity (4. 2) of [1-II].

(iii). M is homeomorphic to E^{2n+1}.

One may define various homeomorphisms by which p corresponds to origin 0 in E^{2n+1}. A simple one is as follows. For any trajectory there is a point y such that the length of $(\gamma_t(y): -\infty < t \leqq 0)$ is equal to 1. As the set of all such y is seen to be homeomorphic to the unit sphere in E^{2n+1}, we choose one homeomorphism. Then any trajectory in M is mapped to an open half line starting at origin 0 in E^{2n+1} by the arc-length. Q.E.D.

2. In this section we assume that a K-contact manifold M is a product $R \times B$ of real line and an almost Kähler manifold $B = M/\xi$ such that a contact form is a connection form. Suppose $\gamma \in \Phi - \mathbf{A}$, then $\gamma^*\eta = \alpha\eta$ and $\gamma\xi = \alpha\xi$ for some constant α. So γ maps a leaf of ξ into another leaf, hence γ induces a transformation $\tilde{\gamma}$ of B such that $\tilde{\gamma}\pi = \pi\gamma$.

(i). $\tilde{\gamma}$ is a homothety.

Proof. An almost Kähler structure $(\tilde{\phi}, \tilde{g})$ in B is defined by

$$\tilde{\phi}\tilde{X} = \pi\phi\tilde{X}^*,$$
$$\tilde{g}(\tilde{X}, \tilde{Y}) = g(\tilde{X}^*, \tilde{Y}^*)$$

for $\tilde{X}, \tilde{Y}$ in $\mathfrak{X}(B)$, [2]. And the relation

$$\gamma\tilde{X}^* = (\tilde{\gamma}\tilde{X})^*$$

follows from the uniqueness of the horizontal lift and $\tilde{\gamma}\pi = \pi\gamma$. Then, from $\gamma^*g = \alpha g + \alpha(\alpha - 1)\eta\cdot\eta$, we can deduce

$$(\tilde{\gamma}^*\tilde{g})(\tilde{X}, \tilde{Y}) = \alpha\tilde{g}(\tilde{X}, \tilde{Y}).$$

(ii). $\tilde{\gamma}$ is almost analytic.

First, utilyzing $\gamma\phi = \phi\gamma$, we can show

$$\tilde{\gamma}\tilde{\phi}\tilde{X} = \pi\phi\gamma\tilde{X}^*,$$

from which we have $\tilde{\gamma}\tilde{\phi} = \tilde{\phi}\tilde{\gamma}$. Summarizing the results we get

THEOREM B. *Suppose that a K-contact manifold M is an R-product bundle over an almost Kähler manifold $B = M/\xi$. If B does not admit any (non-isometric) homothety which is almost analytic, we have $\Phi = \mathbf{A}$.*

It is known that every homothety of a complete Riemannian manifold which is not locally flat is necessarily an isometry [3]. And as the completeness of M/ξ follows from that of M, we can state

THEOREM C. *If a complete K-contact manifold M is an R-product bundle over an almost Kähler manifold M/ξ which is not locally flat, then $\Phi = \mathbf{A}$.*

REFERENCES

[1] S. TANNO, Some transformations on manifolds with almost contact and contact metric structures, I, II, Tôhoku Math. Journ., 15(1963), 140–147, 322–331.

[2] ———, On fiberings of some non-compact contact manifolds, Tôhoku Math. Journ., 15(1963), 289–297.

[3] S. KOBAYASHI, A theorem on the affine transformation group of a Riemannian manifold, Nagoya Math. Journ., 9(1955), 39–41.

TÔHOKU UNIVERSITY.