FINITE GROUPS ADMITTING BRUHAT DECOMPOSITIONS OF TYPE (A_n)

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Introduction. It seems to us that one of the characteristic properties of the algebraic groups is that they admit Bruhat decompositions. In Séminaire Chevalley [1], these properties of algraic groups have been studied in detail. Meantime, postulating these properties for an abstract group, J.Tits [3] has deduced some fundamental properties of Borel subgroups and parabolic subgroups of the group. In this note, we shall prove that, if $n \geqq 2$, the family of finite simple groups $PSL_{n+1}(F_q)$ for any finite field F_q is characterized by the postulation of J. Tits with the symmetric group of degree $n+1$ as Weyl groups. D.G.Higman and J.E.McLaughlin [2] has shown the same result for the finite simple groups $PSL_3(F_q)$ as an application of their results on finite projective planes. Our method is also same as that of them and much indebted to the work of J.Tits on the geometric interpretation of algebraic groups (for example cf. [4]) and theorems of A. Wagner [5] on the collineation groups of the finite projective spaces.

1. Groups admitting Bruhat decompositions. In this section, we shall introduce definition and properties of the groups admitting Bruhat decompositions due to J.Tits (cf. [3]) which are main object of this paper.

We shall say that a group G admits a Bruhat decomposition with Weyl group $\mathfrak{W}$ if it possesses two subgroups B and W satisfying the following conditions (1. 1),$\cdots$, (1. 5).

(1. 1) B and W generate the group G.

(1. 2) $H = B \cap W$ is a normal subgroup of W.

(1. 3) $W/H \cong \mathfrak{W}$ is a finite group generated by reflexions.

Let $\Pi = \{\pi_i; i \in I\}$ be a system of generators of $\mathfrak{W}$ such that π_i^2 is the identity and for each $w \in \mathfrak{W}$, we chose a representative in W which we denote by the same letter w.

(1. 4) For any $i \in I$, $\pi_i B w \subset BwB \cup B\pi_i wB$.

(1. 5) For any $i \in I$, $\pi_i B \pi_i \neq B$.

The group G has following properties. For any subset J of I, let W_J be the subgroup of W generated by H and π_j, $j \in J$. Then $G_J = BW_JB$ is a subgroup of G and in particular $G = BWB$. Moreover, if $BwB = Bw'B$, then $w = w'$. The subgroups G_J $(J \subset I)$ are the only subgroups of G containing B. We say Borel subgroups (resp. parabolic subgroups) the subgroups of G conjugate to B (resp. to G_J for a subset J of I). Any parabolic subgroup is its own normalizer and two distinct parabolic subgroups containing a same Borel subgroup are not conjugate each other. If J and K are two subsets of I, we have

(1.6) $$G_J G_K = BW_J W_K B$$

(1.7) $$G_J \cap G_K = BW_{J \cap K} B$$

When $\mathfrak{W}$ is the symmetric group $\mathfrak{S}_{n+1}$ of degree $n+1$, we shall say that the Bruhat decomposition is of type (A_n) for the reason that the Weyl group of the simple algebraic group of type (A_n) is isomorphic to $\mathfrak{S}_{n+1}$. For any $i \in I$, we set

(1.8) $$A_i = BW_J B \qquad \text{where } J = I - \{i\}.$$

Then the subgroups A_i, $i \in I$, are the only proper maximal subgroups of G containing B. As we show in the following sections, by means of the subgroups $A_1, \cdots, A_n$, we may define a geometry on which the group G acts as a collineation group and if it is of type (A_n), the geometry is the projective geometry of n-dimensions.

2. Definition of the geometry $\Gamma(G; A_1, \cdots, A_n)$. Let G be a group and A, B be two subgroups of G satisfying the following conditions (a), (b) and (c).

(a) $$G = ABA = BAB$$

(b) $$AB \cap BA = A \cup B$$

(c) $$G : A \geqq 2 \text{ and } A : A \cap B \geqq 3.$$

Then G is called a projective ABA-group. By means of the subgroups A and B, an incidence system $\pi(G; A, B)$ can be constructed by taking as points the left cosets Ax of G modulo A and as lines the left cosets By of G modulo B. The point Ax and the line By are to be taken as incident if the cosets Ax and By have an element in common. D. G. Higman and J. E. McLaughlin have proved that for a finite group G, the additional condition

(d) $G = A \cup AxA$ for an element $x \in G$

is necessary and sufficient for $\pi(G; A, B)$ to be Desargusian projective plane and G admits a representation on a collineation group of the plane which contains the little projective group. We shall extend the concept of projective ABA-group to the space of an arbitrary dimension.

Let G be a group and $A_1, \cdots, A_n$ be n subgroups of G. We construct an incidence system $\Gamma(G; A_1, \cdots, A_n)$ by taking as 0-dimensional linear spaces (points), 1-dimensional linear spaces (lines), 2-dimensional linear spaces (planes), $\cdots$, $(n-1)$-dimensional linear spaces (hyperplanes) the cosets A_1x, $A_2x, \cdots, A_nx$ of G modulo $A_1, \cdots, A_n$ respectively. Hereafter, we shall say briefly a i-space for a i-dimensional linear space. The $(i-1)$-space A_ix and the $(j-1)$-space A_jy are to be taken as incident if the cosets A_ix and A_jy have an element in common. We postulate the following conditions on the subgroups

$A_1, \cdots, A_n$.

(2. 1) $G = A_i A_{i+1} A_1$ for each $1 \leqq i \leqq n-1$

(2. 2) $G = A_i A_{i-1} A_n$ for each $2 \leqq i \leqq n$

(2. 3) $A_i A_j \cap A_{i+1} A_1 \subseteqq A_i A_1 \cup A_{i+1} A_j$ for each $1 \leqq i < j \leqq n$

(2. 4) $A_j \subset A_i A_k$ for each $1 \leqq i < j < k \leqq n$

(2. 5) $A_j \subset A_2 A_1 A_{j-1}$ for each $3 \leqq j \leqq n$

(2. 6) $G \neq A_1 A_n$

(2. 7) $A_2 : A_1 \cap A_2 \geqq 3$

For any element $g \in G$, we denote by $\theta(g)$ the transformation of $\Gamma = \Gamma(G; A_1, \cdots, A_n)$ defined by

$$\theta(g) : A_i x \to A_i x g \qquad \text{for each } i \in I \text{ and } x \in G.$$

$\theta(g)$ transforms any i-space to an i-space and preserves incidence relations. Therefore G has a representatian onto a group of collineations $\theta(G)$ of the geometry Γ. The following condition is equivalent to the fact that the group $\theta(G)$ is doubly transitive on the points of Γ.

(2. 8) $G = A_1 \cup A_1 x A_1$ for some $x \in G$.

3. Incidence relations in the geometry $\Gamma(G; A_1, \cdots, A_n)$. We shall prove some properties on the incidence system Γ defined in the preceding section. The followings are trivial by definition.

(3. 1) *Any two distinct linear spaces of the same dimensions are not incident each other.*

(3. 2) *For any linear space, there exists a point which is incident to it.*

From (2. 4) and (2. 6), we have $G \neq A_1 A_i$ for each $1 \leqq i \leqq n$. Therefore we have

(3. 3) *For any p-space S_p, there exists at least one point which is not incident to S_p.*

We see easily that (2. 7) is equivalent to the following.

(3. 4) *Every line is incident to at least three distinct points.*

LEMMA 1. (2. 1) *is equivalent to the following.*

(3. 5) *For any p-space S_p $(p < n)$ and a point S_0 which is not incident to S_p, there is at least one $(p+1)$-space S_{p+1} which is incident to both S_p and S_0.*

PROOF OF (2. 1) $\Rightarrow$ (3. 5): Set $S_p = A_{p+1} x$ and $S_0 = A_1 y$. Then, by (2. 1), $xy^{-1} = a_{p+1} a_{p+2} a_1$ for $a_{p+1} \in A_{p+1}$, $a_{p+2} \in A_{p+2}$ and $a_1 \in A_1$. If we set $S_{p+1} = A_{p+2} a_1 y$, then it is incident to both S_0 and S_p.

Proof of (3. 5) $\Rightarrow$ (2. 1): For any $x \in G$, if $A_i x$ and A_1 are incident to $A_{i+1} y$ for some $y \in G$, then $xy^{-1} \in A_i A_{i+1}$ and $y \in A_{i+1} A_1$. Therefore, we have $x \in A_i A_{i+1} A_1$. Thus $G = A_i A_{i+1} A_1$.

Similarly, we have the following.

LEMMA 2. (2.2) *is equivalent to the following.*
(3. 6) *For any p-space* S_p $(p>0)$ *and a hyperplane* S_{n-1} *which is not incident to* S_p, *there is at least one* $(p-1)$*-space* S_{p-1} *which is incident to both* S_p *and* S_{n-1}.

LEMMA 3. (2.3) *is equivalent to the following.*
(3. 7) *Let* S_{p-1} *be a* $(p-1)$*-space and* S_0 *be a point which is not incident to* S_{p-1}. *If* S_p *and* S_q *are two linear spaces which are incident to both* S_{p-1} *and* S_0, *then* S_p *is incident to* S_q.

PROOF OF (2.3)$\Rightarrow$(3.7): We may suppose that $S_{p-1}=A_px$ where $x \in A_{p+1}A_1$ (cf. (2.1) and Lemma 1) and $S_0=A_1$. Let $S_p=A_{p+1}y$ and $S_q=A_{q+1}z$ be incident to both S_{p-1} and S_0. Then $u=xz^{-1} \in A_pA_{q+1}$ and we may assume that y and z are elements of A_1. Since $x \in A_{p+1}A_1$ and $z \in A_1$, we have also $u \in A_{p+1}A_1$. Therefore $u \in A_pA_{q+1} \cap A_{p+1}A_1$. If $u \in A_pA_1$, then $x \in A_pA_1$. This contradicts to the fact that S_{p-1} and S_0 are not incident. Thus we have $u \in A_{p+1}A_{q+1}$. This shows that S_p and S_q are incident each other.

Proof of (3.7)$\Rightarrow$(2.3): Let $x \in A_iA_j \cap A_{i+1}A_1$ and we shall show that, if $x \notin A_iA_1$, then $x \in A_{i+1}A_j$. We have $x=a_ia_j=a_{i+1}a_1$ where $a_i \in A_i$, $a_j \in A_j$, $a_{i+1} \in A_{i+1}$ and $a_1 \in A_1$. So A_ix and A_1 are not incident each other and incident to both $A_{i+1}x$ and A_j respectively. Therefore $A_{i+1}x$ and A_j are incident each other. Thus we have $x \in A_{i+1}A_j$.

COROLLARY. (2.3) *for* $j=i+1$ *is equivalent to the following.*
(3.7′) *For any p-space* S_p *and a point* S_0 *which is not incident to* S_p, *there is at most one* $(p+1)$*-space which is incident to both* S_p *and* S_0.

LEMMA 4. (2.4) *is equivalent to the following.*
(3. 8) *Let* S_p, S_q *and* S_r *be three linear spaces of dimensions* p, q *and* r *respectively and* $p<q<r$. *If* S_p *and* S_q, S_q *and* S_r *are incident each other respectively, then* S_p *and* S_r *are incident each other.*

PROOF OF (2.4)$\Rightarrow$(3.8): Let $S_p=A_{p+1}x$, $S_q=A_{q+1}y$ and $S_r=A_{r+1}z$. Then by definition, $xy^{-1} \in A_{p+1}A_{q+1}$ and $yz^{-1} \in A_{q+1}A_{r+1}$. Therefore by (2.4), $xz^{-1} \in A_{p+1}A_{r+1}$. Thus we have S_p and S_r are incident.

Proof of (3.8)$\Rightarrow$(2.4): Let $i<j<k$. For any $x \in A_j$, A_ix and A_j, A_j and A_k are incident respectively. Therefore A_ix and A_k are incident. Thus we have $x \in A_iA_k$ and $A_j \subset A_iA_k$.

LEMMA 5. (2.5) *is equivalent to the following.*
(3. 9) *For any* $(p-1)$*-space* S_{p-1} *and a line* S_1 *which are incident to a p-space* S_p, *there is at least one point which is incident to both* S_{p-1} *and* S_1.

PROOF OF (2.5)$\Rightarrow$(3.9): We may suppose $S_p=A_{p+1}$. Let $S_1=A_2x$ and

$S_{p-1} = A_p y$ where we may assume that x and y are elements of A_{p+1}. Then $xy^{-1} = a_2 a_1 a_p$ for $a_2 \in A_2$, $a_1 \in A_1$ and $a_p \in A_p$. If we set $S_0 = A_1 a_p y$, then it is incident to both S_1 and S_{p-1}.

Proof of $(3.9) \Rightarrow (2.5)$: For any $x \in A_j$, $A_2 x$ and A_{j-1} are both incident to A_j. Therefore, there is a point $A_1 y$ which is incident to both $A_2 x$ and A_{j-1}. We have $xy^{-1} \in A_2 A_1$ and $y \in A_1 A_{j-1}$. Thus we have $x \in A_2 A_1 A_{j-1}$.

The properties (3.1), $\cdots$, (3.9) characterize the n-dimensional projective geometry (cf. Appendix). Therefore we have the following.

PROPOSITION 1. *Let G be a group which has n subgroups $A_1, \cdots, A_n$ $(n \geqq 2)$ satisfying* (2.1), $\cdots$, (2.7). *Then the geometry $\Gamma(G; A_1, \cdots, A_n)$ is an n-dimensional projective geometry and G has a representation on a collineation group of the space*

If G is finite, Γ is a finite projective geometry. We shall say that the order of Γ is m when $A_2 : A_1 \cap A_2 = m + 1$. By Theorem 4 of A.Wagner [5], for a doubly transitive collineation group of a finite projective space of n-dimension and of order m with $n \leqq 4$, except when $n = 3$ and $m = 2$, the group contains the little projective group (i.e. the group of collineations generated by all elations of Γ, where an elation is a collineation of Γ fixing all points incident to a hyperplane S_{n-1} and all hyperplanes incident to some point which is incident to S_{n-1}) and when $n = 3$ and $m = 2$, the group is isomorphic to the alternating group on 7 letters or the group of all collineations. Therefore we have the following.

THEOREM 1. *Let G be a finite group which has n, $2 \leqq n \leqq 4$, subgroups $A_1, \cdots, A_n$ satisfying* (2.1), $\cdots$, (2.8). *Then, unless $n = 3$ and $m = 2$, G has a representation on a collineation group of an n-dimensional (Desarguesian) projective space which contains the little projective group. When $n = 3$ and $m = 2$, G has a representation on the group of all collineations or on the alternating group on 7 letters.*

By Theorem 3 of A.Wagner [5], for a collineation group of a finite projective space of n-dimension, $n \geqq 3$, of order m with the following properties

(i) The group is transitive on the space,

(ii) For any hyperplane S_{n-1} of the space, the subgroup of all collineations such that S_{n-1} transforms onto S_{n-1}, as a collineation group of S_{n-1}, contains the little projective group of S_{n-1},

unless $n = 3$ and $m = 2$, the group contains the little projective group of the space. Therefore, if we replace (2.8) by the following

(2. 8′) $G = A_1 \cup A_1 x A_1$ for some $x \in A_2 \cap A_3 \cap \cdots \cap A_n$,
we have the following theorem by induction with respect to n.

THEOREM 2. *Let G be a finite group which has n subgroups $A_1, \cdots, A_n$, $n \geqq 5$, satisfying* (2. 1), $\cdots$, (2. 7) *and* (2. 8′). *Then G has a representation on a collineation group of a n-dimensional projective space which contains the little projective group.*

4. Groups admitting Bruhat decompositions of type (A_n). Let G be a group admits a Bruhat decomposition with the symmetric group of degree $n+1$ as Weyl group, i.e., a group possesses two subgroups B and W satisfying (1.1), $\cdots$, (1. 5) where $\mathfrak{W}$ is the symmetric group of degree $n+1$. Let $\Pi = \{\pi_1, \cdots, \pi_n\}$ be the system of generators of $\mathfrak{W}$ such that

(4. 1) $\pi_i^2 = 1$ for each $1 \leqq i \leqq n$ and $(\pi_i \pi_{i+1})^3 = 1$ for each $1 \leqq i \leqq n-1$.

Let $A_1, \cdots, A_n$ be the subgroups of G defined as (1. 8) for the generators (4. 1). To prove that the groups satisfy (2. 1), $\cdots$, (2. 7) and (2. 8′), we need some lemmas on the Weyl group $\mathfrak{W}$ of type (A_n).

We shall identify $\mathfrak{W}$ with the Weyl group operating on a root system Δ of the type (A_n). Then the element π_i of the system of generators is a reflexion with respect to the root a_i of a fundamental root system $\{a_1, \cdots, a_n\}$ of Δ. We define a linear order in Δ with respect to the fundamental root system. Denote by Σ the set of all postive roots.
Then

$$\Sigma = \{a_{i,j} = a_i + a_{i+1} + \cdots + a_j;\ 1 \leqq i \leqq j \leqq n,\ a_{i,i} \text{ means } a_i\}$$

$$a_{i,j} < a_{k,l} \text{ if and only if } j < l \text{ or } j = l \text{ and } k < i$$

$$\Delta = \Sigma + (-\Sigma)$$

For any root $r \in \Delta$, we denote by $(r) \in \mathfrak{W}$ the reflexion with respect to r, then $(r) = (-r)$. Therefore we use only positive root for the notation (r). For any two roots r and s, (r) and (s) are commutative if and only if $r+s$ is not a root. For any $i, 1 \leqq i \leqq n$, Σ may be partitioned to two subsets Σ_i and Σ_i' such that

$$\Sigma_i = \{a_{j,k};\ j \leqq k < i \text{ or } i < j \leqq k\}$$

$$\Sigma_i' = \{a_{j,k};\ j \leqq i \leqq k\}.$$

Now denote by $\mathfrak{W}_i$ the subgroup of $\mathfrak{W}$ generated by π_j for all j distinct from i. Any element of $\mathfrak{W}_i$ can be expressed by a product of some elements (r) for $r \in \Sigma_i$. We have the following

LEMMA 6. *For any $w \in \mathfrak{W}$ and $i \in I$, if $w \notin \mathfrak{W}_i$, we have*

$$w = w'(r_1) \cdots (r_N) \ (\textit{resp. } w = (r_1) \cdots (r_N)\, w')$$

where $w' \in \mathfrak{W}_i$, $r_j \in \Sigma_i$ for each $1 \leqq j \leqq N$, (r_j) and (r_k) are commutative *to each other for* $1 \leqq j,\ k \leqq N$.

PROOF. For $r \in \Sigma_i'$ and $s \in \Sigma_i$, if (r) and (s) are not commutative, we have $(r)(s) = (s')(r)$ for some $s' \in \Sigma_i$. Therefore $w = w'w''$ where w' is a product of (r)'s for $r \in \Sigma_i$,i.e., an element of $\mathfrak{W}_i$ and w'' is a product of (r)'s for $r \in \Sigma_i'$. Let $w = w'(r_1) \cdots (r_N)$ be an expression of w such that N is a least possible number of such expression. Then (r_j) and (r_k) are commutative to each other for $1 \leqq j,\ k \leqq N$. For, if there is a pair j, k such that (r_j) and (r_k) are not commutative, we can express w as a product of an element w' of $\mathfrak{W}_i$ and $(s_1) \cdots (s_M)$ where $s_j \in \Sigma_i'$ and $M < N$. The second assertion is also proved similary.

COROLLARY. *For any* $w \in \mathfrak{W}$, *if* $w \notin \mathfrak{W}_1$, *we have* $w = w'(r)$ *(resp.* $w = (r)\ w'$*) where* $w' \in \mathfrak{W}_1$ *and* $r \in \Sigma_1'$.

For any two roots where $r,\ r'$ in Σ_1', (r) and (r') are commutative. Therefore we have immediately our assertion from Lemma 6.

LEMMA 7. $\mathfrak{W} = \mathfrak{W}_1 \cup \mathfrak{W}_1(a_1)\mathfrak{W}_1$.

PROOF. Let $w \in \mathfrak{W}$. If $w \notin \mathfrak{W}_1$, then $w = w'(r)$ where $w' \in \mathfrak{W}'_1$ and $r \in \Sigma_1'$. If $r = a_{1,k}$, then $(r) = (a_{2,k})(a_1)(a_{2,k})$. Thus we have $w \in \mathfrak{W}'_1(a_1)\mathfrak{W}_1$.

LEMMA 8. $\mathfrak{W}_j \subset \mathfrak{W}_i\mathfrak{W}'_k$ *for* $i \leqq j \leqq k$.

PROOF. Let $w \in \mathfrak{W}_j$. If $w \notin \mathfrak{W}_i$ we have $w = w'(r_1) \cdots (r_N)$ where $w' \in \mathfrak{W}_i$ and $r_l \in \Sigma_i' \cap \Sigma_j$ for all $1 \leqq l \leqq N$. Since $i \leqq j \leqq k$, $(r_l) \in \mathfrak{W}_k$ for $1 \leqq l \leqq N$. Therefore, we have $w \in \mathfrak{W}_i\mathfrak{W}_k$.

LEMMA 9. $\mathfrak{W} = \mathfrak{W}'_i\mathfrak{W}_{i+1}\mathfrak{W}_1$ *for each* $1 \leqq i \leqq n-1$.

PROOF. For any $w \in \mathfrak{W}$, if $w \notin \mathfrak{W}_1$, we have $w = (r)w'$ where $w' \in \mathfrak{W}_1$ and $r \in \Sigma_1'$. We shall show that $(r) \in \mathfrak{W}_i\mathfrak{W}'_{i+1}\mathfrak{W}_1$. Let $r = a_{1,k}$. If $k \leqq i$, then $(r) \in \mathfrak{W}_{i+1}$. If $k \geqq i+1$, then we have $(r) = (a_{1,k}) = (a_{i+1,k})(a_{1,i})(a_{i+1,k})$ where $(a_{i+1,k})$ is contained in both $\mathfrak{W}_i$ and $\mathfrak{W}_1$ and also $(a_{1,i}) \in \mathfrak{W}_{i+1}$. Thus we have $w \in \mathfrak{W}_i\mathfrak{W}'_{i+1}\mathfrak{W}_1$.

Similarly we have the following.

LEMMA 10. $\mathfrak{W} = \mathfrak{W}_i\mathfrak{W}'_{i-1}\mathfrak{W}_n$ *for each* $2 \leqq i \leqq n$.

COROLLARY. $\mathfrak{W}_j \subset \mathfrak{W}'_i\mathfrak{W}_{i-1}\mathfrak{W}_{j-1}$ *for each* $2 \leqq i < j \leqq n$.

LEMMA 11. $\mathfrak{W}_i\mathfrak{W}'_j \cap \mathfrak{W}_{i+1}\mathfrak{W}_1 \subset \mathfrak{W}_i\mathfrak{W}_1 \cup \mathfrak{W}_{i+1}\mathfrak{W}_j$ *for each* $1 \leqq i < j \leqq n$.

PROOF. Let $w \in \mathfrak{W}_i\mathfrak{W}_j \cap \mathfrak{W}_{i+1}\mathfrak{W}_1$. We shall show that if $w \notin \mathfrak{W}_i\mathfrak{W}_1$, then $w \in \mathfrak{W}_{i+1}\mathfrak{W}_j$. By Lemma 1 and its corollary, since $w \in \mathfrak{W}_{i+1}\mathfrak{W}_1$, we have $w = (s)w''$ where $w'' \in \mathfrak{W}_1$ and $s \in \Sigma_1' \cap \Sigma_{i+1}$ and also, since $w \in \mathfrak{W}_i\mathfrak{W}_j$, we have $w = w'(r_1)\cdots(r_N)$ where $w' \in \mathfrak{W}_i$ and $r_k \in \Sigma_i' \cap \Sigma_j$. Moreover, since $w \notin \mathfrak{W}_i\mathfrak{W}_1$, $s \in \Sigma_i'$ and there is at least one root r_k, say r_1, contained in Σ_1' and yet, since the roots of Σ_1' are commutative each other, r_1 is the only one root contained in Σ_1'. Therefore, we have $(r_N)(r_{N-1})\cdots(r_2) = w_1'' \in \mathfrak{W}_1$. Thus we have $(s)w'(r_1) = w''w_1'' \in \mathfrak{W}_1$ where $w' \in \mathfrak{W}_i$, $s = a_{1,i}$ and $r_1 = a_{1,k}$ $(i \leqq k \leqq j)$. We shall show that $w' \in \mathfrak{W}_{i+1}\mathfrak{W}_j$ which proves $w \in \mathfrak{W}_{i+1}\mathfrak{W}_j$. Assume that $w' \notin \mathfrak{W}_{i+1}\mathfrak{W}_j$. Then we have $w' = w_1'(u)$ where $w_1' \in \mathfrak{W}_{i+1}$ and $u \in \Sigma_{i+1}' \cap \Sigma_i$, for the roots of $\Sigma_{i+1}' \cap \Sigma_i$ are commutative each other. Since $w' \notin \mathfrak{W}_{i+1}\mathfrak{W}_j$, we have $u \in \Sigma_j' \cap \Sigma_{i+1}' \cap \Sigma_i$. Let $u = a_{i+1,l}$, $l \geqq j$. Therefore

$$(s)w'(r_1) = (s)w_1'(u)(r_1) \qquad \text{where } (s)w_1' \in \mathfrak{W}_{i+1}.$$

We shall show that $(s)w'(r_1) = w_3'(r)$ where $w_3' \in \mathfrak{W}_1$ and $(r) \notin \mathfrak{W}_1$. This contradicts to the fact that $(s)w'(u) \in \mathfrak{W}_1$. Thus we shall have $w' \in \mathfrak{W}_{1+1}\mathfrak{W}_j$. First, if $(s)w_1' \in \mathfrak{W}_1$, then $(s)w_1'(u) \in \mathfrak{W}_1$ and $(r_1) \notin \mathfrak{W}_1$. Next, if $(s)w_1' \notin \mathfrak{W}_1$, we have $(s)w_1' = w_2'(v)$ where $w_2' \in \mathfrak{W}_1$ and $v \in \Sigma_1' \cap \Sigma_{i+1}$. Let $v = a_{1,m}$, $1 \leqq m \leqq i$. Since $m \leqq i < k < j \leqq l$, when $m = i$,

$$(v)(u)(r_1) = (a_{1,i})(a_{i+1,l})(a_{1,k}) = (a_{i+1,l})\ (a_{1,l})\ (a_{1,k}) = (a_{i+1,l})(a_{k+1,l})(a_{1,l})$$

and when $m < i$,

$$(v)(u)(r_1) = (a_{1,m})(a_{i+1,l})(a_{1,l}) = (a_{i+1,l})(a_{1,m})(a_{1,l}) = (a_{i+1,l})(a_{m+1,l})(a_{1,m})$$

where $(a_{i+1,l})(a_{k+1,l})$, $(a_{i+1,l})(a_{m+1,l})$ are elements of $\mathfrak{W}_1$ and $(a_{1,l})$, $(a_{1,m})$ are not contained in $\mathfrak{W}_1$.

Now, from (1.6), (1.7) and Lemmas 7, $\cdots$, 11, we see easily that the subgroups $A_1, \cdots, A_n$ satisfy (2.1), $\cdots$, (2.5) and (2.8). (2.6) follows from the fact that $(a_{1,n}) \notin A_1A_n$ and (2.7) follows from the fact $(a_1) \in A_2$ and $\notin A_1$ and from (1.5). Thus we have

PROPOSITION 2. *Let G be a group admitting a Bruhat decomposition of type (A_n), and $A_1, \cdots, A_n$ be the maximal subgroups of G defined as* (1.8) *for a system of generators* (4.1). *Then the subgroups satisfy the conditions* (2.1), $\cdots$, (2.7) *and* (2.8′).

THEOREM 3. *Let G be a finite group admitting a Bruhat decomposition of type (A_n), $n \geqq 2$. Then G has a representation on a collineation group of n-dimensional projective space which contains the little projective group.*

PROOF. From propositions 1 and 2, unless $n = 3$ and $m = 2$, we have the assertion. We shall show that the exceptional case does not happen. Let $n = 3$ and $m = 2$. If G has a representation on the alternating group $\mathfrak{A}_7$ on 7 letters, then $\mathfrak{A}_7$ must have a Bruhat decomposition. Here $\mathfrak{A}_7$ is a collineation group of a finite projective space of 3 dimensions and of order 2. Let S_1, S_2, S_3 be a point, a line and a plane of the space which are incident each other. We denote by A_i, $1 \leqq i \leqq 3$, the subgroups of $\mathfrak{A}_7$ consisting of the collineations which transform S_i onto itself. We have known that $B = A_1 \cap A_2 \cap A_3$ is a 2-Sylow subgroup of $\mathfrak{A}_7$ and of order 2^3. Since the homomorphism θ of $\mathfrak{A}_7$ on to a collineation group of the projective space defined by the decomposition is an isomorphism, the Borel subgroup of $\mathfrak{A}_7$ must be the 2-Sylow subgroups. Therefore the order of $\mathfrak{A}_7$ must be $< 2^3 \cdot 4 \cdot 3 \cdot 2 \cdot 2^3 = 1536$. This is impossible, for the order of $\mathfrak{A}_7$ is $7 \cdot 5 \cdot 3^2 \cdot 2^2 = 2520$. Thus we have that $\mathfrak{A}_7$ has not any Bruhat decomposition with $\mathfrak{S}_4$ as Weyl group.

COROLLARY. *A finite simple group admitting a Bruhat decomposition with the symmetric group of degree $n+1$, $n \geqq 2$, as Weyl group is isomorphic to a special projective group $PSL_{n+1}(F_q)$ for some finite field F_q with q elements.*

REMARKS. Each known simple group of Lie type admits a Bruhat decomposition, but it is not necessary that the Weyl group can be identified with that of a simple algebraic group. Further, in general, the property for simple groups that they admit Bruhat decompositions with a given Weyl group does'nt characterize the family of simple groups. For example, Ree's simple groups associated with the simple Lie algebra of type (F_4) admit Bruhat decomposions with the dihedral group of order 16 as Weyl group which is not isomorphic to any one of a simple algebraic group. Chevalley's simple groups of types (B_n) and (C_n) admit Bruhat decompositions with the same Weyl group and also there are two distinct families of simple groups admitting Bruhat decompositions of type (F_4) (resp. of type (G_2)). However, it may be conjectured that the family of (finite or infinite) Chevalley's simple groups of type (A_n), $n \geqq 2$ (resp. type (D_n), $n \geqq 4$, or type (E_n), $n = 6, 7, 8$) is characterized by the property that they admit Bruhat decompositions with the Weyl group of that type.

Appendix. We shall show that an incidence system Γ satisfying (3. 1),$\cdots$ (3. 9) is a n-dimensional projective geometry. We denote by S_n the set of all points in Γ and we identify the i-space with the set of the points which are incident to the space. Then we have the following:

(*) For any two linear spaces S_p and S_q, $S_p \subseteqq S_q$ if and only if S_p is incident to S_q and $p \leqq q$.

Proof of Sufficiency: If a point S_0 is incident to S_p and S_p is incident to S_q, $p \leqq q$, then by (3.8) we have S_0 is incident to S_q. Proof of necessity: Let $S_p \subseteqq S_q$. We proceed by induction on p. If $p = 0$, then the assertion is trivial. We assume that the assertion is true for any pair p' and q where $p' < p$. For S_p, there is a point S_0 which is incident to S_p(3.2) and for S_0, there is a hyperplane S_{n-1} which is not incident to S_0 (3.3). Then, since S_p and S_{n-1} are not incident, there is a $(p-1)$-space S_{p-1} which is incident to both S_p and S_{n-1} (3.6). Therefore, $S_{p-1} \subseteqq S_p$ and we have $S_{p-1} \subseteqq S_q$ (3.8). By induction hypothesis, S_{p-1} is incident to S_q and $p-1 \leqq q$. If $p - 1 = q$, then by (3.1) $S_{p-1} = S_q$. This is a contradiction, for the point S_0 is incident to S_q and not to S_{p-1}. Therefore we have $p \leqq q$ and, by (3.7), S_p is incident to S_q.

If $P_0, \cdots, P_p$ are any $p+1$ points, we say that they are linearly dependent if there is a q-space S_q such that $P_i \subseteqq S_q$ for all $0 \leqq i \leqq p$ where $q < p$. Otherwise the points are said to be linearly independent.

Now, we shall deduce, for example, the axioms for projective space of n-dimension in Hodge and Peodoe's Method of Algebraic Geometry, Chap VI.

I. If $S_h \subseteqq S_k$ and $S_k \subseteqq S_h$ then $S_k = S_h$.

II. If $S_p \subseteqq S_q$ and $S_q \subseteqq S_r$ then $S_p \subseteqq S_r$.

III. Every line contains at least three distinct points.

These are trivial by definition and (*). (cf. (3.1), (3.8) and (3.4)).

IV. Given any $p+1$ linearly independent points, there is at least one p-space which contains them.

By (3.5), we may construct the space step by step.

V. If $P_0, \cdots, P_p$ are $p+1$ linearly independent points which lie in an S_q, any S_p containing them is contained in S_q.

If $p = 0$, the assertion is trivial. We proceed by induction on p. We assume that the assertion is true for $p' < p$. Let S_{p-1} be a $(p-1)$-space such that $P_0, \cdots, P_{p-1}$ are incident to it. Then by induction hypothesis S_{p-1} is incident to S_q. By (3.7) we have S_p is incident to S_q.

VI. Any p-space contains at least one set of $p+1$ linearly independent points.

For a p-space S_p, we have shown in the proof of (*), there is a $(p-1)$-space S_{p-1} such that it is incident to S_p and there is a point S_0 incident to S_p and not to S_{p-1}. Therefore we have a series of linear spaces $S_0 \subset S_1 \subset \cdots \subset S_{p-1} \subset S_p$ and a set of points $P_0, \cdots, P_p$ such that P_i is incident to S_{i-1} for $1 \leqq i \leqq p$. We shall show by induction on p that the points are linearly independent. P_0 and P_1 are linearly independent. If $P_0, \cdots, P_{p-1}$ are linearly independent, by IV, there is a $(p-1)$-space S'_{p-1} which contains them. By V, the space S'_{p-1} is unique, therefore we have $S'_{p-1} = S_{p-1}$. Assume that $P_0, \cdots, P_p$

are linearly dependent. Then it holds that P_p is contained in S_{p-1}. This is a contradiction.

VII. If $P_0, \cdots, P_p$ are $p+1$ linearly independent points of an S_p, and $Q_0, \cdots, Q_q$ are $q+1$ linearly indepedent points of an S_q, and if the $p+q+2$ points $P_0, \cdots, P_p, Q_0, \cdots, Q_q$ are linearly dependent, there exists at least one point which lies in S_p and S_q.

Suppose that $P_0, \cdots, P_p, Q_0, \cdots, Q_r$ are linearly independent and Q_i, for $r+1 \leqq i \leqq q$, is dependent to them. Let $S_{p+r+1-i}$, $0 \leqq i \leqq r$, be a linear space which contains P_j, $0 \leqq j \leqq p$, and Q_k, $0 \leqq k \leqq r-i$. If $S_1^{(0)}$ is a line which is incident to Q_r and Q_{r+1}, then $S_1^{(0)}$ is incident to S_{p+r+1} (3. 7). S_{p+r} is also incident to S_{p+r+1} (3. 7). Therefore, there is a point $S_0^{(0)}$ which is incident to both $S_1^{(0)}$ and S_{p+r} (3. 9). Now let $S_1^{(1)}$be a line which is incident to $S_0^{(0)}$ and Q_{r-1}. Then $S_1^{(1)}$ and S_{p+r-1} are both incident to S_{p+r} and there is a point $S_0^{(1)}$ which is incident to both $S_1^{(1)}$ and S_{p+r-1} (3. 9). Thus we have point $S_0^{(i)}$, $0 \leqq i \leqq r$, which is incident to S_{p+r-i} and to a line $S_1^{(i)}$ which is incident to S_q. For $i=r$, the point $S_0^{(r)}$ is incident to both S_p and S_q.

VIII. There exists a set of $n+1$ linearly independent points but any set of m points, where $m > n+1$, is linearly dependent.

This follows from (3. 3) and definitions.

ADDED IN PROOF: After completion of this paper, Prof. J. Tits has kindly informed the author that he has obtained that each finite simple group admitting a Bruhat decomposition whose Weyl group is neither a cyclic group of order two nor a dihedral group of order 8, 12 or 16, is derived from a simple algebraic group, i.e., is isomorphic to the subgroup generated by the p-Sylow subgroups of the group of k-rational points of a simple algebraic group defined over a finite field k of characteristic p (cf. [6]). Therefore, in particular, for a finite group, the conjecture stated in Remarks has been solved affirmatively. At the same time, I have known that Prof. C.W.Curtis has also obtained some results in the same direction (cf. [7]).

Mr. T.Tsuzuku has also informed the author that he has obtained the nearly same result as ours (cf. [8]).

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