ON DECOMPOSITION OF WALSH FOURIER SERIES

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(Received December 22, 1964)

Introduction. Let $\{\phi_n(t)\}$ resp. $\{\psi_n(t)\}$ be the systems of Rademacher resp. Walsh functions (see [1] and [5] for detailed properties of these functions) and let $\omega(t)$, $\Omega(k)$ be weight functions introduced by I. I. Hirschman [2], that is

$$\omega(0) = 0,\ \omega(t) = 2^{-n} \quad (2^{-n} \leqq t < 2^{-n+1},\ n = 1, 2, \cdots)$$

$$\Omega(0) = 1,\ \Omega(k) = 2^{n} \quad (2^{n} \leqq k < 2^{n+1},\ n = 0, 1, 2, \cdots).$$

Hirschman [2] extended Paley's inequality, the key theorem for L^p $(p > 1)$ theory of Walsh Fourier series (abbrev. WFS), to weighted L^p-norms and gave also discrete analogues for uniform and weighted norms. The purpose of this article is to study the behavior of the decomposition of WFS in weighted L^1 case, and to reconstruct thereof Hirschman's results by means of the interpolation theorem of J. Marcinkiewicz [7, II, p. 112]. Analogous L^1-estimates for trigonometric Fourier series are found in S. Igari [3], and we borrow basic tools from [2] and [3], though we treat the decomposition in a slightly different form, so that somewhat more tedious inspections are needed.

Let us write, for $f(x)$ and/or $\{c_k\}$ suitably restricted,

$$f(x) = \sum c_k \psi_k(x), \quad c_k = \int f(t)\psi_k(t)\,dt, \quad c_{nk} = \int f(t)\chi_n(t)\,dt \quad (\chi_n(t) = 1,$$

$$t \in [2^{-n}, 2^{-n+1}),\ = 0 \text{ otherwise}), \qquad \delta_0(x; f) = c_0,\ \delta_{n+1}(x; f) = \sum_{k=2^n}^{2^{n+1}-1} c_k \psi_k(x),$$

$$\delta^*(x; f) = \sum_{n=0}^{\infty} \varepsilon_n \delta_n(x; f), \quad c_k^* = \sum_{n=1}^{\infty} \varepsilon_n c_{nk},$$

where $\{\varepsilon_n\}$ is any sequence consisting of 0, 1 and -1.

Our main results are the following:

THEOREM 1. *Let $0 \leqq \alpha < 1$. Then there exists a constant $A_\alpha > 0$, depending on α only, such that for every $y > 0$ and for every $f \in L^1_{-\alpha}$*

$$\mu_{-\alpha}(\{x; |\delta^*(x; f)| > y\}) \leqq A_\alpha \|f\|_{1,-\alpha}/y,$$

where $\mu_{-\alpha}(E)=\int_E \omega^{-\alpha}(t)\,dt$ *and* $L^1_{-\alpha}\equiv\{f\,;\,\|f\|_{1,-\alpha}=\int|f|\,d\mu_{-\alpha}<\infty\}$.

THEOREM 2. *Let* $0\leqq\alpha<1$. *Then there exists a constant* $A_\alpha>0$, *depending on* α *only, such that for every* $y>0$ *and for every* $c=\{c_k\}\in l^1_{-\alpha}\cap l^2$,

$$\mu_{-\alpha}(\{k\,;\,|c_k^*|>y\})\leqq A_\alpha\|c\|_{1,-\alpha}/y$$

where $\mu_{-\alpha}(E)=\sum_{k\in E}\Omega^{-\alpha}(k)$ *and* $l^1_{-\alpha}=\{c\,;\,\|c\|_{1,-\alpha}\equiv\sum|c_k|\Omega^{-\alpha}(k)<\infty\}$.

1. Compact case. We begin with some lemmas needed later.

LEMMA 1. *Let* $-1<\alpha<1$. *Then*

$$\int|\delta^*(x\,;f)|^2\,\omega^\alpha(x)\,dx\leqq A_\alpha\int|f(x)|^2\,\omega^\alpha(x)\,dx,$$

where A_α *depends on* α *only.*

This is proved along the line of [2], and we omit the proof.

LEMMA 2. *Let* $0\leqq\alpha<1$, $f\in L^1_{-\alpha}$ *and let* $y>\|f\|_{1,-\alpha}$ *be given. Then we can decompose* f *as follows*:

(i) $$f=v+w,\quad w=\sum_{i,j}w_{ij}$$

(ii) $$|v|\leqq A_\alpha y\quad \text{a.e.}$$

(iii) $$\|v\|_{1,-\alpha}\leqq A_\alpha\|f\|_{1,-\alpha}$$

(iv) $$\sum_{i,j}\|w_{ij}\|_{1,-\alpha}\leqq A_\alpha\|f\|_{1,-\alpha}$$

(v) *there exists a system* $\{I_{ij}\}$ *of disjoint dyadic intervals* $I_{ij}=[a_{ij},\,a_{ij}+2^{-i})$, $w_{ij}=0$ *outside* I_{ij} *and*

$$\mu_{-\alpha}(E)=\mu_{-\alpha}\Big(\bigcup_{i,j}I_{ij}\Big)=\sum_{i,j}\mu_{-\alpha}(I_{ij})\leqq\frac{1}{y}\|f\|_{1,-\alpha}$$

(vi) *for every* $I_{ij}=I$, $w_{ij}=u$,

$$\int u\,dx=\int_I u\,dx=0\,.$$

This also is essentially known. We refer the reader to S. Igari [3] and [4], omitting the proof. We will give a complete proof to the discrete analogue of this lemma (see Lemma 7 below). Observe that Lemma 1 is used in the proof of Lemma 2.

LEMMA 3. *With the notations in Lemma* 2, *we have*

$$\delta_k(x, w) = 0 \quad \textit{for } x \notin E,\ k = 0, 1, 2, \cdots.$$

This is proved in [6] for the case of $\alpha=0$. The same proof, starting now from Lemma 2, applies for general case $0 \leqq \alpha < 1$.

PROOF OF THEOREM 1. For given $y > 2\,\|f\|_{1,-\alpha}$ decompose f by Lemma 2, obtaining

$$f = v + w, \quad \delta^*(x; f) = \delta^*(x; v) + \delta^*(x; w).$$

$\delta^*(x; w)$ vanishes outside E by Lemma 3. Thus

$$\{x;\ |\delta^*(x; f)| > 2y\} \subset \{x;\ |\delta^*(x; v)| > y\} \cup E.$$

We know that $\mu_{-\alpha}(E) \leqq \|f\|_{1,-\alpha}/y$. On the other hand

$$\mu_{-\alpha}(\{x;\ |\delta^*(x, v)| > y\}) \leqq y^{-2}\int |\delta^*(x; v)|^2\,d\mu_{-\alpha} \leqq A_\alpha y^{-2}\int |v|^2\,d\mu_{-\alpha}$$

$$\leqq \frac{A_\alpha}{y}\int |v(x)|\,d\mu_{-\alpha} \leqq A_\alpha\,\|f\|_{1,-\alpha}/y.$$

Writing y instead of $2y$ we obtain the required result. The truth of the theorem for smaller y is easily verified (see [6]).

With suitable choice of ε_n's, we have

COROLLARY. *Let* $0 \leqq \alpha < 1$, $f \in L^1_{-\alpha}$ *and let* $s_n(x; f)$ *be the n-th partial sum of its* WFS. *Then*

$$\mu_{-\alpha}(\{x;\ |s_n(x; f)| > y\}) \leqq A_\alpha \|f\|_{1,-\alpha}/y,$$

where A_α *depends on* α *only.*

2. Discrete case. Since the underlying measure space is atomic and no longer totally finite, we must argue more carefully and the proofs will be much more complete than the preceding case.

LEMMA 4. *Let $f \in L^2(0,1)$, c_k and c_{nk} be defined as in Introduction. Then we have*

$$\sum_{k=0}^{\infty} c_{mk} c_{nk} = 0 \quad \textit{for} \quad m \neq n .$$

This is a special case of [2, Lemma 3. 1 d], but we include a proof, which seems simpler and more straightforward. In fact we have

$$\begin{aligned}\sum_{k=0}^{N-1} c_{mk} c_{nk} &= \sum_{k=0}^{N-1} \int f(t) \chi_m(t) \psi_k(t)\, dt \int f(u) \chi_n(u) \psi_k(u)\, du \\ &= \int f(t) \chi_m(t)\, dt \int f(u) \chi_n(u) \sum_{k=0}^{N-1} \psi_k(t \dotplus u)\, du \\ &= \int f(t) \chi_m(t) S_N(t; f\chi_n)\, dt \to \int f^2(t) \chi_m(t) \chi_n(t)\, dt = 0,\end{aligned}$$

because $S_N(t; f\chi_n)$ tends in L^2-norm to $f(t)\chi_n(t)$ as $N \to \infty$, q.e.d.

LEMMA 5. *Let f, c_k, c_{nk} be as above. Then*

$$\sum_{k=0}^{\infty} |c_k^*|^2 \leqq \|f\|_2^2 \left(= \sum_{k=0}^{\infty} |c_k|^2 \right).$$

PROOF. For a fixed natural number N, define

$$f_N(t) = f(t) \quad (t \geqq 2^{-N+1}), \ = 0 \quad (0 \leqq t < 2^{-N+1})$$

$$c_{nk}^{(N)} = \int f_N(t) \chi_n(t) \psi_k(t)\, dt, \ c_k^{(N)*} = \sum_{n=1}^{\infty} \varepsilon_n c_{nk}^{(N)} .$$

Our assertion is true for f_N and $\{c_k^{(N)*}\}$: in fact

$$\begin{aligned}\sum_k |c_k^{(N)*}|^2 &= \sum_k \Big(\sum_m \varepsilon_m c_{mk}^{(N)} \Big) \Big(\sum_n \varepsilon_n c_{nk}^{(N)} \Big) \\ &\leqq \sum_k \sum_n |c_{nk}^{(N)}|^2 + 2 \sum_k \sum_{m<n} \varepsilon_m \varepsilon_n c_{mk}^{(N)} c_{nk}^{(N)} \\ &= \sum_k \sum_n |c_{nk}^{(N)}|^2 + 2 \sum_{m<n} \varepsilon_m \varepsilon_n \sum_k c_{mk}^{(N)} c_{nk}^{(N)} = S + T, \quad \text{say,}\end{aligned}$$

the summation over m, n being finite (observe that $c_{nk}^{(N)} = 0$ for $n \geqq N$ and

$c_{nk}^{(N)} = c_{nk}$ for $n < N$). The inner sum in T vanishes by Lemma 4 for every pair (m, n), while S is equal to

$$\sum_n \sum_k |c_{nk}^{(N)}|^2 = \sum_n \int |f_N(t)\chi_n(t)|^2\, dt = \int |f_N(t)|^2\, dt$$

by Parseval's relation.

Since $f_N \to f$ in L^2 as $N \to \infty$, this inequality is easily extended to whole L^2, which completes the proof.

By Hirschman's method, we can generalize Lemma 5 as follows.

LEMMA 6. *Let* $-1 < \alpha < 1$. *There is a constant* A_α, *depending on* α *only, such that, for every* $c = \{c_k\} \in l_\alpha^2 \cap l^2$,

$$\sum_k |c_k^*|^2 \Omega^\alpha(k) \leqq A_\alpha \sum_k |c_k|^2 \Omega^\alpha(k)\,.$$

LEMMA 7. *Let* $0 \leqq \alpha < 1$ *and* $y > 0$ *be given. Then* $c = \{c_k\} \in l_{-\alpha}^1$ *can be decomposed as follows;*

$$\text{(i)} \qquad c = v + w\,, \quad w = \sum_{i,j} w^{(i,j)}, \quad v = \{v_k\} \in l_{-\alpha}^1\,;$$

$$\text{(ii)} \qquad |v_k| \leqq A_\alpha y \quad \textit{for} \quad k = 0, 1, 2, \cdots$$

$$\text{(iii)} \qquad \| v \|_{1,-\alpha} \leqq A_\alpha \| c \|_{1,-\alpha}$$

$$\text{(iv)} \qquad \sum_{i,j} \| w^{(i,i)} \|_{1,-\alpha} \leqq A_\alpha \| c \|_{1,-\alpha}$$

(v) *There exist disjoint "intervals"* I_{ij}:
$u = w^{(ij)} = \{w_k^{(ij)}\}$ *vanishes for* $k \notin I = I_{ij}$

$$\sum_{i,j} \mu_{-\alpha}(I_{ij}) \leqq \frac{1}{y} \| c \|_{1,-\alpha}.$$

$$\text{(vi)} \qquad \sum_k u_k = \sum_{k \in I} u_k = 0, \quad \textit{for every} \quad u = w^{(ij)},\ I = I_{ij}\,.$$

PROOF. We divide the whole sequence $[0, \infty)$ into disjoint "intervals" J_{0j} ($j = 1, 2, \cdots$) as follows.

Let J_{01} be the smallest "interval" $[0, 2^n)$ for which we have $\mu_{-\alpha}([0, 2^{-n})) \geqq \| c \|_{1,-\alpha}/y$. Having defined J_{0j} with right open extremity b_j, let J_{0j+1} be the smallest $[b_j, 2^m)$ for which $\mu_{-\alpha}([b_j, 2^m)) \geqq \| c \|_{1,-\alpha}/y$. Thus J_{0j}'s are all defined, of the form $[2^a, 2^b)$ (except possibly $J_{01} = \{0\} = [0, 1)$).

Divide each J_{0j} into two parts (to be precise, we should confine ourselves to those J's which contain at least two integers, but the reader will not be confused if we omit this trivial remark)

$$J_{0j} = [2^{a(i)}, 2^{b(i)}) = J_{0j}^1 \cup J_{0j}^2 \quad \text{say, where}$$

$$J_{0j}^1 = [2^{a(j)}, 2^{b(j)-1}) \quad \text{if } a(j) < b(j) - 1$$

$$= [2^{a(j)}, 2^{a(j)}+2^{a(j)-1}) \quad \text{if } a(j) = b(j)-1 \text{ i.e. } J_{0j} \text{ consists of } 2^{a(j)} \text{ elements.}$$

$$J_{0j}^2 = J_{0j} - J_{0j}^1$$

Observe that in either case $\mu_{-\alpha}(J_{0j}^l) \leqq \mu_{-\alpha}(J_{0j}) \leqq A_\alpha \mu_{-\alpha}(J_{0j}^l)$, $l = 1, 2$.

If $\mu_{-\alpha}(J_{0j}^l) < \dfrac{1}{y} \sum \{|c_k| \Omega^{-\alpha}(k)\,;\ k \in J_{0j}^l\}$, we call J_{0j}^l an $I_1.$; if not, J_{0j}^l is called a $J_1.$; we enumerate them in two systems from left to right, obtaining $\{I_{1j}\}$ and $\{I_{1j}\}$. Divide each J as above, and repeat this process indefinitely. Observe that the J's are divided until they consist of a single point (integer), and each $I = I_{ij}$ is either "purely dyadic" (by this we mean $I=[2^m, 2^n)$) or is contained in an "elementary dyadic interval" $[2^n, 2^{n+1})$. We obtain two systems of disjoint intervals $\{I_{ij}\}$ $\{J_{ij}\}$, each of the latter consisting of a single point.

Put
$$\begin{aligned} v_k &= \frac{1}{|I|} \sum_{\nu \in I} c_\nu \qquad &&\text{for} \quad k \in I = I_{ij} \\ &= c_k &&\text{for} \quad k \notin E = \cup I \\ u_k &= w_k^{(ij)} = c_k - v_k \quad &&(k \in I = I_{ij}) \\ &= 0 &&(k \notin I = I_{ij}) \\ w_k &= \sum_{i,j} w_k^{(ij)} = \sum_{i,j} u_k\,. \end{aligned}$$

Clearly (i), (v) and (vi) are satisfied. Let us verify (ii). If $k \in I$ for some pair (i, j) then

$$(*) \qquad |v_k| = \frac{1}{|I|} \Big| \sum_{\nu \in I} c_\nu \Big| \leqq \frac{1}{|I|} \sum_{\nu \in I} |c_\nu| \leqq \frac{A_\alpha}{\mu_{-\alpha}(I)} \sum_{\nu \in I} |c_\nu|\, \Omega^{-\alpha}(\nu)\,.$$

In fact, if I is "purely dyadic", $I = [2^m, 2^n)$, we have

$$\mu_{-\alpha}(I) = \sum_{j=m}^{n-1} \sum_{\nu=2^j}^{2^{j+1}-1} \Omega^{-\alpha}(\nu) \geqq A_\alpha 2^{-n(1-\alpha)}$$

and

$$\frac{1}{|I|}\sum_{\nu\in I}|c_\nu| \leqq 2^{-n+1}\sum_{j=m}^{n-1}\sum_{\nu=2^j}^{2^{j+1}-1}|c_\nu| \leqq 2\cdot 2^{-n(1-\alpha)}\sum_{j=m}^{n-1}2^{-j\alpha}\sum_{\nu=2^j}^{2^{j+1}-1}|c_\nu|$$

$$\leqq \frac{A_\alpha}{\mu_{-\alpha}(I)}\sum_{j=m}^{n-1}\sum_{\nu=2^j}^{2^{j+1}-1}|c_\nu|\,\Omega^{-\alpha}(\nu) = \frac{A_\alpha}{\mu_{-\alpha}(I)}\sum_{\nu\in I}|c_\nu|\Omega^{-\alpha}(\nu).$$

and if I is a part of an "elementary dyadic interval", the proof is simpler. Thus, by the definition of the intervals I's, we have

$$|v_k| \leqq \frac{A_\alpha}{\mu_{-\alpha}(I)}\sum_{\nu\in I}|c_\nu|\Omega^{-\alpha}(\nu) \leqq A_\alpha y$$

and (ii) holds for $k\in E$. But, if $k\notin E$, k must be the unique element of a certain J_{ij}, and the inequality distinguishing J's from I's may be interpreted as $|v_k|\leqq y$.

(iii) is directly verified:

$$\|v\|_{1,-\alpha} = \Big(\sum_{k\notin E} + \sum_{k\in E}\Big)|v_k|\Omega^{-\alpha}(k)$$

$$\leqq \sum_{k\notin E}|c_k|\Omega^{-\alpha}(k) + \sum_{i,j}\sum_{k\in I}|v_k|\Omega^{-\alpha}(k)$$

(by (*) above) $$\leqq \sum_{k\notin E}|c_k|\Omega^{-\alpha}(k) + A_\alpha\sum_{i,j}\sum_{k\in I}\frac{\Omega^{-\alpha}(k)}{\mu_{-\alpha}(I)}\sum_{\nu\in I}|c_\nu|\Omega^{-\alpha}(\nu)$$

$$= \sum_{k\notin E}|c_k|\Omega^{-\alpha}(k) + A_\alpha\sum_{i,j}\sum_{\nu\in I}|c_\nu|\Omega^{-\alpha}(\nu)$$

$$\leqq A_\alpha\sum_{k}|c_k|\Omega^{-\alpha}(k) = A_\alpha\|c\|_{1,-\alpha} \qquad \text{q.e.d.}$$

(iv) follows from (iii) and (v):

$$\sum_{i,j}\|w^{(ij)}\|_{1,-\alpha} = \|w\|_{1,-\alpha} \leqq \|c\|_{1,-\alpha} + \|v\|_{1,-\alpha}$$

This completes the proof.

LEMMA 8. *Let $c=\{c_k\}\in l^1_{-\alpha}$, $0\leqq\alpha<1$ be decomposed by the preceding Lemma and let $u=w^{(ij)}$ be a "piece" of it. If the interval $I=I_{ij}$ carrying u is contained in an elementary dyadic interval, then*

$$u_{nk} = \int u(t)\,\chi_n(t)\,\psi_k(t)\,dt = 0 \quad \textit{for} \quad k\notin I,$$

where $u(t)=\sum_{\nu\in I} u_\nu \psi_\nu(t)$.

PROOF. I is of the form

$$[2^{n(1)}+\cdots+2^{n(r-1)},\ 2^{n(1)}+\cdots+2^{n(r-1)}+2^{n(r)})$$

where $n(1)>\cdots>n(r-1)\geqq n(r)\geqq 0$. Thus

$$u_{nk}=\int u(t)\,\chi_n(t)\,\psi_k(t)\,dt$$

$$=\int \psi_N(t)\sum_{\nu=0}^{2^s-1} u_{N+\nu}\psi_\nu(t)\,\chi_n(t)\,\psi_k(t)\,dt$$

where $s=n(r)$ and $N=2^{n(1)}+\cdots+2^{n(r-1)}$. Since $\psi_\nu(t)=1$ for $0\leqq t<2^{-s}$ and $0\leqq \nu<2^s$, (vi) of the preceding Lemma combined with the definition of $\chi_n(t)$ gives

$$u_{nk}=0 \quad \text{for} \quad n>s=n(r)\,.$$

On the other hand, if $n\leqq s$, $\sum_{\nu=0}^{2^s-1} u_{N+\nu}\psi_\nu(t)$ and $\chi_n(t)$ are Walsh polynomials of degree $\leqq 2^s$, and so is their product, while $\psi_N(t)\psi_k(t)$ is a monomial of degree $>2^s$ for $k\notin I$. Thus the integral vanishes by the orthogonality of the Walsh functions.

LEMMA 9. *Let* $u=w^{(ij)}$ *as above Lemma, with* $I=I_{ij}$ *"purely dyadic". and let* u_{nk}, u_k^* *be defined as before. Then*

$$\sum_{k\notin I} |u_k^*|\,\Omega^{-\alpha}(k)\leqq A_\alpha \|u\|_{1,-\alpha}\,.$$

PROOF. We have $I=[2^l,2^m)$ for some integers l, m and

$$u_{nk}=\int u(t)\,\chi_n(t)\,\psi_k(t)\,dt\,.$$

u_{nk} vanishes for $n>m$ as in Lemma 8. If $n\leqq m$, $u(t)\chi_n(t)$ is a Walsh polynomial of degree $\leqq 2^m$, so that u_{nk} again vanishes for $k\geqq 2^m$. If $k<2^l$ and $n\leqq l$, $\psi_k(t)\chi_n(t)$ is a Walsh polynomial of degree $\leqq 2^l$, orthogonal to $u(t)$.

Thus we have only to consider the case $k<2^l$, $l<n\leqq m$. Now $\psi_k(t)=1$ wherever $\chi_n(t)=1$, and since $\chi_n(t)$ is expressed by a difference of Dirichlet kernels,

$$\chi_n(t) = 2^{-n}(2D_{2^{n-1}}(t) - D_{2^n}(t)),$$

we have

$$u_{nk} = 2^{-n}\int u(t)(2D_{2^{n-1}}(t) - D_{2^n}(t))\,dt = 2^{-n}\Big(\sum_{v=2^l}^{2^{n-1}-1} u_v - \sum_{v=2^{n-1}}^{2^n-1} u_v\Big).$$

Consequently

$$|u_k^*| = \Big|\sum_n \varepsilon_n u_{nk}\Big| \leqq \sum_n |u_{nk}| \leqq \sum_{n=l+1}^{m} 2^{-n}\sum_{v=2}^{2^n-1} |u_v|$$

$$= \sum_{n=l+1}^{m} 2^{-n}\sum_{j=l}^{n-1}\sum_{v=2^j}^{2^{j+1}-1} |u_v| \leqq \sum_{j=l}^{m-1}\sum_{v=2^j}^{2^{j+1}-1} |u_v| \sum_{n=j+1}^{\infty} 2^{-n}$$

$$= \sum_{j=l}^{m-1} 2^{-j}\sum_{v=2^j}^{2^{j+1}-1} |u_v|,$$

which is constant for $0 \leqq k < 2^l$. Thus

$$\sum_{k \notin I} |u_k^*|\Omega^{-\alpha}(k) = \sum_{k=0}^{2^l-1} |u_k^*|\Omega^{-\alpha}(k)$$

$$\leqq A_\alpha 2^{l(1-\alpha)}\sum_{j=l}^{m-1} 2^{-j}\sum_{v=2^j}^{2^{j+1}-1} |u_v| \leqq A_\alpha \sum_{j=l}^{m-1} 2^{-j\alpha}\sum_{v=2^j}^{2^{j+1}-1} |u_v| = A_\alpha \|u\|_{1,-\alpha}.$$

Combining Lemmas 8 and 9, we have

LEMMA 10. *Let* $F = \{k \notin E;\ |w_k^*| > y\}$. *Then*

$$\mu_{-\alpha}(F) \leqq A_\alpha \|c\|_{1,-\alpha}/y.$$

PROOF. By Lemmas 8 and 9, $w_k^* = \sum_{i,j} u_k^* = \sum_{i,j}\sum_n \varepsilon_n u_{nk}$ is certainly defined at least for $k \notin E$. Now

$$y\mu_{-\alpha}(F) = y\sum_{k \in F}\Omega^{-\alpha}(k) \leqq \sum_{k \in F} |w_k^*|\Omega^{-\alpha}(k) \leqq \sum_{k \in F}\Omega^{-\alpha}(k)\sum_{i,j} |u_k^*|$$

$$\leqq \sum_{i,j}\sum_{k \notin I} |u_k^*|\Omega^{-\alpha}(k) \leqq A_\alpha \sum_{i,j} \|u\|_{1,-\alpha} \leqq A_\alpha \|c\|_{1,-\alpha}, \qquad \text{q.e.d.}$$

PROOF OF THEOREM 2. Let $G \equiv \{k : |c_k^*| > 2y\}$ and let v, w be the decomposition of c by Lemma 7. It is easy to see that $v \in l^2$.

Clearly

$$\begin{aligned} G &\subset \{k : |v_k^*| > y\} \cup \{k : |w_k^*| > y\} \\ &\subset \{k : |v_k^*| > y\} \cup E \cup F\,, \end{aligned}$$

where F is defined in Lemma 10.

Both E and F are of measure $\leqq \frac{A_\alpha}{y} \|c\|_{1,-\alpha}$ by Lemmas 7 and 10, while

$$\begin{aligned} \mu_{-\alpha}(\{k : |v_k^*| > y\}) &\leqq \frac{1}{y^2} \sum_k |v_k^*|^2 \Omega^{-\alpha}(k) \\ &\leqq \frac{A_\alpha}{y^2} \sum_k |v_k|^2 \Omega_{-\alpha}(k) \qquad \text{(by Lemma 6)} \\ &\leqq \frac{A_\alpha}{y} \sum_k |v_k|\, \Omega_{-\alpha}(k) \leqq \frac{A_\alpha}{y} \|c\|_{1,-\alpha}\,, \end{aligned}$$

q.e.d.

In order to obtain Hirschman's inequalities for $1 < p \leqq 2$, we interpolate Theorem 1 and Lemma 1 for compact case, Theorem 2 and Lemma 6 for discrete case. Now the standard conjugacy argument gives the required result for $p \geqq 2$; this is evidently possible for compact case, and for discrete case, this is assured by the following

LEMMA 11. *Let $a = \{a_k\}$ be a suitably restricted sequence (e.g. $a \in l^2$) and let $b = \{b_k\}$ be a finite sequence. Then we have*

$$\sum_k a_k^* b_k = \sum_k a_k b_k^*$$

PROOF. Write $\sum a_k \psi_k(t) = g(t)$, $\sum b_k \psi_k(t) = h(t)$. Then

$$\begin{aligned} \sum_k a_k^* b_k &= \sum_k b_k \sum_n \varepsilon_n a_{nk} \\ &= \sum_n \varepsilon_n \sum_k \int\int g(t) \chi_n(t) h(u) \psi_k(t \dotplus u)\, du\, dt \\ &= \sum_n \varepsilon_n \int g(t) \chi_n(t) h(t)\, dt \\ &= \sum_n \varepsilon_n \int \Big(\sum_k a_k \psi_k(t) \Big) \chi_n(t) h(t)\, dt \\ &= \sum_n \varepsilon_n \sum_k a_k b_{nk} = \sum_k a_k \sum_n \varepsilon_n b_{nk} = \sum_k a_k b_k^*\,, \end{aligned}$$

q.e.d.

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