AN ALMOST COMPLEX STRUCTURE OF THE TANGENT BUNDLE OF AN ALMOST CONTACT MANIFOLD

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Introduction. In the paper [5], some almost complex structures of the tangent bundle of an almost contact manifold were studied. They are defined by linear connections and an almost contact structure of the base space. Their integrability conditions are closely related to the curvature tensors. "Among the almost complex structures there is one which does not depend on the linear connection. Its integrability is equivalent to the normality of the almost contact structure." But the proof of the latter fact consists of highly complicated computations. In this report, we take different approach and define another kind of almost complex structure which is related essentially to the torsion tensor of the linear connection. By taking special case, we give another verification of the above quoted fact "$\cdots$" which is easier to follow than the former. In the last section, we treat transformations or infinitesimal transformations of a restricted type.

1. Preliminary. TM and π denote (the total space of) the tangent bundle of an almost contact manifold M and the natural projection of TM onto M. Let $U(x^i)$ be a coordinate neighborhood in M with local coordinates x^i, $i=1,2,\cdots, m=\dim M$. Then for a point y of TM, we can take (x^i, y^j) as local coordinates of y where x^i are local coordinates of πy in U and y^j are components of y with respect to the natural frame $\left(\frac{\partial}{\partial x^i}\right)$. The coordinate transformation in $\pi^{-1}(U)\cap\pi^{-1}(U')$, when $U(x^i)\cap U'(x'^i)$ is non-empty, is given by

$$x'^i = x'^i(x^j)\,,\quad y'^i = \frac{\partial x'^i}{\partial x^j}y^j\,.$$

If M has a linear connection ∇, there corresponds a connection map K_y: $(TM)_y \to M_{\pi y}$, $y\in TM$, where $(TM)_y$ (or $M_{\pi y}$) is the tangent space at y (or πy) to TM (or M). For $X=(X^i, X^{m+i})$ the expression of K_yX by the local coordinates is

$$(K_yX)^j = X^{m+j} + y^r\Gamma^j_{rs}X^s\,, \tag{1. 1}$$

where $\Gamma^j_{rs}\dfrac{\partial}{\partial x^j} = \nabla_{\frac{\partial}{\partial x^s}}\dfrac{\partial}{\partial x^r}$. If we take two linear connections ∇ and $\widetilde{\nabla}$ (K and $\widetilde{K}$) and define Q by

$$\widetilde{K}_y X = K_y X + Q(y, \pi_y X), \tag{1. 2}$$

then the relations

$$\widetilde{\nabla}_y \xi = \nabla_y \xi + Q(\xi, y), \tag{1. 3}$$

$$\widetilde{\nabla}_y \eta \cdot u = \nabla_y \eta \cdot u - \eta \cdot Q(u, y), \tag{1. 4}$$

$$\widetilde{\nabla}_y \phi \cdot u = \nabla_y \phi \cdot u + Q(\phi u, y) - \phi \cdot Q(u, y), \tag{1. 5}$$

hold good, where y and u are vector fields on M and ϕ, ξ, η are the tensors which define an almost contact structure on M.

The vector field u° on TM is called the vertical lift of a vector field u on M if u° has components $(0, u^i)$, namely if $Ku^\circ = u$ and $\pi u^\circ = 0$. And we call the vector field $\bar{u}$ on TM the extended vector field or extension of u when $\bar{u}$ has components $\left(u^i, \dfrac{\partial u^i}{\partial x^j} y^j\right)$, this is characterised by $\pi \bar{u} = u$ and

$$K_y \bar{u} = \nabla_y u + T(y, u), \tag{1. 6}$$

where T denotes the torsion tensor of the linear connection ∇. The following relations are useful to calculate the integrability condition of the almost complex structure in §3,

$$[u^\circ, v^\circ] = 0, \tag{1. 7}$$

$$[\bar{u}, v^\circ] = [u, v]^\circ, \tag{1. 8}$$

$$[\bar{u}, \bar{v}] = \overline{[u, v]}. \tag{1. 9}$$

To see these, it is enough to express each vector by the local coordinates, for example, (1. 8) is shown as follows:

$$\begin{aligned} [\bar{u}, v^\circ] &= \left[u^i \frac{\partial}{\partial x^i} + \frac{\partial u^i}{\partial x^r} y^r \frac{\partial}{\partial y^i},\ v^j \frac{\partial}{\partial y^j}\right] \\ &= u^i \frac{\partial v^j}{\partial x^i} \frac{\partial}{\partial y^j} - v^j \frac{\partial u^i}{\partial x^j} \frac{\partial}{\partial y^i} = [u, v]^j \frac{\partial}{\partial y^j}. \end{aligned}$$

The next four tensors are of importance in the theory of an almost contact structure

$$S(u,v)=[u,v]+\phi[\phi u,v]+\phi[u,\phi v]-[\phi u,\phi v]$$
$$+\{v\cdot\eta(u)-u\cdot\eta(v)\}\xi\,,$$
$$S_1(u,v)=\mathfrak{L}(\phi u)\eta\cdot v-\mathfrak{L}(\phi v)\eta\cdot u\,,$$
$$S_2(u)=\mathfrak{L}(\xi)\phi\cdot u\,,$$
$$S_3(u)=\mathfrak{L}(\xi)\eta\cdot u\,,$$

where $\mathfrak{L}$ denotes the operator of Lie derivation. If $\eta(u)$ and $\eta(v)$ are constant, $S(u,v)$ is equal to the sum of the first four terms and

$$S_1(u,v)=-\eta\{[\phi u,v]-[\phi v,u]\}\,.$$

2. Definition. We define $(1,1)$-tensor J on TM using a connection map K by

$$\pi_yJX=\phi\pi_yX+[\eta(K_yX)+\nabla_y\eta\cdot\pi_yX]\xi\,, \tag{2. 1}$$

$$K_yJX=\phi K_yX-\eta(\pi_yX)\xi+\nabla_y\phi\cdot\pi_yX \tag{2. 2}$$
$$+[\eta(K_yX)+\nabla_y\eta\cdot\pi_yX]\nabla_y\xi\,,$$

where $y\in TM$, X is any tangent vector field on TM. We have omitted subscripts πy in the right hand side for brevity, we adopt these abbreviations in the following.

PROPOSITION 2-1. *J defines an almost complex structure on TM. For two linear connections ∇ and $\widetilde{\nabla}$, J and $\widetilde{J}$ are equal if and only if*

$$Q(u,v)-Q(v,u)+\phi\cdot Q(u,\phi v)-\phi\cdot Q(\phi v,u)=0 \tag{2. 3}$$

for any vector fields u,v on M.

PROOF. First $\pi JJX=-\pi X$ and $KJJX=-KX$ follow by direct calculation from (2. 1) and (2. 2). Next, if we utilize (1. 2) and(1. 4), we have

$$\pi_y\widetilde{J}X=\phi\pi_yX+[\eta(\widetilde{K}_yX)+\widetilde{\nabla}_y\eta\cdot\pi_yX]\,\xi$$
$$=\phi\pi_yX+[\eta(K_yX)+\eta\cdot Q(y,\pi_yX)+\nabla_y\eta\cdot\pi_yX-\eta\cdot Q(\pi_yX,y)]\,\xi\,.$$

Therefore we get

$$(2.\ 4)\qquad \pi_y\tilde{J}X - \pi_y JX = \eta[Q(y, \pi_y X) - Q(\pi_y X, y)]\,\xi\,.$$

On the other hand

$$\tilde{K}_y\tilde{J}X = K_y\tilde{J}X + Q(y, \pi_y\tilde{J}X)\,.$$

And here we assume that $\pi_y\tilde{J}X = \pi_y JX$, then

$$\begin{aligned}(2.\ 5)\qquad K_y\tilde{J}X &= \phi\tilde{K}_y X - \eta(\pi_y X)\xi + \tilde{\nabla}_y\phi\cdot\pi_y X\\ &\quad + [\eta(\tilde{K}_y X) + \tilde{\nabla}_y\eta\cdot\pi_y X]\,\tilde{\nabla}_y\xi - Q(y, \pi_y JX)\\ &= K_y JX + \phi\cdot Q(y, \pi_y X) + Q(\phi\pi_y X, y) - \phi\cdot Q(\pi_y X, y) - Q(y, \phi\pi_y X)\\ &\quad + [\eta(K_y X) + \nabla_y\eta\cdot\pi_y X]\,[Q(\xi, y) - Q(y, \xi)]\,,\end{aligned}$$

where we have used (1. 3), (1. 4), (1. 5) and (2. 4). Thus, if $\tilde{J} = J$, it follows from (2. 4) that

$$(2.\ 6)\qquad \eta\,[Q(y, \pi_y X) - Q(\pi_y X, y)] = 0$$

and, further from (2. 5)

$$\begin{aligned}(2.\ 7)\qquad &\phi\cdot Q(y, \pi_y X) - \phi\cdot Q(\pi_y X, y) + Q(\phi\pi_y X, y) - Q(y, \phi\pi_y X)\\ &+ [\eta(K_y X) + \nabla_y\eta\cdot\pi_y X]\,[Q(\xi, y) - Q(y, \xi)] = 0\,.\end{aligned}$$

We put $X = \xi^{\circ}$ in (2. 7), then we have

$$(2.\ 8)\qquad Q(\xi, y) - Q(y, \xi) = 0\,,$$

Operating ϕ to (2. 7), and utilizing (2. 6) and (2. 8), we have (2. 3).

Conversely we assume (2. 3). If we put $v = \xi$, $u = y$, we have (2. 8). If we operate η and ϕ to (2. 3) replaced u, v by $y, \pi_y X$, we have (2. 6) and (2. 7). Thereby $\pi JX = \pi\tilde{J}X$ and $KJX = K\tilde{J}X$ hold good.

THEOREM 2-2. *The tangent bundle TM of an almost contact manifold M has an almost complex structure J associating with a linear connection ∇. J depends only on the torsion of ∇. Therefore, if we take a symmetric connection as ∇, J does not depend on the connection.*

PROOF. If we denote by T and $\tilde{T}$ the torsion tensors of the two linear connections ∇ and $\tilde{\nabla}$, we have

$$Q(u,v)-Q(v,u)=\widetilde{T}(u,v)-T(u,v).$$

And so by Proposition 2-1, we see that $T=\widetilde{T}$ means $J=\widetilde{J}$. Next we suppose that the linear connection ∇ which is used in the definition of J is symmetric. Let $\widetilde{\nabla}$ be a flat connection (here we can assume that a flat connection exists, by restricting ourselves to a small coordinate neighborhood, if necessary). As the torsion tensors of ∇ and $\widetilde{\nabla}$ are equal (zero), J and $\widetilde{J}$ coincide completely. Clearly the expression of $\widetilde{J}$ by local coordinates includes only ϕ, ξ, η and their partial derivatives. And so does J. We call such an almost complex structure the natural almost complex structure.

3. Integrability of J. For the convenience of the calculation, we suppose that, in this section, the torsion tensor T of ∇ satisfies

$$T(u,v)+\phi\cdot T(u,\phi v)=0 \tag{3.1}$$

for any vector fields u and v on M.

LEMMA 3-1. *If $\eta(u)=\alpha$ is constant, then*

$$J\bar{u}=\overline{\phi u}-\alpha\xi^{\circ}. \tag{3.2}$$

PROOF. By virtue of (1.6) and the definition (2.1) and (2.2), we have

$$\pi_y J\bar{u}=\phi u+(\eta\cdot T(y,u))\xi,$$

$$K_y J\bar{u}=\nabla_y(\phi u)+\phi\cdot T(y,u)-\alpha\xi+(\eta\cdot T(y,u))\nabla_y\xi.$$

Under the assumption (3.1), $\eta\cdot T(y,u)=0$ and $\phi\cdot T(y,u)=T(y,\phi u)$ hold. Therefore we get $\pi_y J\bar{u}=\pi_y\overline{\phi u}$ and

$$\begin{aligned} K_y J\bar{u} &= \nabla_y(\phi u)+T(y,\phi u)-\alpha\xi \\ &= K_y(\overline{\phi u}-\alpha\xi^{\circ}). \end{aligned}$$

LEMMA 3-2. *If $\eta(u)=\alpha$ is constant, then*

$$Ju^{\circ}=(\phi u)^{\circ}+\alpha\bar{\xi}. \tag{3.3}$$

PROOF. In the similar fashion, we have

$$\pi_y Ju^{\circ}=\alpha\xi=\pi_y(\alpha\bar{\xi}),$$

$$K_y Ju^{\circ}=\phi u+\alpha\nabla_y\xi=K_y((\phi u)^{\circ}+\alpha\bar{\xi}).$$

This proves Lemma 3-2. Now, we denote by N the Nijenhuis tensor of J, i.e., for any vector fields X, Y on TM

$$N(X, Y) = [X, Y] + J[JX, Y] + J[X, JY] - [JX, JY].$$

LEMMA 3-3. *If $\eta(u)=\alpha$ and $\eta(v)=\beta$ are constant, we have*

$$\pi_y N(u^\circ, v^\circ) = -[\alpha S_3(v) - \beta S_3(u)]\,\xi, \tag{3.4}$$

$$K_y N(u^\circ, v^\circ) = -\alpha S_2(v) + \beta S_2(u) - [\alpha S_3(v) - \beta S_3(u)]\,\nabla_y \xi, \tag{3.5}$$

$$\pi_y N(\bar{u}, v^\circ) = -[S_1(u, v) - \beta \nabla_y(S_3(u))]\,\xi + \beta S_2(u), \tag{3.6}$$

$$\begin{aligned} K_y N(\bar{u}, v^\circ) = {} & S(u, v) - [S_1(u, v) - \beta \nabla_y(S_3(u))]\,\nabla_y \xi \\ & + \beta T(y, S_2(u)) + \beta\,[\nabla_y(S_2(u)) - S_3(u)\xi], \end{aligned} \tag{3.7}$$

$$\pi_y N(\bar{u}, \bar{v}) = S(u, v) - \nabla_y(S_1(u, v)) \cdot \xi + [\alpha S_3(v) - \beta S_3(u)]\,\xi, \tag{3.8}$$

$$\begin{aligned} K_y N(\bar{u}, \bar{v}) = {} & \nabla_y(S(u, v)) + T(y, S(u, v)) + S_1(u, v) \cdot \xi \\ & + \alpha S_2(v) - \beta S_2(u) - \nabla_y(S_1(u, v)) \cdot \nabla_y \xi \\ & + [\alpha S_3(v) - \beta S_3(u)]\,\nabla_y\, \xi. \end{aligned} \tag{3.9}$$

PROOF. By virtue of (1.7), (1.8), (1.9), Lemma 3-1 and 3-2, we get

$$\begin{aligned} \pi_y N(u^\circ, v^\circ) &= \pi_y[u^\circ, v^\circ] + \pi_y J[Ju^\circ, v^\circ] + \pi_y J[u^\circ, Jv^\circ] - \pi_y[Ju^\circ, Jv^\circ] \\ &= \pi_y J[\alpha\bar{\xi}, v^\circ] + \pi_y J[u^\circ, \beta\bar{\xi}] - \pi_y[(\phi u)^\circ, \beta\bar{\xi}] - \pi_y[\alpha\bar{\xi}, (\phi v)^\circ] \\ &= \alpha \pi_y J[\xi, v]^\circ + \beta \pi_y J[u, \xi]^\circ \end{aligned}$$

and by the definition (2.1)

$$\begin{aligned} &= (\alpha\eta[\xi, v] + \beta\eta[u, \xi])\,\xi \\ &= -[\alpha\mathfrak{L}(\xi)\eta \cdot v - \beta\mathfrak{L}(\xi)\eta \cdot u]\,\xi. \end{aligned}$$

Thus we obtain (3.4). In the next place

$$\begin{aligned} K_y N(u^\circ, v^\circ) &= K_y J[\alpha\bar{\xi}, v^\circ] + K_y J[u^\circ, \beta\bar{\xi}] - K_y[(\phi u)^\circ, \beta\bar{\xi}] - K_y[\alpha\bar{\xi}, (\phi v)^\circ] \\ &= \alpha\phi[\xi,\ v] + \alpha\eta([\xi, v]) \cdot \nabla_y \xi \\ &\quad + \beta\phi[u, \xi] + \beta\eta([u, \xi]) \cdot \nabla_y \xi - \beta[\phi u, \xi] - \alpha[\xi, \phi v] \\ &= -[\alpha\mathfrak{L}(\xi)\eta \cdot v - \beta\mathfrak{L}(\xi)\eta \cdot u]\,\nabla_y \xi - \alpha\mathfrak{L}(\xi)\phi \cdot v + \beta\mathfrak{L}(\xi)\phi \cdot u, \end{aligned}$$

this proves (3.5). The verifications for (3.6)~(3.9) may be done similarly, and so we shall omit them.

THEOREM 3-4. *The natural almost complex structure J of TM is integrable if and only if the almost contact structure is normal.*

PROOF. If J is integrable, by (3.4) and (3.8) we see that $S(u,v) =\nabla_y(S_1(u,v))\cdot\xi$. Since y is an arbitrary point of TM, $S(u,v)=0$, namely the almost contact structure is normal. Conversely if $S=0$ we have $S_1=S_2 =S_3=0$, consequently, $N=0$.

REMARK. So far as J is concerned, the assumption (3.1) is equivalent to $T=0$ by Proposition 2-1.

4. **Transformations**. Let $\mu:(x^i)\rightarrow(x'^i)$ be a transformation of M, then an extended transformation $\bar{\mu}$ of TM is characterized by $(x^i,y^i)\rightarrow\left(x^{i'},\dfrac{\partial x'^i}{\partial x^j}y^j\right)$. If we write its differential by the same letter, we have

$$\bar{\mu}\bar{u}=\overline{\mu u}\,, \tag{4. 1}$$

$$\bar{\mu}u^\circ=(\mu u)^\circ\,, \tag{4. 2}$$

for the extension or vertical lift of a vector field u on M. To see these, let $X=(X^i, X^{m+i})$ be any vector field on TM, then the image $\bar{\mu}X$ by the differential of $\bar{\mu}$ is given by

$$\left(\frac{\partial x'^i}{\partial x^s}X^s,\ \frac{\partial^2 x'^i}{\partial x^r\partial x^s}y^rX^s+\frac{\partial x'^i}{\partial x^r}X^{m+r}\right).$$

From this, (4.1) and (4.2) follow.

LEMMA 4-1. *Suppose that $\eta(u)=\alpha$ is constant and J is the natural almost complex structure (∇ : symmetric connection). Then for an extended transformation $\bar{\mu}$ we have*

$$\pi_yJ\bar{\mu}\bar{u}=\phi\mu u+\nabla_y(\eta\cdot\mu u)\cdot\xi\,, \tag{4. 3}$$

$$K_yJ\bar{\mu}\bar{u}=\nabla_y(\phi\mu u)+\nabla_y(\eta\cdot\mu u)\cdot\nabla_y\xi-(\eta\cdot\mu u)\,\xi\,, \tag{4. 4}$$

$$\pi_yJ\bar{\mu}u^\circ=(\eta\cdot\mu u)\,\xi\,, \tag{4. 5}$$

$$K_yJ\bar{\mu}u^\circ=\phi\mu u+(\eta\cdot\mu u)\,\nabla_y\xi\,. \tag{4. 6}$$

On the other hand, as for $\bar{\mu}J$, we have

$$\pi_y \bar{\mu} J \bar{u} = \mu \phi u \, , \tag{4. 7}$$

$$K_y \bar{\mu} J \bar{u} = \nabla_y (\mu \phi u) - \alpha \mu \xi \, , \tag{4. 8}$$

$$\pi_y \bar{\mu} J u^{\circ} = \alpha \mu \xi \, , \tag{4. 9}$$

$$K_y \bar{\mu} J u^{\circ} = \mu \phi u + \alpha \nabla_y (\mu \xi) \, , \tag{4. 10}$$

where $\eta(\mu u) = \mu^{-1*}(\mu^{*}\eta \cdot u)$.

Proofs are very easy because of (4. 1) and (4. 2).

THEOREM 4-2. *An extended transformation $\bar{\mu}$ is almost analytic with respect to the natural almost complex structure J of TM if and only if two relations $\phi\mu = \mu\phi$ and $\mu^{*}\eta = \sigma\eta$ hold for some constant σ.*

PROOF. If $J\bar{\mu} = \bar{\mu}J$, from (4. 3) and (4. 7) we get

$$\phi \mu = \mu \phi \, , \qquad \nabla_y (\mu^{-1*}(\mu^{*}\eta \cdot u)) = 0 \, .$$

The relation $\phi\mu = \mu\phi$ implies $\mu^{*}\eta = \sigma\eta$ for some scaler field σ on M. And so by $\nabla_y(\alpha \mu^{-1*}\sigma) = 0$, $d\sigma = 0$ holds. That is to say σ is constant. Conversely if $\phi\mu = \mu\phi$ and $\mu^{*}\eta = \sigma\eta$ are valid for constant σ, we have $\mu\xi = \sigma\xi$. Then (4.3) $\sim$(4.10) mean $J\bar{\mu} = \bar{\mu}J$.

COROLLARY 4-3. *An extension $\bar{u}$ of a vector field u on M is almost analytic with respect to the natural almost complex structure if and only if $\mathfrak{L}(u)\phi = 0$ and $\mathfrak{L}(u)\eta = c\eta$ hold for some constant c.*

Analogously to the preceeding Corollary, we have

THEOREM 4-4. *A vertical lift u° is almost analytic with respect to the natural almost complex structure if and only if u is an infinitesimal automorphism of the almost contact structure.*

PROOF. Let v be any vector field on M such that $\eta(v)$ is constant, then

$$\begin{aligned} \pi_y \mathfrak{L}(u^{\circ}) J \cdot v &= \pi_y [u^{\circ}, J\bar{v}] - \pi_y J [u^{\circ}, \bar{v}] \\ &= - \eta[u, v] \cdot \xi \\ &= (\mathfrak{L}(u)\eta \cdot v)\, \xi \, . \end{aligned}$$

And

$$K_y \mathfrak{L}(u^\circ)J \cdot \bar{v} = [u, \phi v] - \phi[u, v] - \eta[u, v] \cdot \nabla_y \xi$$
$$= \mathfrak{L}(u)\phi \cdot v + (\mathfrak{L}(u)\eta \cdot v)\nabla_y \xi .$$

Similarly we have

$$\pi_y \mathfrak{L}(u^\circ)J \cdot v^\circ = 0$$
$$K_y \mathfrak{L}(u^0)J \cdot v^\circ = \eta(v)[u, \xi] .$$

From these we see that $\mathfrak{L}(u^\circ)J=0$ if and only if $\mathfrak{L}(u)\phi=0$, $\mathfrak{L}(u)\eta=0$ and $[u, \xi] = 0$.

REMARK 1. If the almost contact structure is normal, the words 'almost complex' and 'almost analytic' turn to 'complex' and 'analytic' in Theorem 4-2, Corollary 4-3 and Theorem 4-4.

REMARK 2. With respect to the local coordinates, the natural almost complex structure J has following components ([5] §5)

$$\begin{pmatrix} \phi^i_j + \xi^i y^r \dfrac{\partial \eta_j}{\partial x^r} & \xi^i \eta_j \\ y^r \dfrac{\partial \phi^i_j}{\partial x^r} + y^r \dfrac{\partial \xi^i}{\partial x^r} y^s \dfrac{\partial \eta_j}{\partial x^s} - \xi^i \eta_j & \phi^i_j + y^r \dfrac{\partial \xi^i}{\partial x^r} \eta_j \end{pmatrix} .$$

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