SATURATION OF LOCAL APPROXIMATION BY LINEAR POSITIVE OPREATORS

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1. Introduction and inverse theorem. Let $f(x)$ be an integrable function, with period 2π and let its Fourier series be

$$(1) \qquad S[f] \equiv \sum_{k=0}^{\infty} A_k(x) \equiv \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx).$$

If the positivity of $f(x)$ implies the positivity of a linear operator $L_n(f, x)$, the operator is called a linear positive operator.

Let $\rho_k^{(n)}$ $(k=0, 1, 2, \cdots, \rho_0^{(n)}=1)$ be the "summating" function and consider a family of linear positive operators

$$(2) \qquad L_n(f, x) = \sum_{k=0}^{\infty} \rho_k^{(n)} A_k(x).$$

Let us suppose that for a positive constant C, we have

$$(3) \qquad \lim_{n\to\infty} \frac{1-\rho_k^{(k)}}{1-\rho_1^{(n)}} = Ck^2 \quad (k = 1, 2, \cdots).$$

The purpose of the present paper lies in considering local saturation by linear positive operators. Throughout the paper the norms should be taken with respect to the variable x and the subscript p $(1 \leqq p \leqq \infty)$ to L^p-norm will be generally omitted. Another convention is that the space (C) is meant by the notation L^∞, and the interval $[a, b]$ is an arbitrary subinterval of $[0, 2\pi]$. Thus the class $\mathrm{Lip}(\alpha, p)$ with $p = \infty$ reduces to $\mathrm{Lip}\,\alpha$. Also, let us write

$$\| L_n(f, x) - f(x) \|_{(a,b)} \equiv \left(\int_a^b | L_n(f, x) - f(x) |^p \, dx \right)^{\frac{1}{p}}$$

and

$$\mathrm{Lip}(1, p; a, b) \equiv \{ f(x) | \sup_{|h| \leqq \delta} \| f(x+h) - f(x) \|_{(a,b)} = O(\delta) \}.$$

THEOREM 1. (1°) *If* $\| L_n(f,x) - f(x)\|_{(a,b)} = O(1-\rho_1^{(n)})$, *then* $f(x)$ *is a linear function in* $[c,d]$, *where* $[c,d]$ *is any fixed subinterval of* $[a,b]$.

(2°) *If* $\| L_n(f,x) - f(x)\|_{(a,b)} = O(1-\rho_1^{(n)})$, *then* $f'(x)$ *belongs to the class* $\mathrm{Lip}\,(1,p;c,d)$.

THEOREM A. (G.Sunouchi [5]). *A necessary and sufficient condition for* $f''(x)$ *to exist and belong to the class B over* (a,b) *is the uniform boundedness of* $\sigma_m^2[x,S'']$ *over* $[a,b]$, *where* $\sigma_m^2[x,S'']$ *means the* $(C,2)$*-means of the second derived series of* (1).

THEOREM B. (G. Sunouchi [5]). *A necessary and sufficient condition for* $f''(x)$ *to exsist and belong to the class* L^p $(p>1)$ *over* (a,b) *is*

$$\int_a^b |\sigma_m^2[x,S'']|^p\,dx = O(1)\,.$$

THEOREM C. (G.Sunouchi [5]). *A necessary and sufficient condition for* $f'(x)$ *to exsist and belong to the class BV over* (a,b) *is*

$$\int_a^b |\sigma_m^1[x,S'']|\,dx = O(1)\,.$$

PROOF OF THEOREM 1. (1°) The proofs of the proposition (1°) and (2°) are almost the same. So we shall only give the proof of the proposition (2°) with respect to (C)-norm. The proofs of the propositions (1°) and (2°) in L^p-space are analogous to the case (C)-space.

(2°) Since

$$L_n(f,x) - f(x) = O(1-\rho_1^{(n)}) \text{ uniformly over } (a,b)\,,$$

we have

$$\sigma_m^2\left[x, \frac{1}{1-\rho_1^{(n)}}\{f(x) - L_n(x,f)\}\right] = O(1)$$

for every m and uniformly in x in any fixed interval subinterval of $[a,b]$, because

$$\frac{1}{1-\rho_1^{(n)}}\{L_n(f,x) - f(x)\} \sim \sum_{k=0}^{\infty} \frac{1-\rho_k^{(n)}}{1-\rho_1^{(n)}} A_k(x)\,,$$

(see Zygmund [6, p. 367, Th. 9.20]). Letting $n\to\infty$, we get by (3)

$$\sigma_m^3\left[x, \sum_{k=0}^{\infty} k^2 A_k(x)\right] = O(1).$$

Hence we have $f''(x) \in B$ in $[c, d]$ from Theorem A.

REMARK 1. We have only to apply Theorem B or C in order to verify the facts (1°) and (2°) in L^p-space ($p \geqq 1$).

2. Direct theorem. Let us suppose that the linear positive operator (2) can be represented in the following form:

$$L_n(f, x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)\, U_n(t)\, dt,$$

where

$$(4) \qquad U_n(t) = \frac{1}{2} + \sum_{k=1}^{\infty} \rho_k^{(n)} \cos kt \geqq 0\,.$$

THEOREM 2. *If $f'(x)$ belongs to the class* $\mathrm{Lip}(1, p; a, b)$ *and*

$$(*) \quad L_n(t^2, 0) = O(1-\rho_1^{(n)})$$

then

$$\| L_n(f, x) - f(x) \|_{(c,d)} = O(1-\rho_1^{(n)})\,.$$

REMARK 2. If the constant in (3) is 1, the condition (*) can be omitted (see, [1]).

PROOF OF THEOREM 2. Let us write $\delta \equiv \min(c-a, b-d)$. By generalized Minkowski inequality, we have

$$\left(\int_c^d | L_n(f, x) - f(x)|^p\, dx\right)^{\frac{1}{p}}$$

$$= \frac{1}{2\pi}\left\{\int_c^d \left|\int_0^{\pi} [f(x+t) - 2f(x)+f(x-t)]\, U_n(t)\, dt\right|^p dx\right\}^{\frac{1}{p}}$$

$$= \frac{1}{2\pi}\left\{\int_c^d \left|\left(\int_0^{\delta} + \int_{\delta}^{\pi}\right) [f(x+t) + f(x-t) - 2f(x)]\, U_n(t)\, dt\right|^p dx\right\}^{\frac{1}{p}}$$

$$\leqq \frac{1}{2\pi}\left\{\int_c^d \left|\int_0^\delta [f(x+t)+f(x-t)-2f(x)]\,U_n(t)\,dt\right|^p dx\right\}^{\frac{1}{p}}$$

$$+\frac{1}{2\pi}\left\{\int_c^d \left|\int_\delta^\pi [f(x+t)+f(x-t)-2f(x)]\,U_n(t)\,dt\right|^p dx\right\}^{\frac{1}{p}}$$

$$= I_1 + I_2\,, \text{ say.}$$

Then

$$I_1 \leqq \frac{1}{2\pi}\int_0^\delta U_n(t)\,dt\left(\int_c^d |f(x+t)+f(x-t)-2f(x)|^p\,dx\right)^{\frac{1}{p}}$$

$$\leqq \frac{M}{2\pi}\int_0^\delta t^2 U_n(t)\,dt$$

$$\leqq \frac{M}{2\pi}\int_{-\pi}^\pi t^2\,U_n(t)\,dt = \frac{M}{2}\,L_n(t^2,0) \qquad (5)\,.$$

$$I_2 = \frac{1}{2\pi}\left\{\int_c^d \left|\int_\delta^\pi [f(x+t)+f(x-t)-2f(x)]\,U_n(t)\,dt\right|^p dx\right\}^{\frac{1}{p}}$$

$$\leqq \frac{1}{2\pi}\left\{\int_c^d dx\left(\int_\delta^\pi |f(x+t)+f(x-t)-2f(x)|\,U_n(t)\,dt\right)^p\right\}^{\frac{1}{p}}$$

$$\leqq \frac{1}{2\pi}\left\{\int_c^d dx\left(\int_\delta^\pi |f(x+t)|\,U_n(t)\,dt + \int_\delta^\pi |f(x-t)|\,U_n(t)\,dt\right.\right.$$

$$\left.\left. + 2\int_\delta^\pi |f(x)|\,U_n(t)\,dt\right)^p\right\}^{\frac{1}{p}}$$

Since

$$\int_\delta^\pi |f(x+t)|\,U_n(t)\,dt \leqq \frac{1}{(1-\cos\delta)^2}\int_\delta^\pi (1-\cos t)^2\,U_n(t)\,|f(x+t)|\,dt\,,$$

we get

$$I_2 \leqq \frac{1}{2\pi}\,\frac{1}{(1-\cos\delta)^2}\left\{\int_c^d dx\left(\int_\delta^\pi (1-\cos t)^2\,U_n(t)\,|f(x+t)|\,dt\right.\right.$$

$$\left.\left. + \int_\delta^\pi (1-\cos t)^2\,U_n(t)|f(x-t)|\,dt + 2\int_\delta^\pi (1-\cos t)^2\,U_n(t)|f(x)|\,dt\right)^p\right\}^{\frac{1}{p}}$$

$$\leqq \frac{1}{2\pi}\frac{1}{(1-\cos\delta)^2}\left[\left\{\int_c^d dx\left(\int_\delta^\pi (1-\cos t)^2 U_n(t)\,|f(x+t)|\,dt\right)^p\right\}^{\frac{1}{p}}\right.$$

$$+\left\{\int_c^d dx\left(\int_\delta^\pi (1-\cos t)^2 U_n(t)\,|f(x-t)|\,dt\right)^p\right\}^{\frac{1}{p}}$$

$$\left.+\left\{\int_c^d dx\left(\int_\delta^\pi (1-\cos t)^2 U_n(t)\,|f(x)|\,dt\right)^p\right\}^{\frac{1}{p}}\right]$$

$$\leqq \frac{1}{2\pi}\frac{1}{(1-\cos\delta)^2}\left[\int_\delta^\pi (1-\cos t)^2 U_n(t)\,dt\left(\int_c^d |f(x+t)|^p\,dx\right)^{\frac{1}{p}}\right.$$

$$+\int_\delta^\pi (1-\cos t)^2 U_n(t)\,dt\left(\int_c^d |f(x-t)|^p\,dx\right)^{\frac{1}{p}}$$

$$\left.+\,2\int_\delta^\pi (1-\cos t)^2 U_n(t)\,dt\left(\int_c^d |f(x)|^p\,dx\right)^{\frac{1}{p}}\right]$$

$$\leqq \frac{M'}{2\pi}\frac{1}{(1-\cos\delta)^2}\int_\delta^\pi (1-\cos t)^2 U_n(t)\,dt$$

$$\leqq \frac{M'}{2\pi(1-\cos\delta)^2}\int_{-\pi}^\pi (1-\cos t)^2 U_n(t)\,dt$$

$$= \frac{M'(1-\rho_1^{(n)})}{4\pi(1-\cos\delta)^2}\left(4-\frac{1-\rho_2^{(n)}}{1-\rho_1^{(n)}}\right) = O(1-\rho_1^{(n)}) \qquad (6),$$

where we apply the condition (3) and the fact that

$$L_n(\psi_k, 0) = \frac{1}{\pi}\int_{-\pi}^\pi (1-\cos kt)\,U_n(t)\,dt = 1-\rho_k^{(n)},$$

$$\psi_k = 1-\cos kx \quad (k=1,2,\cdots).$$

Hence, by (5) and (6), for any function $f'(x) \in \mathrm{Lip}(1, p; a, b)$ we have

$$\|L_n(f, x)-f(x)\|_{(c,d)} = \frac{M}{2}L_n(t^2, 0)+O(1-\rho_1^{(n)}).$$

That is,

$$\|L_n(f, x)-f(x)\|_{(c,d)} = O(1-\rho_1^{(n)}) \qquad \text{q.e.d.}$$

3. Determination of the class of local saturation by some linear positive operators.

3.1 The integral of de la Vallée Poussin is defined by

$$V_n(x) = \frac{h_n}{2\pi}\int_{-\pi}^{\pi} f(x+t)\cos^{2n}\frac{t}{2}\,dt$$

$$= \sum_{k=0}^{n} \frac{(n!)^2}{(n-k)!\,(n+k)!} A_k(x)\,, \quad h_n = \frac{2n(2n-2)\cdots 4\cdot 2}{(2n-1)(2n-3)\cdots 3.1}\,,$$

$$\rho_k^{(n)} = \frac{(n!)^2}{(n-k)!\,(n+k)!} = 1 - \frac{k^2}{n} + O\left(\frac{1}{n^2}\right).$$

THEOREM 3. (R. G. Mamedov [2]). *For space L^p $(1 \leqq p \leqq \infty)$, the method of de la Vallée Poussin $V_n(x)$ is saturated locally; its order of saturation is n^{-1}, its class of saturation is the class of functions $f(x)$ for which*

$$f''(x) \in B[c, d] \quad (p = \infty)$$

$$f''(x) \in L^p[c, d] \quad (1 < p < \infty)$$

$$f'(x) \in BV[c, d] \quad (p = 1)\,.$$

PROOF. The proofs of inverse problem are easily verified, and so we may confine ourselves to the proof of direct problem. But, from the fact

$$\lim_{n\to\infty} \frac{1-\rho_k^{(n)}}{1-\rho_1^{(n)}} = k^2$$

we have the proof of the direct theorem by Theorem 2.

We may write this result by the notation

$$\text{L.Sat.}\ [V_n] = [\{f \mid f' \in \text{Lip}(1, p; a, b)\}\,,\ n^{-1},\ \text{linear function}\}]\,.$$

3.2 The integral of Jackson-de la Vallée Poussin is defined by

$$I_n(x) = \frac{1}{2\pi\tau_4}\int_{-\infty}^{\infty} f\left(x + \frac{2t}{n}\right)\left(\frac{\sin t}{t}\right)^4 dt \quad \left(\tau_4 = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{\sin t}{t}\right)^4 dt\right)$$

$$= \sum_{k=0}^{2n-1} h\left(\frac{k}{n}\right) A_k(x)\,,$$

$$h(x) = \begin{cases} 1 - \dfrac{3}{2} x^2 + \dfrac{3}{4} |x|^3 & \text{if } \ |x| \leqq 1\,, \\ \dfrac{1}{4} (2 - |x|)^3 & \text{if } \ 1 \leqq |x| \leqq 2\,, \\ 0 & \text{if } \ |x| \geqq 2\,. \end{cases}$$

THEOREM 4. (R. G. Mamedov [2]).

$$\text{L. Sat. } [I_n] = [\{f | f' \in \mathrm{Lip}(1, p;\, a, b)\}, n^{-2}, \textit{linear function}]\,.$$

PROOF. We have only to consider

$$\lim_{n \to \infty} \frac{1 - \rho_k^{(n)}}{1 - \rho_1^{(n)}} = k^2\,.$$

REMARK 3. M. G. Mamedov states that in Theorem 3 and 4

$$\| L_n(f, x) - f(x) \|_{(a,b)} = O(1 - \rho_1^{(n)})$$

implies $f(x) =$ constant, but a careful inspection of his method will tell that $f(x)$ is linear.

3.3 The Gauss-Weierstrass integral of $f(x)$ is

$$\begin{aligned} W(x, \xi) &= \sum_{k=0}^{\infty} \exp(-k^2 \xi / 4)\, A_k(x) \\ &= \sqrt{\frac{\pi}{\xi}} \int_{-\pi}^{\pi} f(x+t) \exp(-t^2/\xi)\, dt\,, \end{aligned}$$

$\rho_k^{(\xi)} = \exp(-k^2 \xi/4)$, the parameter ξ tending to zero.

We have

THEOREM 5.

$$\text{L. Sat. } [W_\xi] = [\{f | f' \in \mathrm{Lip}(1, p;\, a, b), \ \xi, \ \textit{linear function}]\,.$$

PROOF. Since

$$\lim_{\xi \to 0} \frac{1 - \rho_k^{(\xi)}}{1 - \rho_1^{(\xi)}} = k^2\,,$$

the proof is trivial.

4. Local saturation by generalized Jackson operators. P. P. Korovkin proved that the order of approximation by linear positive operators $L_n(f, x)$ was not better than n^{-2} in the following theorem.

THEOREM D (P. P. Korovkin [1]). *If $L_n(f, x)$ is a sequence of linear positive polynomial operators defined on the set of continuous and 2π-periodic functions, then at least one of the two sequences of numbers*

$$n^2 \max_{-\pi \leqq x \leqq \pi} L_n \left\{ \sin^2 \frac{t-x}{2}, x \right\},$$

$$n \max_{-\pi \leqq x \leqq \pi} |L_n(1, x) - 1|, \quad (n=1, 2, \cdots)$$

does not tend to zero.

The purpose of this section is determining the class and order of local saturation by generalized Jackson operators.

Let us write

$$L_{n,m}(f, x) = \frac{1}{2\pi \tau_m} \int_{-\infty}^{\infty} f\left(x + \frac{2t}{n}\right) \left(\frac{\sin t}{t}\right)^m dt,$$

$$\tau_m = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\frac{\sin t}{t}\right)^m dt$$

If, especially, m is even, the operator is a positive operator. From the view point of asymptotic approximation, Y. Matsuoka [3] and F. Schurer [4] studied the case of $m=4, 6$; $m=8, 10, 12$, respectively. We prove the following theorem. For the sake of simplicity, we state only the uniform norm.

THEOREM 6. *If $m \geqq 6$,*
then

$$\text{L. Sat. } [L_{n,m}(f, x)] = [\{f \mid f' \in \text{Lip}(1, \infty; a, b), n^{-2}, \textit{linear function}\}].$$

For the proof of Theorem 6, we need a lemma.

LEMMA. *If we denote by $C_0^{(4)}[a, b]$, the class of functions $g(x)$ such that $g(x)=0$ outside of $[a, b]$ and its 4-th derivative $g^{(4)}(x)$ is continuous in $[0, 2\pi]$. For any function $f(x) \in C_0^{(4)}[a, b]$ and for any point $x \in [a, b]$, we have*

$$L_{n,m}(f,x)-f(x)=M\cdot\frac{f''(x)}{n^2}+o\left(\frac{1}{n^2}\right),$$

where the order $o(n^{-2})$ *is independent of the point* x.

PROOF. Let M_i^n $(i=1,\cdots,4)$ be absolute constants. By Taylor's formula, for any point $x\in[a,b]$, we have

$$L_{n,m}(f,x)-f(x)=\frac{n}{2\pi\tau_m}\int_0^\infty[f(x+2t)+f(x-2t)-2f(x)]\left(\frac{\sin nt}{nt}\right)^m dt$$

$$=\frac{n}{2\pi\tau_m}\left(\int_0^\delta+\int_\delta^\infty\right)=I_1+I_2,\ \text{say},$$

where $\delta\equiv\min\left(\frac{a}{2},\frac{2\pi-b}{2}\right)$.

$$I_1\leqq\frac{2nf''(x)}{\pi\tau_m}\int_0^\delta t^2\left(\frac{\sin nt}{nt}\right)^m dt+\frac{M_1}{2\pi}\cdot\frac{n}{\tau_m}\int_0^\delta t^4\left(\frac{\sin nt}{nt}\right)^m dt$$

$$=\frac{M_2f''(x)}{n^{m-1}}\int_0^\delta\left(\frac{\sin nt}{t}\right)^{m-2}\sin^2 nt\,dt+\frac{M_3}{n^{m-1}}\int_0^\delta\left(\frac{\sin nt}{t}\right)^{m-4}\sin^4 nt\,dt$$

$$\leqq\frac{M_2f''(x)n^{m-2}}{n^{m-1}\cdot n}\int_0^\infty\left(\frac{\sin t}{t}\right)^{m-2}dt+\frac{M_3n^{m-4}}{n^{m-1}\cdot n}\int_0^\infty\left(\frac{\sin t}{t}\right)^{m-4}dt$$

$$=Mf''(x)\cdot\frac{1}{n^2}+M_4\frac{1}{n^4}=M\frac{f''(x)}{n^2}+o\left(\frac{1}{n^2}\right).$$

$$I_2\leqq\frac{n}{2\pi\tau_m}\int_\delta^\infty|f(x+2t)+f(x-2t)-2f(x)|\left(\frac{\sin nt}{nt}\right)^m dt$$

$$\leqq\frac{n}{2\pi\tau_m}\cdot\frac{1}{n^m}\int_\delta^\infty|f(x+2t)+f(x-2t)-2f(x)|\,t^{-m}\,dt$$

$$=O\left(\frac{1}{n^{m-1}}\right)=o\left(\frac{1}{n^4}\right).$$

Hence $$I_1+I_2=M\frac{f''(x)}{n^2}+o\left(\frac{1}{n^2}\right).$$

Thus we complete the proof of Lemma.

PROOF OF THEOREM 6. (i) If

$$\lim_{n\to\infty} n^2\{L_{n,m}(f,x)-f(x)\}=0 \text{ uniformly in } [a,b],$$

then, for any $g(x)\in C_0^{(4)}$, we have

$$\lim_{n\to\infty}\int_0^{2\pi} n^2\{L_{n,m}(f,x)-f(x)\}\,g(x)\,dx=0.$$

Since $L_{n,m}(f,x)$ has a symmetric kernel, we can interchange $f(x)$ and $g(x)$, that is

$$\int_0^{2\pi} n^2\{L_{n,m}(f,x)-f(x)\}\,g(x)\,dx=\int_0^{2\pi} n^2\{L_{n,m}(g,x)-g(x)\}\,f(x)\,dx.$$

On the other hand, Lemma gives

$$\lim_{n\to\infty} n^2\{L_{n,m}(g,x)-g(x)\}=Mg''(x), \text{ boundedly.}$$

Thus we get

$$\int_0^{2\pi} f(x)\,g''(x)\,dx=0$$

Hence by the well-known lemma, $f(x)$ is a polynomial of the first degree over $[c,d]$.

(ii) If $n^2\{L_{n,m}(f,x)-f(x)\}=O(1)$ uniformly in $[a,b]$, by the weak* compactness of the unit ball of the space $B[a,b]$, we can take a subsequence n_ν and a function $h(x)\in B[a,b]$ such that

$$\lim_{\nu\to\infty}\int_0^{2\pi} n_\nu^2\{L_{n_\nu,m}(f,x)-f(x)\}\,g(x)\,dx=\int_0^{2\pi} h(x)\,g(x)\,dx.$$

But the left-hand side is equal to

$$M\int_0^{2\pi} f(x)\,g''(x)\,dx$$

and the right hand side is equal to

$$\int_0^{2\pi} H_2(x)\,g''(x)\,dx$$

where $H_2(x)$ is a second integral of $h(x)$. Hence $H_2(x)-Mf(x)$ is at most a polynomial of the first degree in $[c, d]$ and $f''(x)$ is bounded in $[c, d]$.

(iii) Let us set $\delta \equiv \min\left(\frac{c-a}{2}, \frac{b-d}{2}\right)$. For any $x \in [a, b]$, we have

$$L_{n,m}(f(t), x) - f(x) = \frac{1}{2\pi\tau_m}\int_{-\infty}^{\infty}\left[f\left(x+\frac{2t}{n}\right)-f(x)\right]\left(\frac{\sin t}{t}\right)^m dt$$

$$= \frac{n}{2\pi\tau_m}\int_{-\infty}^{\infty}[f(x+2t)-f(x)]\left(\frac{\sin nt}{nt}\right)^m dt$$

$$= \frac{n}{2\pi\tau_m}\int_0^{\infty}[f(x+2t)+f(x-2t)-2f(x)]\left(\frac{\sin nt}{nt}\right)^m dt$$

$$= \frac{n}{2\pi\tau_m}\left(\int_0^{\delta}+\int_{\delta}^{\infty}\right) = I_1 + I_2, \text{ say}.$$

Then

$$I_2 = \frac{n}{2\pi\tau_m}\int_{\delta}^{\infty}[f(x+2t)+f(x-2t)-2f(x)]\left(\frac{\sin nt}{nt}\right)^m dt$$

$$\leqq \frac{n}{2\pi\tau_m}\int_{\delta}^{\infty}|f(x+2t)+f(x-2t)-2f(x)|\left(\frac{\sin nt}{nt}\right)^m dt$$

$$\leqq \frac{n}{2\pi\tau_m}\cdot\frac{1}{n^m}\int_{\delta}^{\infty}|f(x+2t)+f(x-2t)-2f(x)|\,t^{-m}\,dt$$

$$= O\left(\frac{1}{n^{m-1}}\right). \tag{7}$$

$$I_1 = \frac{n}{2\pi\tau_m}\int_0^{\delta}[f(x+2t)+f(x-2t)-2f(x)]\left(\frac{\sin nt}{nt}\right)^m dt$$

$$\leqq \frac{1}{2\pi\tau_m}\cdot\frac{1}{n^{m-1}}\int_0^{\delta}|f(x+2t)+f(x-2t)-2f(x)|\left(\frac{\sin nt}{t}\right)^m dt$$

$$\leqq \frac{1}{2\pi\tau_m}\cdot\frac{1}{n^{m-1}}\int_0^{\delta}t^2(\sin^{m-2} nt)\,t^{-m}\,dt$$

$$= \frac{1}{2\pi\tau_m}\cdot\frac{1}{n^{m-1}}\int_0^{\delta}\left(\frac{\sin nt}{t}\right)^{m-2} dt$$

$$= \frac{1}{2\pi\tau_m}\cdot\frac{1}{n^{m-1}}\int_0^{n\delta} n^{m-3}\left(\frac{\sin t}{t}\right)^{m-2} dt$$

$$= O\left(\frac{1}{n^{m-1}}\, n^{m-3}\cdot\int_0^{\infty}\left(\frac{\sin t}{t}\right)^{m-2} dt\right)$$

$$= O(n^{-2})\,. \qquad (8)$$

Hence, by (7) and (8), for any function $f'(x) \in \mathrm{Lip}(1\,; a, b)$ we have

$$L_{n,m}(f(t), x) - f(x) = I_1 + I_2 = O\Big(\frac{1}{n^{m-1}}\Big) + O\Big(\frac{1}{n^2}\Big)$$

$$= O\Big(\frac{1}{n^2}\Big) \quad \text{if} \quad m \geqq 4\,.$$

Thus we get the complete proof of the Theorem 6.

REMARK 4. From the Theorem 4, Theorem 6 and P. P. Korovkin's theorem, if follows that if $m(\geqq 4)$ is even, the method of generalized Jackson operators attains to the order of local best approximation by linear positive polynomial operators.

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