SOME REMARKS ON A REPRESENTATION OF A GROUP, II

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1. This note is a continuation of [5] and two examples of II_1-factors are constructed. The first example shows the following proposition that is an analogy of an example of [3] and that of [1].

PROPOSITION 1. *Let* $\boldsymbol{M}$ *be a hyperfinite continuous von Neumann algebra. Then there exist a regular maximal abelian subalgebra* $\boldsymbol{A}$ *and an abelian subalgebra* $\boldsymbol{B}$ *of* $\boldsymbol{M}$ *with the following properties.*

(1) $\boldsymbol{B}' \cap \boldsymbol{M} = \boldsymbol{A}$

(2) $\boldsymbol{A}$ *is a unique maximal abelian subalgebra of* $\boldsymbol{M}$ *which contains* $\boldsymbol{B}$ *and* $\boldsymbol{A} \neq \boldsymbol{B}$.

(3) $(\boldsymbol{B}' \cap \boldsymbol{M})' \cap \boldsymbol{M} \neq \boldsymbol{B}$.

The second example reproduces the following result of [2].

PROPOSITION 2. *There exists a group* G *of outer automorphisms of a hyperfinite continuous factor* $\boldsymbol{M}$ *such that the crossed product* $(\boldsymbol{M}, G)$ *does not have property* P *in the sense of* [6].

2. For convenience sake, we shall summerize the result of [7]. Let G be an arbitrary countably infinite group. Let Δ be the set of all functions $\alpha(g)$ on G: $\alpha(g)=1$ on a finite subset of G and $=0$ elsewhere, and Δ is an additive group under the addition $[\alpha+\beta](g) = \alpha(g)+\beta(g)$ (mod 2), $0(g)=0$ for all $g \in G$. Let Δ' be the set of all functions $\varphi(\gamma)$ on Δ: $\varphi(\gamma) = 1$ on a finite subset of Δ and $=0$ elsewhere. Δ' is an additive group under the addition $[\varphi+\psi](\gamma) = \varphi(\psi)+\psi(\gamma)$ (mod 2) and $0(\gamma) = 0$ for all $\gamma \in \Delta$. For every $\alpha \in \Delta$, $\varphi \to \varphi^\alpha$: $\varphi^\gamma(\gamma) = \varphi(\gamma+\alpha)$ is an automorphism of Δ'. Defining the product $(\varphi, \alpha)(\psi, \beta) = (\varphi^\beta+\psi, \alpha+\beta)$, we have a locally finite countably infinite group $\mathfrak{G}$ of all elements $(\varphi, \alpha) \in (\Delta', \Delta)$ with the identity $(0, 0)$ and $(\varphi, \alpha)^{-1} = (\varphi^\alpha, \alpha)$. Let $\boldsymbol{H}$ be the Hilbert space $l_2(\mathfrak{G})$, and for each $(\varphi, \alpha) \in \mathfrak{G}$ let $V_{(\varphi,\alpha)}$ be the unitary operator on $\boldsymbol{H}$ defined by $[V_{(\varphi,\alpha)}f]((\psi, \beta)) = f((\psi, \beta)(\varphi, \alpha))$. Then the ring of operators $\boldsymbol{M}=\boldsymbol{R}\ (V_{(\varphi,\alpha)} | (\varphi, \alpha) \in \mathfrak{G})$ is a hyperfinite continuous factor.

Next, define an operator T_g (resp. T'_g) on Δ (resp. Δ') for each $g \in G$ as follows:

$$[T_g\alpha](h) = \alpha(gh), \quad [T'_g\varphi](\gamma) = \varphi(T_g^{-1}\gamma) \quad \text{for} \quad \alpha \in \Delta, \ \varphi \in \Delta'.$$

Then, for each $g \in G$ we define a unitary operator U_g on $\boldsymbol{H}$ by $[U_g f]((\varphi, \alpha)) = f((T'_g\varphi, T_g\alpha))$, and $g \to U_g$ is a faithful unitary representation of G on $\boldsymbol{H}$ and for each $g \in G$ $(\neq e)$

$$V_{(\varphi,\alpha)} \to U_g^{-1} V_{(\varphi,\alpha)} U_g = V_{(T'_g\varphi, T_g\varphi)}$$

defines an outer automorphism of $\boldsymbol{M}$. Thus we can construct the crossed product $(\boldsymbol{M}, G)$ in the sense of [8] and $(\boldsymbol{M}, G)$ is a factor of type II_1.

3. In this section we shall prove Proposition 1. In §3 and §4 we use the notations used in §2. Let φ_0 be the element of Δ' which takes value 1 only at $0 \in \Delta$. Let $\boldsymbol{A} = \boldsymbol{R}(V_{(\varphi,0)} | \varphi \in \Delta')$ and $\boldsymbol{B} = \boldsymbol{R}(V_{(\varphi_0,0)})$. Then it is obvious that $\boldsymbol{A}$ and $\boldsymbol{B}$ are abelian subalgebras of $\boldsymbol{M} = \boldsymbol{R}(V_{(\varphi,\alpha)} | (\varphi, \alpha) \in \mathfrak{G})$ which is a hyperfinite continuous von Neumann algebra. We shall prove that these $\boldsymbol{A}$ and $\boldsymbol{B}$ satisfy the assertion of Proposition 1.

LEMMA 1. *$\boldsymbol{A}$ is a regular maximal abelian subalgebra of $\boldsymbol{M}$.*

PROOF. Let (φ, α) be an element of $\mathfrak{G}$ such that $(\varphi, \alpha)(\psi, 0) = (\psi, 0)(\varphi, \alpha)$ for all $\psi \in \Delta'$. Then, by the law of multiplication in $\mathfrak{G}$ we have $\psi = \psi^\alpha$ for all $\psi \in \Delta'$, and so $\alpha = 0$. Hence we have $\boldsymbol{A}' \cap \boldsymbol{M} = \boldsymbol{A}$. Let A be an element of $\boldsymbol{A}$. According to [4], there is a unique family of scalars $\{\lambda_\varphi\}_{\varphi \in \Delta'}$ such that $A = \sum_{\varphi \in \Delta'} \lambda_\varphi V_{(\varphi,0)}$ where $\sum$ is taken in the sense of metric convergence in $\boldsymbol{M}$. Thus we have

$$\begin{aligned} V^*_{(\psi,\beta)} A V_{(\psi,\beta)} &= \sum_{\varphi \in \Delta'} \lambda_\varphi V_{(\psi^\beta,\beta)(\varphi,0)(\psi,\beta)} \\ &= \sum_{\varphi \in \Delta'} \lambda_\varphi V_{(\varphi^\beta,0)} \in \boldsymbol{A}. \end{aligned}$$

Hence $\boldsymbol{P} \equiv \boldsymbol{R}$ $(U \in \boldsymbol{M}$, unitary$|$ $U^*\boldsymbol{A}U \subseteqq \boldsymbol{A}) = \boldsymbol{M}$, that is $\boldsymbol{A}$ is a regular maximal abelian subalgebra of $\boldsymbol{M}$.

LEMMA 2. *$\boldsymbol{A} \neq \boldsymbol{B}$, $\boldsymbol{B}' \cap \boldsymbol{M} = \boldsymbol{A}$ and so $\boldsymbol{A}$ is a unique maximal abelian subalgebra of $\boldsymbol{M}$ which contains $\boldsymbol{B}$.*

PROOF. It is obvious that $\boldsymbol{A}\neq\boldsymbol{B}$ and $\boldsymbol{B}'\cap\boldsymbol{M}\supseteqq\boldsymbol{A}$. If $(\varphi,\alpha)(\varphi_0,0)=(\varphi_0,0)(\varphi,\alpha)$, $(\varphi+\varphi_0,\alpha)=(\varphi+\varphi_0^\alpha,\alpha)$ and we have $\varphi_0=\varphi_0^\alpha$. Hence $\alpha=0$ by the definition of φ_0 and $(\varphi,\alpha)=(\varphi,0)$. Therefore $\boldsymbol{B}'\cap\boldsymbol{M}\subseteqq\boldsymbol{A}$. Let $\boldsymbol{C}$ be a maximal abelian subalgebra of $\boldsymbol{M}$ which contains $\boldsymbol{B}$. Then we have $\boldsymbol{A}=\boldsymbol{B}'\cap\boldsymbol{M}\supseteqq\boldsymbol{C}'\cap\boldsymbol{M}=\boldsymbol{C}$, and $\boldsymbol{A}=\boldsymbol{C}$ by the maximality of $\boldsymbol{C}$.

By Lemma 2, we have

$$(\boldsymbol{B}'\cap\boldsymbol{M})'\cap\boldsymbol{M}=\boldsymbol{A}'\cap\boldsymbol{M}=\boldsymbol{A}\neq\boldsymbol{B}$$

and the assertion of Proposition 1 is proved.

4. In this section, we shall prove Proposition 2. For each $g\in G$, $(\varphi,\alpha)\to(T_g'\varphi,T_g\alpha)$ defines an automorphism of $\mathfrak{G}$, and the collection of all pair $(g,(\varphi,\alpha))\in(G,\mathfrak{G})$ is a countably infinite group by the law of composition:

$$(g,(\varphi,\alpha))\,(h,(\psi,\beta))=(gh,(\varphi,\alpha)(T_{g^{-1}}'\psi,T_{g^{-1}}\beta))\,,$$

$$(g,(\varphi,\alpha))^{-1}=(g^{-1},(T_g'\varphi^\alpha,T_g\alpha))\,,$$

$$(e,(0,0))(g,(\varphi,\alpha))=(g,(\varphi,\alpha))(e,(0,0))=(g,(\varphi,\alpha))\,.$$

By $\widetilde{\boldsymbol{M}}$ we mean the ring of operators generated by $V_{(g,(\varphi,\alpha))}$ on $l_2((G,\mathfrak{G}))$: $[V_{(g,(\varphi,\alpha))}f]((h,(\psi,\beta)))=f((h,(\psi,\beta))(g,(\varphi,\alpha)))$.
Then the following lemma is easily seen.*)

LEMMA 3. *The crossed product $(\boldsymbol{M},G)$ is isomorphic to $\widetilde{\boldsymbol{M}}$.*

By Lemma 3 and the result of [6] we have

PROPOSITION 2'. *If G is a free group with two generators, the crossed product $(\boldsymbol{M},G)$ does not have property P.*

PROOF. To prove this proposition, it is sufficient to show that $(G,\mathfrak{G})$ does not admit a non-negative, right invariant, finitely additive measure μ such that $\mu((G,\mathfrak{G}))=1$ by Lemma 3 and [6: Lemma 7]. Suppose that such a μ exist. Let p be the projection of $(G,\mathfrak{G})$ to G, that is, $p((g,(\varphi,\alpha)))=g$. For each $E\subset G$, we define $\mu_1(E)=\mu(p^{-1}(E))$, and then μ_1 is a non-negative, right invariant, finitely additive measure on G. This contradicts to [6: p. 24] and the assertion is proved.

*) N. Suzuki has pointed out that more general discussion can be done in the context of crossed extension of a group.

REFERENCES

[1] J. DIXMIER, Sous-annaux abélien maximaux dans les facteurs de type fini, Ann. of Math., 59(1954), 279-286.
[2] H. CHODA AND M. ECHIGO, A remark on a construction of finite factors I, Proc. Japan Acad., 40(1964), 474-478.
[3] R. V. KADISON, Normalcy in operator algebras, Duke Math. Journ., 29(1962), 459-464.
[4] F. J. MURRAY AND J. VON NEUMANN, On rings of operators, IV, Ann. of Math., 44 (1943), 716-808.
[5] T. SAITÔ, Some remarks on a representation of a group, Tôhoku Math. Journ., 12(1960), 383-388.
[6] J. SCHWARTZ, Two finite, non-hyperfinite, non-isomorphic factors, Comm. Pure Appl. Math., 16(1963), 19-26.
[7] N. SUZUKI, A linear representation of a countably infinite group, Proc. Japan Acad., 34(1958), 575-579.
[8] N. SUZUKI, Crossed products of rings of operators, Tôhoku Math. Journ., 11(1959), 113-124.

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