ALMOST ANALYTIC VECTOR FIELDS IN ALMOST COMPLEX MANIFOLDS

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Introduction. S. Tachibana ([6], [7])[1] introduced the notion of covariant almost analytic vector field in certain almost Hermitian manifold by generalizing the following fact: "In a compact Kählerian manifold, the inner product of a contravariant analytic vector field and a covariant analytic vector field is constant over the manifold (K. Yano [9])." Further, by several geometors this notion was generalized in general almost complex manifold and to one of almost analytic tensor field.

Let (M, J) and (M', J') be almost complex manifolds of class C^∞ where J and J' are almost complex structures on M and M' respectively. Let f be a differentiable mapping of M into M'; the mapping f is called *almost analytic* if

$$df_P \circ J_P = J'_{f(P)} \circ df_P \qquad \text{for} \quad P \in M.$$

It is well-known that an almost analytic mapping of a complex manifold into another complex manifold is holomorphic (S. Helgason [1]).

Now, if we adopt a tensor bundle of M as the second manifold M' and define a suitable almost complex structure on the tensor bundle of M, then almost analytic tensor fields on M can be defined by differentiable cross-section of M in the tensor bundle of M. In the present paper we shall consider almost analytic vector fields from this point of view and discuss the relation between the usual almost analytic vector fields and ours. We shall also obtain the generalization of the usual almost analytic vector fields.

1. Contravariant almost analytic vectors. Let M be a $2n$-dimensional almost complex manifold with structure tensor φ. The set of all tangent vectors of M constitutes, with a natural topology, the so-called tangent bundle of M. We denote it by $T(M)$ and the natural projection $T(M) \to M$ by π. When $T(M)$ admits an almost complex structure Φ, a differentiable cross-section f is called *contravariant almost analytic* vector field with recpect to (φ, Φ) over M if

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1) See the bibliography at the end of the paper.

(1.1) $df_P \circ \varphi_P = \Phi_{f(P)} \circ df_P$ for $P \in M$.

Let $x^{i\ 2)}$ be local coordinates in a neighborhood U of a point P of M, then a tangent vector y at P, which is an element of $T(M)$, is expressible in the form (x^i, y^i) where y^i's are components of y with respect to the natural frame $\partial/\partial x^i$. We may consider (x^i, y^i) local coordinates in a neighborhood $\pi^{-1}(U) = U \times E^n$ of $T(M)$. To a transformation of local coordinates of M

$$x^{i'} = x^{i'}(x^1, x^2, \cdots, x^{2n})$$

there corresponds in $T(M)$ the coordinate transformation

(1.2) $\begin{cases} x^{i'} = x^{i'}(x^1, x^2, \cdots, x^{2n}), \\ y^{i'} = (\partial_r x^{i'}) y^r. \end{cases}$

If we put

$$x^{\bar{i}} = y^i \quad x^{\bar{i'}} = y^{i'},\ ^{3)}$$

then we may write (1.2) as

(1.3) $x^{I'} = x^{I'}(x^1, x^2, \cdots, x^{4n}).\ ^{4)}$

Let φ be a tensor field of type $(1,1)$ on M and $\varphi_j{}^i$ be its components with respect to a coordinate neighborhood $U(x^i)$. If we put

(1.4) $\begin{cases} \Phi_j{}^i = \varphi_j{}^i, & \Phi_j{}^{\bar{i}} = y^r \partial_r \varphi_j{}^i, \\ \Phi_{\bar{j}}{}^i = 0, & \Phi_{\bar{j}}{}^{\bar{i}} = \varphi_j{}^i, \end{cases}$

then we have a tensor field Φ of type $(1,1)$ on $T(M)$ whose components are $\Phi_J{}^I$ with respect to the coordinate neighborhood $\pi^{-1}(U)(x^i, y^i)$. Φ is called the *extension* of φ (S. Sasaki [4]). We can easily see that if $\varphi_j{}^i$ is an almost complex structure, so is also $\Phi_J{}^I$.

Now, cross-section f can be locally expressed by

$$\bar{x}^I = \bar{x}^I(x^1, x^2, \cdots, x^{2n}),$$

namely

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2) Indices i, j, k ; a, b, c ; r, s, t run over the range $1, 2, \cdots, 2n$.

3) We put $\bar{i} = 2n + i$, $\bar{i'} = 2n + i'$.

4) $I, J, K = 1, 2, \cdots, 4n$.

(1.5)
$$\begin{cases} \overline{x}^i = x^i, \\ \overline{y}^i = u^i \ (x^1, x^2, \cdots, x^{2n}). \end{cases}$$

By virtue of (1.1), (1.4) and (1.5), locally the condition that f be almost analytic is

$$\varphi_j{}^r \partial_r \overline{x}^I = \Phi_R{}^I \partial_j \overline{x}^R .$$

In the case $I = i$ we get the identity $\varphi_j{}^i = \varphi_j{}^i$ and in the case $I = \overline{i}$

$$\varphi_j{}^r \partial_r u^i = u^r \partial_r \varphi_j{}^i + \varphi_r{}^i \partial_j u^r ,$$

or

(1.6)
$$\mathfrak{L}(u) \varphi_j{}^i = 0 .$$

As the last equation is nothing but the defining equation of usual contravariant almost analytic vector field in the almost complex manifold M, our contravariant almost analytic vector field coincides with the usual one. Consequently, the usual contravariant almost analytic vector field is regarded as the contravariant almost analytic vector field with respect to φ and its extension Φ which is considered from the point of φ only. We denote it by the contravariant almost analytic vector field with respect to φ.

THEOREM 1.1. *In order that an extension $\overline{u}^I = (u^i, y^r \partial_r u^i)$ of a contravariant vector field u^i of M be contravariant almost analytic with respect to Φ on $T(M)$, it is necessary and sufficient that u^i itself is contravariant almost analytic with respect to φ on M.*

PROOF. Let u^i be contravariant almost analytic on M, i.e. $\mathfrak{L}(u)\varphi_j{}^i = 0$. After some calculations we get

$$\mathfrak{L}(\overline{u}) \Phi_j{}^i = \mathfrak{L}(\overline{u}) \Phi_{\overline{j}}{}^{\overline{i}} = \mathfrak{L}(u) \varphi_j{}^i = 0,$$

$$\mathfrak{L}(\overline{u})\Phi_{\overline{j}}{}^i = 0, \quad \mathfrak{L}(\overline{u})\Phi_j{}^{\overline{i}} = y^r \partial_r(\mathfrak{L}(u) \varphi_j{}^i) = 0,$$

from which we have $\mathfrak{L}(\overline{u})\Phi_J{}^I = 0$ i.e. $\overline{u}$ is contravariant almost analytic on $T(M)$. Converse is evident.

Next, we shall introduce an almost complex structure on $T(M)$ different from $\Phi_J{}^I$. Let Γ_{ji}^h be an affine connection on M. If we define $\widetilde{\Phi}_J{}^I$ by

(1.7)
$$\begin{cases} \widetilde{\Phi}_j{}^i = \Gamma_j{}^i, \quad \widetilde{\Phi}_j{}^{\overline{i}} = -(\delta_j{}^i + \Gamma_j{}^r \Gamma_r{}^i), \\ \widetilde{\Phi}_{\overline{j}}{}^i = \delta_j{}^j, \quad \widetilde{\Phi}_{\overline{j}}{}^{\overline{i}} = -\Gamma_j{}^i \end{cases}$$

where $\Gamma_j{}^i = \Gamma_{jr}{}^i y^r$, then it is an almost complex structure on $T(M)$ (C. T. Hsu [2]). By virtue of $\widetilde{\Phi}$, we can find the contravariant almost analytic vector field with respect to $(\varphi, \widetilde{\Phi})$. In the same way as before, by (1.1), (1.5) and (1.7), we have

$$\varphi_j{}^r \partial_r \bar{x}^I = \widetilde{\Phi}_h{}^I \partial_j \bar{x}^h .$$

Putting $I = i$ and $I = \bar{i}$, we have

(1. 8) $$\varphi_j{}^i = \widetilde{\nabla}_j u^i$$

and

$$\varphi_j{}^r \partial_r u^i = -(\delta_j{}^i + \Gamma_j{}^r \Gamma_r{}^i + \Gamma_r{}^i \partial_j u^r),$$

where $\widetilde{\nabla}_j$ denotes the operator of covariant derivative with respect to the affine connection $\Gamma_{ji}{}^h$. However, the last equation is identically satisfied by (1.8). In fact, we have the following

$$\varphi_j{}^r \partial_r u^i + \delta_j{}^i + \Gamma_j{}^r \Gamma_r{}^i + \Gamma_r{}^i \partial_j u^r$$

$$= \varphi_j{}^r (\varphi_r{}^i - \Gamma_r{}^i) + \delta_j{}^i + \Gamma_j{}^r \Gamma_r{}^i + \Gamma_r{}^i (\varphi_j{}^r - \Gamma_j{}^r) = 0 .$$

Thus contravariant almost analytic vector field with respect to $(\varphi, \widetilde{\Phi})$ is given by (1.8). In general, contravariant almost analytic vector field satisfying (1.8) does not satisfy (1.6) i.e. not contravariant almost analytic in the sense of the usual one. Therefore, we shall call the contravariant almost analytic vector field with respect to (φ, Φ) and $(\varphi, \widetilde{\Phi})$ as the contravariant almost analytic vector fields of the *first kind* and the *second kind* of M respectively.

In particular, if M is almost Hermitian, (1.8) may be written as

$$\varphi_j{}^i = \nabla_j u^i$$

or

(1. 9) $$\nabla_j u_i = \varphi_{ji} \quad (\varphi_{ji} = \varphi_j{}^r g_{ri}) ,$$

where ∇_j denotes the operator of covariant derivative with respect to the Christoffel's symbols $\left\{ \begin{matrix} h \\ j \ i \end{matrix} \right\}$. From (1.9) we see that contravariant almost analytic vector field of the second kind u^i is a Killing vector field since φ_{ji} is skew-symmetric. Therefore almost Hermitian manifolds admitting contravariant almost analytic vector field of the second kind coincides with certain almost Kählerian manifolds introduced by Y. Mutô (Y. Mutô [3]). Thus almost

Hermitian manifold with contravariant almost analytic vector field of the second kind becomes necessarily almost Kählerian one and contravariant almost analytic vector field of the second kind the one of the first kind (Y. Muto [3]).

THEOREM 1.2. *In a compact almost Hermitian manifold M there does not exist any contravariant almost analytic vector field of the second kind.*

PROOF. Let u^i be contravariant almost analytic vector field of the second kind. Then we get

$$(\nabla^j u^i)\nabla_j u_i = \nabla_j((\nabla^j u^i)u_i) - (\nabla_j \nabla^j u^i)u_i = \nabla_j((\nabla^j u^i)u_i)$$

because $\nabla_j \nabla^j u^i = \nabla_j \varphi^{ji} = 0$.

By integration it follows that

$$\int_M (\nabla^j u^i)\nabla_j u_i\, d\sigma = \int_M \nabla_j((\nabla^j u^i)u_i)\, d\sigma = 0,$$

where $d\sigma$ is the volume element of M.   From the last equation we have $\nabla_j u_i = 0$ i.e. $\varphi_{ji} = 0$.   This is a contradiction.   Q.E.D.

THEOREM 1.3. *In an almost Hermitian manifold with parallel Rcici tensor field R_{ji} ($R_{ji} \neq 0$), there does not exist any contravariant almost analytic vector field of the second kind.*

PROOF. Let u^i be contravariant almost analytic vector field of the second kind. As u^i is a Killing vector field we get

$$\nabla^r \nabla_r u^i + R_r{}^i u^r = 0\,.$$

By virtue of (1.9) and $\nabla^r \varphi_r{}^i = 0$ we have $R_{ir} u^r = 0$.   Operating ∇_k to $R_{ir} u^r = 0$ and making use of $\nabla_k R_{ir} = 0$, we have $R_{ir} \nabla_k u^r = 0$.   Again by (1.9), we get $R_{ir} \varphi_k{}^r = 0$ i.e. $R_{ij} = 0$.   This is a contradiction.

THEOREM 1.4. *If an almost Hermitian manifold admitting a contra-avriant almost analytic vector field of the second kind is flat, then the almost complex structure is integrable.*

PROOF. It follows immediately by

$$N_{ji}{}^h = 2\varphi_j{}^s R_{sir}{}^h u^r\,.$$

2. Cotangent bundles. Let M be an n-dimensional differentiable mani-fold of class C^∞. The set of all non-zero covectors of M constitutes, with natural topology, the so-called cotangent bundle of M. We denote it by $^cT(M)$ and the natural projection $^cT(M) \to M$ by π. Let x^i be local coordi-nates in a neighborhood U of a point of M, then a cotangent vector y at P which is an element of $^cT(M)$ is expressible in the form (x^i, y_i)[5] where y_i are components of y with respect to the natural coframe dx^i. We may consider (x^i, y_i) local coordinates in a neighborhood $\pi^{-1}(U) = U \times E^n$ of $^cT(M)$.

To a transformation of local coordinates of M

$$x^{i'} = x^{i'}(x^1, x^2, \cdots, x^n),$$

there corresponds in $^cT(M)$ the coordinate transformation

(2. 1)
$$\begin{cases} x^{i'} = x^{i'}(x^1, x^2, \cdots, x^n) \\ y_{i'} = (\partial_{i'} x^i) y_i. \end{cases}$$

If we put

$$x^{\bar{i}} = y_i \qquad x^{\bar{i}'} = y_{i'},$$

then we may write (2.1) as

(2. 2)
$$x^{I'} = x^{I'}(x^1, x^2, \cdots, x^{2n}).$$

So we may define tensors of $^cT(M)$ by making use of a matrix $\left(\dfrac{\partial x^{I'}}{\partial x^I}\right)$ i.e.

(2. 3)
$$\begin{bmatrix} \dfrac{\partial x^{i'}}{\partial x^i} & 0 \\[3ex] \dfrac{\partial}{\partial x^i}\left(\dfrac{\partial x^r}{\partial x^{i'}}\right) y_r & \dfrac{\partial x^i}{\partial x^{i'}} \end{bmatrix},$$

its transposed matrix and their Kronecker products. Making use of (2.3), we have after some complicated calculations, the following lemmas.

LEMMA 1. *Let t be a tensor field of type $(1,1)$ on M and $t_j{}^i$'s are its components with respect to a coordinate neighborhood $U(x^i)$. If we put*

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5) In this section, the indices run over as follows,

$i, j, k = 1, 2, \cdots, n$; $\bar{i} = n+i$, $\bar{i}' = n+i'$; $I, J, K = 1, 2, \cdots, 2n$.

$$(2.4) \quad \begin{cases} T_j{}^i = t_j{}^i, \quad T_j{}^{\bar{i}} = (\partial_j t_i{}^r - \partial_i t_j{}^r) y_r, \\ T_{\bar{j}}{}^i = 0, \quad T_{\bar{j}}{}^{\bar{i}} = t_i{}^j, \end{cases}$$

then we get a tensor field T of type (1,1) on $^cT(M)$ whose components are $T_J{}^I$ with respect to the coordinate neighborhood $\pi^{-1}(U)(x^i, y_i)$.

LEMMA 2. *Let M be an almost complex manifold with the structure tensor φ and $\varphi_j{}^i$'s are its components with respect to a coordinate neighborhood $U(x^i)$. If we put*

$$(2.5) \quad \begin{cases} F_j{}^i = \varphi_j{}^i, \quad F_j{}^{\bar{i}} = \dfrac{1}{2} \{\partial_j \varphi_i{}^r - \partial_i \varphi_j{}^r + \varphi_j{}^t \varphi_i{}^s (\partial_t \varphi_s{}^r - \partial_s \varphi_t{}^r)\} y_r, \\ F_{\bar{j}}{}^i = 0, \quad F_{\bar{j}}{}^{\bar{i}} = \varphi_i{}^j, \end{cases}$$

then we get a tensor field F of type (1,1) on $^cT(M)$ whose components are $F_J{}^I$ with respect to the coordinate neighborhood $\pi^{-1}(U)(x^i, y_i)$. Moreover it is an almost complex structure on $^cT(M)$.

Next, take a homogeneous contact form $\eta = y_i dx^i$ of M which is a global 1-form over $^cT(M)$ and consider the exterior differential $d\eta$ of the contact form η (S. Sasaki [5]). In every neighborhood, it is given by

$$(2.6) \quad d\eta = \frac{1}{2} \widetilde{F}_{IJ} dx^I \wedge dx^J ,$$

where we have put

$$\widetilde{F}_{IJ} = \partial_I \eta_J - \partial_J \eta_I \qquad \eta_I = (0, y_i).$$

We can easily see that $(\widetilde{F}_{IJ})$ has the following numerical components:

$$(2.7) \qquad \widetilde{F}_{ij} = 0 \quad \widetilde{F}_{i\bar{j}} = -\delta_i{}^j \quad \widetilde{F}_{\bar{i}j} = \delta_j{}^i \quad \widetilde{F}_{\bar{i}\bar{j}} = 0.$$

which is called the *fundamental null system of $^cT(M)$.*

Now, let g be the fundamental metric tensor of M and g_{jk}'s are the components of g with respect to a coordinate neighborhood $U(x^i)$ in M. We shall define a line element in a coordinate neighborhood $\pi^{-1}(U)(x^i, y_i)$ of $^cT(M)$ by

$$(2.8) \qquad d\sigma^2 = g_{jk}(x)\, dx^j dx^k + g^{jk}(x)\, Dy_j Dy_k ,$$

where Dy_j means the covariant differential of y_j, i.e.

$$Dy_j = dy_j - \left\{ {i \atop j\,k} \right\} y_i dx^k,$$

$\left\{ {i \atop j\,k} \right\}$ being Christoffel's symbols. It is clear that the right hand side of (2.8) is invariant under transformations of $^cT(M)$ and positive definite. In the following we shall consider the cotangent bunble $^cT(M)$ as a Riemannian manifold endowed with the metric (2.8).

The components of the fundamental metric tensor of $^cT(M)$ with respect to the coordinate neighborhood $\pi^{-1}(U)(x^i, y_i)$ can be obtained by putting (2.8) in the form

(2.9) $d\sigma^2 = G_{JK} dx^J dx^K \quad (dx^{\bar{j}} = dy_j).$

We can easily see that

(2.10) $G_{jk} = g_{jk} + \gamma_j{}^i \gamma_{ik}, \quad G_{j\bar{k}} = -\gamma_j{}^k, \quad G_{\bar{j}\bar{k}} = g^{jk},$

where we have put

$$\gamma_{jk} = \left\{ {i \atop j\,k} \right\} y_i.$$

We can explicitly give the contravariant components of the fundamental metric tensor of $^cT(M)$. This is easily done by solving the equation

(2.11) $G_{JK} G^{KI} = \delta_J{}^I$

thinking G^{KI} as unknown variables. The results is as follows:

(2.12) $G^{jk} = g^{jk}, \quad G^{j\bar{k}} = \gamma_k{}^j, \quad G^{\bar{j}\bar{k}} = g_{jk} + \gamma_j{}^i \gamma_{ik}.$

Thus we have

THEOREM 2.1. *Let $^cT(M)$ be a cotangent bundle of Riemannian manifold M and $\widetilde{F}_{IJ}$ and G_{IJ} the components of the fundamental null-system and the fundamental metric tensor of $^cT(M)$ respectively. If we define $\widetilde{F}_J{}^I$ by*

(2.13) $\widetilde{F}_J{}^I = \widetilde{F}_{JK} G^{KI},$

then it is an almost complex structure on $^cT(M)$. Moreover G_{IJ} is an almost

Hermitian metric with respect to the almost complex structure $\widetilde{F}_J{}^I$ and the almost Hermitian manifold with the structure tensors $\widetilde{F}_J{}^I$ and G_{IJ} becomes necessarily an almost Kählerian manifold. (P. Tondeur [8]).

We can easily see that $\widetilde{F}_J{}^I$ has the following components:

(2. 14) $\qquad \widetilde{F}_j{}^i = -\gamma_j{}^i, \quad \widetilde{F}_{\bar{j}}{}^i = g^{ij}, \quad \widetilde{F}_j{}^{\bar{i}} = -(g_{ij}+\gamma_i{}^k\gamma_{kj}), \quad \widetilde{F}_{\bar{j}}{}^{\bar{i}} = \gamma_i{}^j .$

THEOREM 2.2. *In order that an infinitesimal transformation of Riemannian manifold M is an infinitesimal isometry, it is necessary and sufficient that its extension in ${}^cT(M)$ is a contravariant almost analytic vector field of the first kind.*

PROOF. By extension $\bar{\xi}^I$ in ${}^cT(M)$ of a vector field ξ^i on M we understand a vector field whose components are $(-\xi^i, y_r\partial_i\xi^r)$. In fact, we can easily verify that the extension is a vector field in ${}^cT(M)$. After some complicated calculations we have the following equations:

$$\mathfrak{L}(\bar{\xi})\widetilde{F}_j{}^i = \{g^{is}(\nabla_s\nabla_j\xi^r+R_{tsj}{}^r\xi^t) - \begin{Bmatrix} r \\ j\,s \end{Bmatrix}(\nabla^s\xi^i+\nabla^i\xi^s)\}\,y_r,$$

$$\mathfrak{L}(\bar{\xi})\widetilde{F}_{\bar{j}}{}^i = \nabla^j\xi^i + \nabla^i\xi^j,$$

$$\mathfrak{L}(\bar{\xi})\widetilde{F}_j{}^{\bar{i}} = \{\gamma_i{}^s(\nabla_j\nabla_s\xi^t+R_{tjs}{}^r\xi^t) + \gamma_j{}^s(\nabla_i\nabla_s\xi^r+R_{tis}{}^r\xi^t)\}\,y_r$$

$$\qquad\qquad - \gamma_{jr}\gamma_{is}(\nabla^r\xi^s+\nabla^s\xi^r) + (\nabla_j\xi_i+\nabla_i\xi_j),$$

$$\mathfrak{L}(\bar{\xi})\widetilde{F}_{\bar{j}}{}^{\bar{i}} = \{-g^{js}(\nabla_s\nabla_i\xi^r+R_{tsi}{}^r\xi^t) + \begin{Bmatrix} r \\ i\,s \end{Bmatrix}(\nabla^j\xi^s+\nabla^s\xi^j)\}\,y_r .$$

From these equations the theorem follows.

3. Covariant almost analytic vectors.

Let M be a $2n$-dimensional almost complex manifold with the structure tensor φ. When the cotangent bundle ${}^cT(M)$ of M admits an almost complex structure F, a differentiable cross-section f is called *covariant almost analytic* vector field with respect to (φ, F) over M, if

(3. 1) $\qquad\qquad df_P \circ \varphi_P = F_{f(P)} \circ df_P \qquad\qquad$ for $P \in M$.

Now, a cross-section f can be locally expressed by

$$\bar{x}^I = \bar{x}^I\,(x^1, x^2, \cdots, x^{2n})$$

or

$$(3.2) \qquad \begin{cases} \bar{x}^i = x^i, \\ \bar{y}_i = w_i \ (x^1, x^2, \cdots, x^{2n}). \end{cases}$$

Since φ and F defined in Lemma 2 in §2 are both almost complex structure on M and $'T(M)$ respectively, by the same way as the contravariant case, the condition that the vector field w_i is covariant almost analytic with respect to (φ, F) is given by

$$(3.3) \qquad \varphi_j{}^r \partial_r w_i = \frac{1}{2} \{ \partial_j \varphi_i{}^r - \partial_i \varphi_j{}^r + \varphi_j{}^b \varphi_i{}^a (\partial_b \varphi_a{}^r - \partial_a \varphi_b{}^r) \} y_r + \varphi_i{}^r \partial_j w_r$$

which is equivalent to the following equation

$$(3.4) \qquad \partial_j (\varphi_i{}^r w_r) = w_r \partial_i \varphi_j{}^r + \varphi_j{}^r \partial_r w_i + \frac{1}{2} N_{ji}{}^s \varphi_s{}^r w_r,$$

where $N_{ji}{}^s$ is Nijenhuis' tensor of the structure tensor φ.

If M is almost Hermitian, then (3.4) may be written as

$$(3.5) \qquad \nabla_j (\varphi_i{}^r w_r) = w_r \nabla_i \varphi_j{}^r + \varphi_j{}^r \nabla_r w_i + \frac{1}{2} N_{ji}{}^s \varphi_s{}^r w_r.$$

From (3.4) we have

$$(3.6) \qquad (\partial_j \varphi_i{}^r - \partial_i \varphi_j{}^r) w_r = \varphi_j{}^r \partial_r w_i - \varphi_i{}^r \partial_j w_r + \frac{1}{2} N_{ji}{}^s \varphi_s{}^r w_r.$$

Making use of (3.6), we get

$$N_{ji}{}^h w_h = \{ \varphi_j{}^a (\partial_a \varphi_i{}^h - \partial_i \varphi_a{}^h) - \varphi_i{}^a (\partial_a \varphi_j{}^h - \partial_j \varphi_a{}^h) \} w_h$$

$$= -2 {}^* O_{ji}^{sr} (\partial_s w_r - \partial_r w_s) + N_{ji}{}^h w_h$$

i.e.

$$ {}^* O_{ji}^{sr} (\partial_s w_r - \partial_r w_s) = 0.$$

Thus we have the following

THEOREM 3.1. *In an almost complex manifold M with the structure tensor φ the tensor field $\partial_{[j} w_{i]}$ is pure for a covariant almost analytic vector field w.*

REMARK. In the equation (3.4) if we take away the third term in the right hand side, then we get the equation defining the usual covariant almost analytic vector field in M. (S. Tachibana [6], [7]).

If in the Lemma 1 of §2 we use an almost complex structure $\varphi_j{}^i$ instead of general tensor field $t_j{}^i$, then we have the tensor $E_J{}^I$ defined by

$$\begin{cases} E_j{}^i = \varphi_j{}^i, & E_j{}^{\bar{i}} = (\partial_j \varphi_i{}^r - \partial_i \varphi_j{}^r) y_r, \\ E_{\bar{j}}{}^i = 0, & E_{\bar{j}}{}^{\bar{i}} = \varphi_i{}^j. \end{cases}$$

As we have

$$E_j{}^R E_R{}^i = -\delta_j{}^i, \quad E_{\bar{j}}{}^R E_R{}^i = 0, \quad E_j{}^R E_R{}^{\bar{i}} = N_{ji}{}^h y_h, \quad E_{\bar{j}}{}^R E_R{}^{\bar{i}} = -\delta_{\bar{j}}{}^{\bar{i}}.$$

the tensor field $E_J{}^I$ is not always an almost complex structure of ${}^c T(M)$. However, if we use the tensor field $E_J{}^I$ in the place of the almost complex structure $F_J{}^I$ in (3.1), then we have

$$(3.7) \qquad \partial_j(\varphi_i{}^r w_r) = w_r \partial_i \varphi_j{}^r + \varphi_j{}^r \partial_r w_i$$

which is the equation defining the usual covariant almost analytic vector field. Although this shows that the usual covariant almost analytic vector field is not appropriate to call covariant almost analytic in the sense of our standpoint, we can easily find, by comparing the equations (3.4) and (3.7), that the usual one is a special covariant almost analytic one with the property $N_{ji}{}^h w_h = 0$, since it is well-known that the usual covariant almost analytic vector field has the identity $N_{ji}{}^h w_h = 0$. (K. Yano and M. Ako [10]).

Next, since we have seen that the tensor field $\widetilde{F}_J{}^I$ is an almost complex structure in ${}^c T(M)$, we can define covariant almost analytic vector field with respect to $(\varphi, \widetilde{F})$. In the same way as before, we get, in an almost Hermitian manifold with the structure tensor $(G, \widetilde{F})$, the relation

$$(3.8) \qquad \nabla_j w_i = \varphi_{ji}$$

by (2.14), (3.1) and (3.2). From (3.8) we can see that
  (i) w is a Killing vector field,
  (ii) w satisfying (3.8) does not satisfy (3.4),
  (iii) w is also contravariant almost analytic vector field of the second kind by virtue of (1.9).
Because of the property (ii) above, we shall call the covariant almost analytic vector fields with respect to (φ, F) and $(\varphi, \widetilde{F})$ as the covariant almost analytic vector fields of the *first kind* and the *second kind* of M respectively.

THEOREM 3.2. *In an almost Hermitian manifold, let u^i be a contravariant almost analytic vector field of the second kind. Then $g_{ji}u^i$ is covariant almost analytic one of the second kind.*

4. Properties of covariant almost analytic vectors in *O-space.

By *O-space we mean a $2n$-dimensional differentiable manifold M with a fixed almost Hermitian structure $(\varphi_j{}^i, g_{ji})$ such that

$$(4.1) \qquad\qquad *O_{ji}^{ba} \nabla_b \varphi_{ah} = 0 \,,$$

where $\varphi_{ji} = \varphi_j{}^r g_{ri}$. It is well-known that an almost Kählerian manifold and a K-space are *O-spaces. In the present section by covariant almost analytic vector field we shall mean covariant almost analytic one of the first kind.

In an *O-space, the following identities

$$(4.2) \qquad\qquad \nabla^r \varphi_{ri} = 0$$

and

$$(4.3) \qquad\qquad N_{jih} = 2\varphi_j{}^r(\nabla_r \varphi_{ih} - \nabla_i \varphi_{rh})$$

hold good. On the other hand, in a general almost Hermitian manifold

$$(4.4) \qquad\qquad *O_{ih}^{ba} \nabla_j \varphi_{ba} = 0$$

is always valid.

Now, we shall deform the equation (3.5) which gives the definition of a covariant almost analytic vector field. From (3.5), we have

$$(4.5) \quad w^r \nabla_i \varphi_{jr} - w^r \nabla_j \varphi_{ir} + \frac{1}{2} N_{ji}{}^s \varphi_s{}^r w_r + \varphi_j{}^r \nabla_r w_i - \varphi_i{}^r \nabla_j w_r = 0 \,.$$

On taking account of (4.1), (4.3) and (4.4), (4.5) becomes the following form

$$\varphi_j{}^r \nabla_r w_i - \varphi_i{}^r \nabla_j w_r = 0 \,.$$

Hence, we have the following

THEOREM 4.1. *A necessary and sufficient condition for a vector field w_i in an *O-space to be a covariant almost analytic one of the first kind is that*

$$(4.6) \qquad\qquad \varphi_j{}^r \nabla_r w_i - \varphi_i{}^r \nabla_j w_r = 0 \,.$$

We notice that the equation (4.6) is the same form as the one in Kählerian manifold (K. Yano [9]).

Next, applying $\varphi_k{}^i \nabla^j$ to (4.6), we have

$$(4.7) \qquad \nabla^j \nabla_j w_k - R^*{}_{kr} w^r - \varphi_k{}^i (\nabla^j \varphi_i{}^r) \nabla_j w_r = 0 \,,$$

where we have put

$$R^*{}_{kr} = \frac{1}{2} \varphi^{pq} R_{pqsr} \varphi_k{}^s$$

and

$$\varphi^{pq} = \varphi_s{}^q g^{sp} \,.$$

Let w_k be a vector field and define $N(w)_k$ by the equation

$$(4.8) \qquad N(w)_k = \varphi_k{}^r (\nabla_j \varphi_{ir}) \nabla^j w^i \,,$$

where $\nabla^j w^i = g^{jr} \nabla_r w^i$. On taking account of (4.8), (4.7) may be written as

$$(4.9) \qquad \nabla^j \nabla_j w_k - R^*{}_{kr} w^r + N(w)_k = 0 \,.$$

Now, let w_k be a vector field and introduce a tensor field

$$(4.10) \qquad t(w)_{jk} = T(w)_{ji} g^{il} \varphi_{lk} \,,$$

where

$$T(w)_{ji} = \varphi_j{}^r \nabla_r w_i - \varphi_i{}^r \nabla_j w_r \,.$$

For simplicity, we shall denote t_{ji} instead of $t(w)_{ji}$ in the following. $t_{ji} = 0$ is equivalent to the fact that the vector field w_k is a covariant almost analytic one. Applying ∇^j to (4.10), we obtain

$$\nabla^j t_{jk} = (\nabla^j \varphi_j{}^r)(\nabla_r w^l)\varphi_{lk} + \varphi_j{}^r(\nabla^j \nabla_r w^l)\varphi_{lk} + \varphi_j{}^r(\nabla_r w^l)\nabla^j \varphi_{lk} - \nabla^j \nabla_j w_k \,.$$

Making use of (4.1) and (4.3), the last equation becomes

$$(4.11) \qquad \nabla^j t_{jk} = -\{\nabla^r \nabla_r w_k - R^*{}_{kr} w^r + N(w)_k\} \,.$$

In the next place, we shall compute

$$\nabla^j(t_{jk} w^k) = (\nabla^j t_{jk}) w^k + t_{jk} \nabla^j w^k \,.$$

Substituting (4.10) and (4.11) into the equation above, we have

(4. 12) $\nabla^j(t_{jk}w^k) = -\{\nabla^r\nabla_r w_k - R^*{}_{kr}w^r + N(w)_k\}w^k$

$$+ \varphi^{lj}\varphi^{ki}(\nabla_l w_k)\nabla_j w_i + (\nabla_j w_i)\nabla^j w^i .$$

Now, if we put $t^2(w) = t_{jk}t^{jk}$, then

(4. 13) $t^2 = 2\{(\nabla_j w_k)\nabla^j w^k + \varphi^{lj}\varphi^{ki}(\nabla_l w_k)\nabla_j w_i\} .$

Consequently, from (4.12) and (4.13), we have

$$\nabla^j(t_{jk}w^k) + \frac{1}{2}t^2 = -\{\nabla^r\nabla_r w_k - R^*{}_{kr}w^r + N(w)_k\}w^k .$$

If M is compact, by integration of the last equation, we have the following

THEOREM 4.2.   *In a compact *O-space M, the integral formula*

(4. 14) $\displaystyle\int_M\left[(\nabla^j\nabla_j w_i - R^*{}_{ir}w^r + N(w)_i)w^i + \frac{1}{2}t^2\right]d\sigma = 0$

is valid for any vector field w_i.

We have seen that a covariant almost analytic vector field w_i satisfies the equation (4.9). Conversely, consider a vector field w_i satisfying (4.9). Then if M is compact, we have $t^2 = 0$ i.e. $t_{ji} = 0$ by virtue of (4.14), so w_i is covariant almost analytic. Thus we have

THEOREM 4.3.   *A necessary and sufficient condition for a vector field* w_i *in a compact *O-space to be a covariant almost analytic vector field of the first kind is that the equation (4.9) is satisfied.*

In general, $\widetilde{w}^i = \varphi_r{}^i w^r$ is not necessarily a contravariant almost analytic vector field even though w^i is a contravariant almost analytic one, but about covariant almost analytic vector fields we have the following

THEOREM 4.4.   *In an *O-space, if a vector field* w_i *is a covariant almost analytic vector field of the first kind, then* $\widetilde{w}_i = \varphi_i{}^r w_r$ *is also covariant almost analytic one of the same kind.*

PROOF.

$$\varphi_j{}^r \nabla_r \widetilde{w}_i - \varphi_i{}^r \nabla_j \widetilde{w}_r = \{\varphi_j{}^r (\nabla_r \varphi_{is}) - \varphi_i{}^r (\nabla_j \varphi_{rs})\} w^s + \varphi_j{}^r \varphi_i{}^s \nabla_r w_s + \nabla_j w_i$$

$$= \varphi_j{}^t (*O_{ti}^{kr} \nabla_k \varphi_{rs}) w^s + *O_{ji}^{rs} \nabla_r w_s .$$

From (4.6), we have

(4. 15) $$\nabla_j w_i + \varphi_j{}^r \varphi_i{}^s \nabla_r w_s = 0 \quad \text{i.e.} \quad *O_{ji}^{rs} \nabla_r w_s = 0 .$$

By (4.1) and (4.15), we have

$$\varphi_j{}^r \nabla_r \widetilde{w}_i - \varphi_i{}^r \nabla_j \widetilde{w}_r = 0 ,$$

from which the Theorem follows.

Moreover we can easily see that in an *O-space a covariant almost analytic vector field w_i is an imcompressible vector field. In fact, transvecting g^{ij} to (4.15), we get $\nabla^i w_i = 0$.

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