ON THE INTEGRABILITY OF A STRUCTURE DEFINED BY TWO SEMI-SIMPLE O-DEFORMABLE VECTOR 1-FORMS WHICH COMMUTE WITH EACH OTHER

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Introduction. Recently, C. S. Houch considered a structure defined by two vector 1-forms (tensor fields of type $(1,1)$) h and k on a differentiable manifold satisfying the following conditions,

1) $h^2 = \lambda^2 E,\ k^2 = \mu^2 E$,

where E is a vector 1-form defined by the identity matrix, and $\lambda^2 = \pm 1$, $\mu^2 = \pm 1$,

2) $hk = kh$,

and proved that this structure is integrable if and only if the vector 2-forms (tensor fields of type $(1,2)$, skew symmetric in their covariant part) $[h,h]$, $[k,k]$ and $[h,k]$ vanish identically (Cf. [2][1]).

In this paper, we consider the case when h and k are vector 1-forms defined on a differentiable manifold M satisfying some algebraic equations without multiple roots, and prove that the structure defined by h and k is integrable, i.e., for any point $x \in M$, we can always find a local coordinate system around x with respect to which the vector 1-forms h and k have constant components, if and only if $[h,h]$, $[k,k]$ and $[h,k]$ vanish identically. We use a Theorem of E. T. Kobayashi (Cf. [3]). Although he supposes that the real vector 1-form h is of class C^∞, by virtue of A. Nijenhuis and W. H. Woolf's theorem (Cf. [4]), we get the same result if we suppose the vector 1-form h is of class C^2. So, throughout this paper, we suppose that all the structure and the tensor fields are of class C^2 when h and k are real vector 1-forms, and analytic when h and k are complex. Moreover, let $f(x)$ be a polynomial of x, i.e.,

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n ,$$

and let h be a vector 1-form, then $f(h)$ means a vector 1-form defind by

1) The number in brackets refers to Bibliography at the end of the paper,

$$f(h) = a_0E + a_1h + a_2h^2 + \cdots + a_nh^n .$$

1. The square bracket of two vector 1-forms. Let h and k be two vector 1-forms on M. The square bracket $[h, k]$ of h and k is a vector 2-form defined by

$$(1.1)\qquad [h, k](u, v) = [hu, kv] - h[u, kv] - k[hu, v] + hk[u, v] + [ku, hv] - k[u, hv] - h[ku, v] + kh[u, v] ,$$

where u and v are vector fields over M and the square brackets $[hu, kv]$ etc. of two vector fields are the usual Poisson brackets (Cf. [1]).

With respect to a local coordinate system (x^i), the components $([h, k])^i{}_{jk}$ of $[h, k]$ are given by

$$(1.2)\qquad ([h, k])^i{}_{jk} = h^m{}_j \partial_m k^i{}_k - h^m{}_k \partial_m k^i{}_j - h^i{}_m \partial_j k^m{}_k + h^i{}_m \partial_k k^m{}_j + k^m{}_j \partial_m h^i{}_k - k^m{}_k \partial_m h^i{}_j - k^i{}_m \partial_j h^m{}_k + k^i{}_m \partial_k h^m{}_j ,$$

where $h^i{}_j$ and $k^i{}_j$ are the components of h and k respectively, and ∂_j denotes the differential operator $\partial/\partial x^j$.

From the definition (1.1), it follows immediately

PROPOSITION 1. *For arbitrary vector 1-forms h,k and l, the following relations hold good.*

1) $[h, E] = [E, h] = 0$.

2) $[h, k] = [k, h]$.

3) $[h+k, l] = [h, l] + [k, l]$, $[h, k+l] = [h, k] + [h, l]$.

4) $[ch, k] = c[h, k]$,

where c is a constant.

5) $[h, kl](u, v) + [hl, k](u, v)$

$$= [h, k](lu, v) + [h, k](u, lv) + h([k, l](u, v)) + k([h, l](u, v)) ,$$

where u and v are vector fields over M.

Making use of 5) in Proposition 1, we obtain

COROLLARY. *If* $[h, h] = [k, k] = [h, k] = 0$, *then* $[hk, k] = 0$.

Moreover, when there is a linear relation with constant coefficients between hk and kh, we get the following Proposition by direct calculations.

PROPOSITION 2. *If h and k are two vector 1-forms satisfying*

$$hk + \mu kh = 0 \quad (\mu : \textit{const.}),$$

then the relation

$$[hk, hk](u, v) = [h, h](ku, kv) - \mu(h[k, k](hu, v) + h[k, k](u, hv)) - 2\mu hk[h, k](u, v) - h^2[k, k](u, v)$$

holds good for arbitrary vector fields u and v.

COROLLARY. *If h and k are two vector 1-forms satisfying*

$$hk + \mu kh = 0 \quad \textit{and} \quad [h, h] = [k, k] = [h, k] = 0,$$

then $[hk, hk]$ vanishes identically.

2. On the structure defined by a semi-simple 0-deformable vector 1-form. Theorem of E. T. Kobayashi. Let h be a vector 1-form defined on M. We consider the case when the Jordan canonical form of h_x for $x \in M$ is equal to a fixed diagonal matrix, and we call such a vector 1-form a semi-simple 0-deformable vector 1-form (Cf. [3]). Moreover, let $I_h[x]$ be the set of polynomials which vanish identically for $x=h$, then $I_h[x]$ is an ideal of the polynomial ring of x, and we call the generator of $I_h[x]$ the *minimal polynomial* of h. Then we can easily see that h is a semi-simple 0-deformable vector 1-form if and only if the minimal polynomial of h admits only simple roots.

For the structure defined by such a vector 1-form, E. T. Kobayashi (Cf. [3]) proved the following

THEOREM. *The structure defined by a semi-simple 0-deformable vector 1-form h is integrable if and only if the vector 2-form $[h, h]$ vanishes identically.*

Next, we introduce some complex vector 1-forms which we use in §3. Let $f(x) = \prod_{i=1}^{n} (x-\lambda_i)$ be the minimal polynomial of h. Then, since h is semi-simple, $f'(\lambda_i) \neq 0$ for all i. If we set

$$f_i(x) = \frac{1}{f'(\lambda_i)} \frac{f(x)}{x-\lambda_i},$$

$f_i(x)$ is a polynomial of x, and for any polynomial $g(x)$ of degree smaller than n, we have

$$g(x) = \sum_{i=1}^{n} g(\lambda_i) f_i(x).$$

In particular, we have

$$1 = \sum_{i=1}^{n} f_i(x) \quad \text{and} \quad x = \sum_{i=1}^{n} \lambda_i f_i(x).$$

So, if we put $p_i = f_i(h)$, we get the following relations,

$$\sum_{i=1}^{n} p_i = E, \tag{2. 1}$$

$$\sum_{i=1}^{n} \lambda_i p_i = h. \tag{2. 2}$$

On the other hand, we have

$$p_i p_j = 0 \quad (i \neq j), \tag{2. 3}$$

for $f_i(x) f_j(x)$ is a multiple of $f(x)$. So, if we multiply both sides of (2.1) by p_i, we get

$$p_i^2 = p_i.$$

Summarizing these, we get the following

PROPOSITION 3. *If h is a semi-simple 0-deformable vector 1-form with minimal polynomial $\prod_{i=1}^{n}(x-\lambda_i)$, then we can find n polynomials p_i $(i = 1, \cdots, n)$ of h which satisfy the following relations.*

1) $p_i p_j = 0,$ *if* $i \neq j$, $p_i^2 = p_i$.

2) $h = \sum_{i=1}^{n} \lambda_i p_i$.

We call the tensor p_i the *projection tensor* for λ_i.

REMARK. In fact, we can easily see that the tensor p_i is the projection tensor of the distribution D_i which consists of the vectors u satisfying $hu = \lambda_i u$.

3. The integrability of the structure defined by two semi-simple 0-deformable vector 1-forms which commute with each other. In this section, we shall study the condition for the structure defined by two semi-simple 0-deformable vector 1-forms h and k which commute with each other to be integrable. From (1.2), it follows immediately that if the structure is integrable, we have

$$[h, h] = [k, k] = [h, k] = 0 .$$

Now we consider the converse problem. For this purpose, we shall find a vector 1-form l such that h and k can be expressed as polynomials of l.

Let $f(x) = \prod_{i=1}^{m} (x-\lambda_i)$ and $g(x) = \prod_{\alpha=1}^{n} (x-\mu_\alpha)$ be the minimal polynomial of h and k respectively. If we take a real number λ different from $-\lambda_i$'s, and next we choose a real number μ different from the following $\frac{m(m-1)}{2} \times n^2$ numbers

$$-\mu_\alpha - \frac{(\lambda+\lambda_j)(\mu_\alpha-\mu_\beta)}{\lambda_i-\lambda_j} \qquad (i<j),$$

then the mn numbers $(\lambda_i+\lambda)(\mu_\alpha+\mu)$'s are all different from one another. We arrange these mn numbers in a certain order, and denote them by ν_ρ ($\rho = 1, \cdots, mn$). Now we put

$$l = (h + \lambda E)(k + \mu E),$$

i.e.,

$$l = \sum_{i=1}^{m} (\lambda_i+\lambda)\, p_i \sum_{\alpha=1}^{n} (\mu_\alpha+\mu)\, p_\alpha ,$$

where p_i and p_α denote the projection tensors given by Proposition 3 associated with h and k respectively. Then l is a real vector 1-form when h and k are real. If we arrange $p_i p_\alpha$'s in the same order as $(\lambda_i+\lambda)(\mu_\alpha+\mu)$'s and denote them by p_ρ $(\rho=1,\cdots,mn)$, then we have

$$l=\sum_{\rho=1}^{mn}\nu_\rho p_\rho .$$

And by virtue of the fact that h and k commute with each other, we have

$$p_\rho{}^2=p_\rho \quad \text{and} \quad p_\rho p_{\rho'}=0 \quad \text{if} \quad \rho\neq\rho' .$$

So we get

$$a(l)=\sum_{\rho=1}^{mn}a(\nu_\rho)\,p_\rho ,$$

where $a(x)$ is a polynomial of x. Therefore, the polynomial $\prod_{\rho=1}^{mn}(x-\nu_\rho)$ vanishes identically for $x=l$, which shows that l is a semi-simple 0-deformable vector 1-form.

On the other hand, by virtue of Corollaries of Propositions 1 and 2, if the relation

$$[h,h]=[k,k]=[h,k]=0$$

holds good, we have

$$[l,l]=0 .$$

So, from E. T. Kobayashi's theorem, we can find a local coordinate system (x^i) around each $x\in M$, with respect to which the components of l are constant.

Next, we consider the square matrix (ν_ρ^p) $(p=0,1,\cdots,mn-1;\ \rho=1,\cdots,mn)$. Since

$$\det(\nu_\rho^p)=(-1)^{\frac{1}{2}mn(mn-1)}\triangle(\nu_1,\cdots,\nu_{mn})\neq 0$$

(Vandermonde's determinant), it is a regular matrix. So, if we denote the inverse matrix of it by (a_p^ρ), we have

$$a_p^{\rho'}l^p=a_p^{\rho'}\sum_{\rho=1}^{mn}\nu_\rho^p p_\rho=\sum_{\rho=1}^{mn}\delta_\rho^{\rho'}p_\rho=p_{\rho'} ,$$

which shows that p_ρ is expressed as a polynomial of l for all ρ. But it is evident that h and k can be expressed as linear combinations of p_ρ's. So we see that h and k can be expressed as polynomials of l. Therefore, the components of h and k with respect to (x^i) are constant.

Summarizing these, we get

THEOREM. *A structure defined by two semi-simple* 0*-deformable vector* 1*-forms* h *and* k *which commute with each other is integrable if and only if the vector* 2*-forms* $[h, h]$, $[k, k]$ *and* $[h, k]$ *vanish identically.*

If we proceed this argument step by step, we obtain the following

THEOREM. *A structure defined by several semi-simple* 0*-deformable vector* 1*-forms* $h_1, h_2, \cdots, h_p$ *which commute with each other is integrable if and only if the vector* 2*-forms* $[h_i, h_j]$ *vanish identically for all pairs* i,j.

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