ON A CONJECTURE OF BERBERIAN

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1. In [1], S. K. Berberian conjectured that the closure of the numerical range of a hyponormal operator coincides with the convex hull of its spectrum. The purpose of this note is to give an affirmative answer to his conjecture.

Throughout this paper, operator means a bounded linear operator on a Hilbert space. The spectrum of an operator T is denoted by $\sigma(T)$, and its convex hull is denoted by $\sum(T)$. The numerical range of an operator T, denoted by $W(T)$, is the set $W(T) = \{(Tx, x) : \|x\| = 1\}$. We write $\overline{W}(T)$ for the closure of $W(T)$. An operator T is called normaloid if $\|T\| = \sup\{|\lambda| : \lambda \in W(T)\}$. For a compact convex subset X of the plane, a point $\lambda \in X$ is bare if there is a circle through λ such that no points of X lie outside this circle. A closed subset X of the plane is a spectral set for an operator T if $\| u(T) \| \leqq \sup\{|u(z)| : z \in X\}$ for every rational function $u(z)$ having no poles in X.

2. In this section, we shall prove the following theorem.

THEOREM. *Let T be an operator such that $T - \lambda I$ is normaloid for every complex number λ, then we have $\overline{W(T)} = \sum(T)$.*

A key of our proof is the following lemma.

LEMMA 1. *Let T be an operator and $\lambda \in \overline{W(T)}$ a bare point of $\overline{W(T)}$, then there exists a complex number λ_0 satisfying $|\lambda - \lambda_0| = \sup\{|\mu - \lambda_0| : \mu \in \overline{W(T)}\}$.*

PROOF. By the definition of bare point, there is a circle through λ such that no points of $\overline{W(T)}$ lie outside this circle. The center λ_0 of this circle satisfies our requirement.

For convenience we state the following known result as a lemma ([4: Corollary to Theorem 4]).

LEMMA 2. *For an operator T, $\lambda \in \overline{W(T)}$ and $|\lambda| = \|T\|$ imply $\lambda \in \sigma(T)$.*

PROOF OF THEOREM. It is sufficient to show that each bare point of $\overline{W(T)}$ belongs to $\sigma(T)$ ([4: Lemma 3]). Let λ be a bare point of $\overline{W(T)}$, there is a λ_0 satisfying $|\lambda-\lambda_0| = \sup\{|\mu-\lambda_0|: \mu \in \overline{W(T)}\}$ by Lemma 1. Thus, by the hypothesis on T and the fact $\overline{W(T)}-\lambda_0 = \overline{W(T-\lambda_0 I)}$, we have $\|T-\lambda_0 I\| = |\lambda-\lambda_0|$. Since $\lambda-\lambda_0 \in \overline{W(T-\lambda_0 I)}$, $\lambda-\lambda_0 \in \sigma(T-\lambda_0 I)$ by Lemma 2 and so we have $\lambda \in \sigma(T)$. Hence the proof is completed.

As a corollary, Berberian's conjecture is solved affirmatively.

COROLLARY 1. *For a hyponormal operator* T, $\overline{W(T)} = \sum(T)$.

PROOF. If T is hyponormal, i.e. $TT^* \leqq T^*T$, $T-\lambda I$ is also a hyponormal operator for every complex number λ. Thus $T-\lambda I$ is normaloid for every λ by [1: Corollary 4] and so $\overline{W(T)} = \sum(T)$ by our theorem.

M. Schreiber [5] has shown that if $\sum(T)$ is a spectral set for T, $\sum(T) = \overline{W(T)}$.

COROLLARY 2. *If* $\overline{W(T)}$ *is a spectral set for a bounded operator* T, $\overline{W(T)} = \sum(T)$.

In fact, since $W(T)$ is a spectral set for T, we have

$$\| u_\lambda(T) \| = \| T-\lambda I \| \leqq \sup\{|\mu-\lambda| : \mu \in \overline{W(T)}\} \leqq \| T-\lambda I \|$$

for each rational function $u_\lambda(z) = z-\lambda$, and the conclusion follows.

It is obvious that the spectrality of $\sum(T)$ for T implies the spectrality of $\overline{W(T)}$ for T, but by Corollary 2 the converse implication holds.

The following result is proved in [3] and [5].

COROLLARY 3. *For a Toeplitz operator* T_ϕ, $\overline{W(T_\phi)} = \sum(T_\phi)$.

PROOF. Let L_ψ be a Laurent operator corresponding to a Toeplitz operator T_ψ, it is known $\sigma(L_\psi) \subset \sigma(T_\psi)$. Thus we have

$$\|T_\phi-\lambda I\| = \|T_{(\phi-\lambda)}\| \leqq \|L_{(\phi-\lambda)}\| = \sup\{|\mu-\lambda|: \mu \in \sigma(L_\phi)\}$$

$$\leqq \sup\{|\mu-\lambda|: \mu \in \overline{W(T_\phi)}\} \leqq \|T_\phi-\lambda I\|,$$

and the assertion is true by our theorem.

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