THE TENSOR PRODUCT OF FUNCTION ALGEBRAS

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1. Introduction. We consider the tensor product of two function algebras A and B on compact Hausdorff spaces X and Y, respectively, where by function algebras we shall mean uniformly closed subalgebras of the continuous complex-valued functions which contain the constants and separate the points.

Let $A \odot B$ be the algebraic tensor product of A, B and let $\sum_{i=1}^{n} f_i \otimes g_i \in A \odot B$, then, by $\left(\sum_{i=1}^{n} f_i \otimes g_i\right)(x,y) = \sum_{i=1}^{n} f_i(x)\, g_i(y)$ for $(x,y) \in X \times Y$, $\sum_{i=1}^{n} f_i \otimes g_i$ belongs to $C(X \times Y)$. Let $A \hat{\otimes} B$ be the completion of $A \odot B$ under the λ-norm.*) The λ-norm is identical with the usual uniform norm on $X \times Y$ and $C(X) \hat{\otimes} C(Y) = C(X \times Y)$. Thus, $A \hat{\otimes} B$ is a Banach algebra. Further, it is easily seen that $A \hat{\otimes} B$ becomes a function algebra on $X \times Y$, which we shall denote by $\mathfrak{A}$. Now, it will be natural to ask what properties of A and B are inherited to $\mathfrak{A}$, or conversely. We shall show that the Šilov boundary and the Choquet boundary of $\mathfrak{A}$ are represented exactly as Cartesian products of such subsets. Each Gleason part of the maximal ideal space of $\mathfrak{A}$ is also a Cartesian product of parts of maximal ideal spaces of A, B respectively. Even if both A and B are dirichlet algebras ($A \neq C(X)$, $B \neq C(Y)$), $\mathfrak{A}$ is far from being dirichlet. However, the maximal ideal space of $\mathfrak{A}$ remains to have an analytic structure, where analytic functions of two complex variables on the open unit bicylinder are involved.

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2. Tensor product $A \hat{\otimes} B$. In what follows, we denote by ∂_A, $\mathfrak{M}(A)$ and $M(A)$ the Šilov boundary, the maximal ideal space and the Choquet boundary of A, respectively. The closed unit disk $\{z : |z| \leqq 1\}$ of the complex plane is denoted by D and its interior $\{z : |z| < 1\}$ by D^i. We use the symbol T for the unit circle.

*) Let A^*, B^* be the conjugate spaces of A, B respectively. For $\sum f_i \otimes g_i \in A \odot B$, the λ-norm is defined by $\|\sum f_i \otimes g_i\|_\lambda = \sup|\sum \varphi(f_i)\psi(g_i)|$ where φ, ψ run over the unit balls of A^*, B^* respectively ([13]).

It is known that $\mathfrak{M}(\mathfrak{A})$ is homeomorphic to $\mathfrak{M}(A)\times\mathfrak{M}(B)$, that is, for every $h\in\mathfrak{M}(\mathfrak{A})$ there corresponds a unique $(\varphi,\psi)\in\mathfrak{M}(A)\times\mathfrak{M}(B)$ such that $h=\varphi\otimes\psi$, which means that if $F=\sum_{i=1}^{n} f_i\otimes g_i\in A\odot B$ then $h(F)=\sum_{i=1}^{n}\varphi(f_i)\psi(g_i)$ ([9]). For $F\in\mathfrak{A}$ and for a fixed $y_0\in Y$, we define F_{y_0} by $F_{y_0}(x)=F(x,y_0)$. F_{x_0} is similarly defined. The following is nothing but Lemma 2 of [9].

LEMMA 1. *Let $F\in\mathfrak{A}$. Then every F_{y_0} belongs to A and every F_{x_0} to B.*

PROOF. Let φ_{y_0} be the functional which associates with $g\in B$ the value $g(y_0)$. Then $\varphi_{y_0}\in B^*$. We define a mapping T_{y_0} of $A\odot B$ into A by $T_{y_0}(\Sigma f_i\otimes g_i)=\Sigma g_i(y_0)f_i$ for $\Sigma f_i\otimes g_i\in A\odot B$. T_{y_0} is continuous, so extended to $\mathfrak{A}$. Let $\{\Sigma f_i^{(n)}\otimes g_i^{(n)}\}$ be a sequence such that $\Sigma f_i^{(n)}\otimes g_i^{(n)}\to F$. Then, $T_{y_0}(F)\in A$, and $(T_{y_0}(F))(x)=\lim\Sigma f_i^{(n)}(x)g_i^{(n)}(y_0)=F_{y_0}(x)$ for $x\in X$, which completes the proof.

THEOREM 1.*) $\partial_{\mathfrak{A}}=\partial_A\times\partial_B$.

PROOF. Let $(x_0,y_0)\,\bar{\in}\,\partial_A\times\partial_B$. We assume that $x_0\,\bar{\in}\,\partial_A$. There then exists a neighborhood U of x_0 such that for every $f\in A$ the inequality $\sup_{x\in U}|f(x)|\leqq\sup_{x\in U^c}|f(x)|$ holds. $U\times Y$ is a neighborhood of (x_0,y_0), and for every $F\in\mathfrak{A}$ we have

$$\sup_{U\times Y}|F(x,y)|=\sup_{y\in Y}\sup_{x\in U}|F_y(x)|\leqq\sup_{Y}\sup_{U^c}|F(x,y)|=\sup_{(U\times Y)^c}|F(x,y)|,$$

so $(x_0,y_0)\,\bar{\in}\,\partial_{\mathfrak{A}}$. Conversely, let $(x_0,y_0)\in\partial_A\times\partial_B$. Then for every neighborhood $U\times V$ of (x_0,y_0) we can choose $f\in A$ and $g\in B$ such that $\sup_U|f(x)|>\sup_{U^c}|f(x)|$, $\sup_V|g(y)|>\sup_{V^c}|g(y)|$. We put $F=f\otimes g$. Then, we have $\sup_{U\times V}|F(x,y)|>\sup_{U^c\times V^c}|F(x,y)|$. Also we have $\sup_{U\times V}|F(x,y)|>\sup_{U\times V^c}|F(x,y)|$ and $\sup_{U\times V}|F(x,y)|>\sup_{U^c\times V}|F(x,y)|$. Thus, $\sup_{U\times V}|F(x,y)|>\sup_{(U\times V)^c}|F(x,y)|$, which shows that $(x_0,y_0)\in\partial_{\mathfrak{A}}$.

For the Choquet boundary of a function algebra, the following are equivalent ([2]), p. 325):

*) Some of our results (Theorem 1 and parts of Corollary 1 and Theorem 4) are contained in [3]. We shall state them for the sake of completeness.

1. $x_0 \in M(A)$.

2. *For every neighborhood* U *of* x_0, *there exists* $f \in A$ *such that* $\|f\| \leqq 1$, $|f(x_0)| > 3/4$ *and* $|f(x)| < 1/4$ *for all* $x \bar{\in} U$.

For our present use, we prove the following lemma which seems to be interesting for its own sake.

LEMMA 2. *The following are equivalent*:

1. $x_0 \in M(A)$.

2. *Let* U *be a neighborhood of* x_0. *Then there exists a sequence* $\{f_n\}$ *in* A *such that* $\|f_n\| \leqq 1$, $\lim |f_n(x_0)| = 1$ *and* $\lim |f_n(x)| = 0$ *uniformly for* $x \bar{\in} U$.

PROOF. It is sufficient to show that 1 implies 2. Let $g \in C_R(X)$ be such that $0 \leqq g \leqq 1$, $g(x_0) = 1$ and $g(x) = 0$ for $x \bar{\in} U$. By Lemma 5.1 in [2] applied in this case, we have $\sup\{f(x_0) \mid f \in \Re_e A, f \leqq g\} \geqq 1$. Let $\{\rho_n\}$ be an increasing sequence of positive numbers such that $\rho_n \to 1$ and $\{\varepsilon_n\}$ a decreasing sequence of positive numbers tending to 0. Since $1-(\log \rho_n/\log \varepsilon_n) < 1$ for $n = 1, 2, 3, \cdots$, there exist $h_n' \in \Re_e A$ for which $h_n' \leqq g$ and $h_n'(x_0) > 1 -(\log \rho_n/\log \varepsilon_n)$ hold. We put $h_n = \log \varepsilon_n \cdot (1-h_n')$. Since $h_n \in \Re_e A$, $h_n + ik_n \in A$ for some $k_n \in \Re_e A$, so $f_n = \exp(h_n + ik_n) \in A$. It is easily seen that $\|f_n\| \leqq 1$, $|f_n(x_0)| > \rho_n$ and $|f_n(x)| \leqq \varepsilon_n$ for $x \bar{\in} U$, which completes the proof.

THEOREM 2. $M(\mathfrak{A}) = M(A) \times M(B)$.

PROOF. Let $(x_0, y_0) \in M(\mathfrak{A})$ and let U, V be neighborhoods of x_0, y_0 respectively. Then there exists $\{F^{(n)}\} \subset \mathfrak{A}$ such that $\|F^{(n)}\| \leqq 1$, $|F^{(n)}(x_0, y_0)| \to 1$ and $F^{(n)}(x, y) \to 0$ uniformly for $(x, y) \bar{\in} U \times V$. Put $f_n = F^{(n)}_{y_0}$ and $g_n = F^{(n)}_{x_0}$. f_n and g_n satisfy the condition of Lemma 2, so $x_0 \in M(A)$ and $y_0 \in M(B)$. Let conversely $x_0 \in M(A)$, $y_0 \in M(B)$ and let $U \times V$ be a neighborhood of (x_0, y_0). Let $\{f_n\}$, $\{g_n\}$ be sequences in A, B satisfying the condition of Lemma 2 for U, V. Putting $F_n = f_n \otimes g_n$, it is clear that $\{F_n\}$ also satisfies the condition for $U \times V$, hence $(x_0, y_0) \in M(\mathfrak{A})$.

Given a function algebra A on X, X is always decomposed into maximal antisymmetric closed subsets, that is, $X = \bigcup_\alpha J_\alpha$, where the restriction of A to J_α is an antisymmetric subalgebra of $C(J_\alpha)$ and J_α is the maximal set having this property ([1]).

THEOREM 3.*) *Let* $X=\bigcup_{\alpha} J_{\alpha}$ *and* $Y=\bigcup_{\beta} K_{\beta}$ *be maximal antisymmetric decompositions, then* $X\times Y=\bigcup_{\alpha,\beta}(J_{\alpha}\times K_{\beta})$ *is also the maximal antisymmetric decomposition of* $X\times Y$.

PROOF. First, every $J_{\alpha}\times K_{\beta}$ is an antisymmetric set. In fact, let $F\in\mathfrak{A}$, $F(x,y)=$real for $(x,y)\in J_{\alpha}\times K_{\beta}$, and let (x_0,y_0), (x_1,y_1) be arbitrary points in $J_{\alpha}\times K_{\beta}$. Then $F_{x_0}(y)=$ real for $x\in K_{\beta}$ and $F_{y_0}(x)=$ real for $x\in J_{\alpha}$, therefore they are constant on K_{β}, J_{α}, respectively, and $F(x_0,y_0)=F_{x_0}(y_1)=F_{y_1}(x_1)=F(x_1,y_1)$. This implies that F is constant on $J_{\alpha}\times K_{\beta}$. Hence, there exists a maximal antisymmetric set $\mathfrak{K}$ in $X\times Y$ such that $J_{\alpha}\times K_{\beta}\subset\mathfrak{K}$. Let $p_x\mathfrak{K}$ denote the projection of $\mathfrak{K}$ into X and let $f\in A$, then $f(p_x\mathfrak{K})=(f\otimes 1)(\mathfrak{K})$. If $f|p_x\mathfrak{K}=$real, then f takes on a constant value on $p_x\mathfrak{K}$, so $p_x\mathfrak{K}$ is an antisymmetric set. Similarly, $p_y\mathfrak{K}$ is antisymmetric. From $J_{\alpha}\times K_{\beta}\subset\mathfrak{K}\subset p_x\mathfrak{K}\times p_y\mathfrak{K}$ and from the fact that J_{α}, K_{β} are maximal, we see that $J_{\alpha}=p_x\mathfrak{K}$, $K_{\beta}=p_y\mathfrak{K}$, so $\mathfrak{K}=J_{\alpha}\times K_{\beta}$.

COROLLARY 1. $\mathfrak{A}$ *is antisymmetric if and only if each component is antisymmetric.*

COROLLARY 2. $\mathfrak{A}$ *is an essential algebra if and only if at least one of the components is essential.*

PROOF. Let P and Q be the collections of all one-point antisymmetric subsets of X, Y, respectively. Then, by Theorem 3 in [10], for A or B to be essential it is necessary and sufficient that $X-P^i=X$ or $Y-Q^i=Y$, which is equivalent to $X\times Y-(P\times Q)^i=X\times Y$.

THEOREM 4. $\mathfrak{A}$ *is analytic if and only if each component is analytic.*

PROOF. Let A and B be analytic. If $F\in\mathfrak{A}$, and F vanishes on an open set $U\times V$, then for every $x_0\in U$, we have $F_{x_0}(y)=0$ on V, so $F_x(y)=0$ for all $y\in Y$. Next, let $y\in Y$. For any $x\in U$, we have $F_{y_0}(x)=F_x(y_0)=0$, hence $F_{y_0}(x)=0$ for all $x\in X$. Thus, F vanishes identically. Let, conversely, $\mathfrak{A}$ be analytic, and let U be an open subset of X. If $f\in A$ and $f=0$ on U, then, since $(f\otimes 1)(U\times Y)=f(U)$, we have $f\otimes 1\equiv 0$, so $f=0$ on X. Hence, A is analytic.

Let G, H be compact abelian groups, and let Γ_+, Λ_+ be subsemigroups of $\widehat{G}, \widehat{H}$, such that each contains identity and generates $\widehat{G}, \widehat{H}$ respectively.

*) Results from Theorem 3 to Theorem 4 are due to J. Tomiyama.

We denote by $A(G, \Gamma_+)$, or briefly by $A(G)$, the algebra of all generalized analytic functions on G with respect to Γ_+, that is, $A(G,\Gamma_+) = \{f \mid f \in C(G),$ $\hat{f}(\gamma) = \int_G f(x)\overline{\gamma(x)}\,dx = 0$ for $\gamma \bar{\in} \Gamma_+\}$, where dx denotes the normalized Haar measure on G. $A(H,\Lambda_+)$ is similarly defined. They are function algebras.

THEOREM 5. $A(G,\Gamma_+) \hat{\otimes} A(H,\Lambda_+) = A(G\times H, \Gamma_+ \times \Lambda_+)$.

PROOF. It is clear that $\Gamma_+ \times \Lambda_+$ is a subsemigroup of $(G\times H)^\wedge$ containing the identity of $(G\times H)^\wedge$ and generates $(G\times H)^\wedge$, so $A(G\times H, \Gamma_+\times\Lambda_+)$ is defined as above. For $f\otimes g \in A(G) \odot A(H)$, $(f\otimes g)^\wedge(\gamma,\lambda) = \hat{f}(\gamma)\,\hat{g}(\lambda)$. If $F\in A(G)\odot A(H)$ and $(\gamma,\lambda) \bar{\in} \Gamma_+\times\Lambda_+$, we have $\hat{F}(\gamma,\lambda)=0$, thus $F \in A(G\times H,$ $\Gamma_+\times\Lambda_+)$. Conversely, any $F\in A(G\times H)$ is the uniform limit of trigonometric polynomials consisting of members of $\Gamma_+\times\Lambda_+$ ([8]) and these belong to $A(G)$ $\odot A(H)$, so we have $A(G)\hat{\otimes} A(H) = A(G\times H)$ as desired.

3. Dirichlet algebras and analytic structure. It is well known that, for $\varphi, \varphi' \in \mathfrak{M}(A)$, the relation $\|\varphi-\varphi'\| < 2$ is an equivalence relation ([5]), which we shall denote by $\varphi\sim\varphi'$. Equivalence classes in $\mathfrak{M}(A)$ are called (Gleason) parts. If a part does not reduce to a single point, it is said to be non-trivial. If A is a dirichlet algebra, every non-trivial part P of $\mathfrak{M}(A)$ is the image of a continuous one-to-one mapping τ of D^i and every $f \in A$ has the property that $\hat{f}\circ\tau$ is an analytic function on D^i. In this sense, a sort of analytic structure is shared by $\mathfrak{M}(A)$, and this structure may be considered as complex one dimensional. But, in general, when A is not dirichlet, this is not true, as is easily seen from considering algebras consisting of analytic functions of several complex variables defined on a suitable region.

LEMMA 3. *Let P, Q be parts of $\mathfrak{M}(A)$, $\mathfrak{M}(B)$, respectively, then $P\times Q$ is a part of $\mathfrak{M}(\mathfrak{A})$. Conversely, every part of $\mathfrak{M}(\mathfrak{A})$ is of this form.*

PROOF. For $h, h' \in \mathfrak{M}(\mathfrak{A})$, we have $h = \varphi\otimes\psi$, $h' = \varphi'\otimes\psi'$ where φ,φ' $\in \mathfrak{M}(A)$, $\psi,\psi' \in \mathfrak{M}(B)$. For the proof, it is sufficient to show that $h\sim h'$ if and only if $\varphi\sim\varphi'$ and $\psi\sim\psi'$. Let first $\varphi\sim\varphi'$ and $\psi\sim\psi'$. Then, clearly $\varphi\otimes\psi$ $\sim\varphi\otimes\psi'$, $\varphi\otimes\psi' \sim \varphi'\otimes\psi'$, so $\varphi\otimes\psi\sim\varphi'\otimes\psi'$. We suppose conversely that $\varphi \nsim \varphi'$, say. Since $\|\varphi-\varphi'\| = 2$, there exists a sequence $\{f_n\}\subset A$ such that $\|f_n\| \leqq 1$ and $|\varphi(f_n) - \varphi'(f_n)| \to 2$. Put $F_n = f_n\otimes 1$. Then $\|F_n\| \leqq 1$ and $|h(F_n) - h'(F_n)| \to 2$, which implies that $h \nsim h'$.

THEOREM 6. *Let $\mathfrak{A} = A\hat{\otimes} B$ in which both A and B are dirichlet algebras. If $\mathfrak{P}$ is a non-trivial part of $\mathfrak{M}(\mathfrak{A})$, then $\mathfrak{P}$ is either the image of a one-to-one* *continuous mapping* Φ *of* D^i *or* $D^i \times D^i$ *and for every* $F \in \mathfrak{A}$, $\widehat{F} \circ \Phi$ *is an analytic function on such a region.*

PROOF. By Lemma 3, $\mathfrak{P} = P \times Q$ where P, Q are parts of $\mathfrak{M}(A), \mathfrak{M}(B)$, respectively. In the case where one of P, Q is trivial, $\mathfrak{P}$ is clearly an image of D^i. If both P and Q are non-trivial, there are one-to-one continuous mappings τ and σ which map D^i onto P, Q respectively, in such a manner that $\widehat{f} \circ \tau$, $\widehat{g} \circ \sigma$ are analytic for $f \in A$, $g \in B$. We define Φ by $\Phi(z, w) = (\tau(z), \sigma(w))$ for $(z, w) \in D^i \times D^i$. Φ is clearly a one-to-one continuous mapping onto $\mathfrak{P}$. For $F = \Sigma f_i \otimes g_i \in A \odot B$, we have $(\widehat{F} \circ \Phi)(z, w) = \Sigma(\widehat{f_i} \circ \tau)(z)(\widehat{g_i} \circ \sigma)(w)$, which is an analytic function on $D^i \times D^i$. Since, for every $F \in \mathfrak{A}$, $\widehat{F} \circ \Phi$ is uniformly approximated by such functions on $D^i \times D^i$, $\widehat{F} \circ \Phi$ is also analytic.

THEOREM 7. *If* A *is dirichlet,* $A \widehat{\otimes} C(Y)$ *is dirichlet.*

PROOF. Since $C_R(X) \odot C_R(Y)$ is dense in $C_R(X \times Y)$ as a real algebra, it is sufficient to show that every $\Sigma u_i \otimes v_i$ in $C_R(X) \odot C_R(Y)$ can be approximated by members of $\mathfrak{Re}(A \widehat{\otimes} C(Y))$. But this is easily seen from the fact that if $f_i \in \mathfrak{Re} A$ are chosen to be near to u_i then $\Sigma f_i \otimes v_i$ is in $\mathfrak{Re}(A \widehat{\otimes} C(X))$ and near to $\Sigma u_i \otimes v_i$.

REMARK 1. If $\mathfrak{A}$ is dirichlet, each component is dirichlet. In fact, suppose that A, say, is not dirichlet. There then exists a non-zero real measure μ on X which annihilates A. Choose a non-zero real measure ν on Y. Then, $\mu \times \nu$ annihilates $\mathfrak{A}$, because, for $\Sigma f_i \otimes g_i \in A \odot B$, we have $\int_{X \times X} (\Sigma f_i \otimes g_i) d(\mu \times \nu)$ $= \sum \int_X f_i \, d\mu \cdot \int_Y g_i \, d\nu = 0$. But, clearly $\mu \times \nu \neq 0$, which contradicts the assumption that $\mathfrak{A}$ is dirichlet.

REMARK 2. In connection with Theorem 7, it will be of interest to note that, in the case when $A = A(G)$ and $B = A(H)$, $A = C(X)$ or $B = C(Y)$ provided that $\mathfrak{A}$ is dirichlet. In fact, let $\Gamma_+ \subsetneqq \widehat{G}$, $\Lambda_+ \subsetneqq \widehat{H}$ and let (γ_0, λ_0) be such that $\gamma_0 \overline{\in} \Gamma_+$, $\lambda_0 \overline{\in} \Lambda_+$, then $(\gamma_0, \lambda_0^{-1}) \overline{\in} (\Gamma_+ \times \Lambda_+) \cup (\Gamma_+ \times \Lambda_+)^{-1}$. Thus, $A(G \times H)$ cannot be dirichlet ([8]).

4. Examples. From Theorem 7 we see that many dirichlet algebras are constructed the parts of maximal ideal spaces of which are of the form $P \times$(one-point set), because every part of the algebra $C(Y)$ is a single point.

An example is the algebra $\mathfrak{A}$ generated by polynomials in z, t where (z, t) lies on the cylindrical surface $X = \{(z, t) : |z| = 1, 0 \leqq t \leqq 1\}$. We denote by A_0 the disk algebra which consists of all continuous functions on T which are extended analytically on D^i. It is clear that $\mathfrak{A} = A_0 \hat{\otimes} C(0,1)$. $\mathfrak{M}(\mathfrak{A})$ is the solid cylinder $\{(z, t) : |z| \leqq 1, 0 \leqq t \leqq 1\}$; every point of X is a trivial part and $D^i \times \{t\}$ is another type of part for each t, $0 \leqq t \leqq 1$ (See also [11], p. 88).

Let $\mathfrak{A}$ be the algebra which consists of the continuous functions F on T^2 such that, for every integer n, the function f_n on T defined by $f_n(e^{i\theta}) = \int_0^{2\pi} F(e^{i\theta}, e^{i\varphi}) e^{in\varphi} d\varphi$ has Fourier transform vanishing on the negative integers (([6]), p. 303). Then, we have $\mathfrak{A} = A_0 \hat{\otimes} C(T)$.

Next, let $\boldsymbol{A}_0 = \{F | F \in C(T^2), \int_0^{2\pi} \int_0^{2\pi} F(e^{i\theta}, e^{i\varphi}) e^{im\theta} e^{in\varphi} d\theta\, d\varphi = 0$, for $m > 0$ or $n > 0\}$, and let $\boldsymbol{A}$ be the *bicylinder algebra*, i.e., the algebra of all functions which are continuous on $D \times D$ and analytic on $D^i \times D^i$ ([7], p. 230). $\boldsymbol{A}_0 = A_0 \hat{\otimes} A_0$, and $\hat{\boldsymbol{A}}_0 \cong \hat{A}_0 \hat{\otimes} \hat{A}_0 \subset \boldsymbol{A}$. Conversely, $\boldsymbol{A} | T^2 \subset \boldsymbol{A}_0$ which is seen by considering the limit of Fourier coefficients of $F(re^{i\theta}, re^{i\varphi})$, $F \in \boldsymbol{A}$, for $r \to 1$. Thus $\boldsymbol{A}_0 = \boldsymbol{A} | T^2$ and, since every function of $\boldsymbol{A}$ attains its maximum on the skeleton T^2, $\boldsymbol{A}_0 \cong \boldsymbol{A}$. By Lemma 3, the parts are identified as follows: $D^i \times D^i$ is a part; for every z_0, $|z_0| = 1$, $\{(z_0, w) : |w| < 1\}$ constitutes one part and, similarly, $\{(z, w_0) : |z| < 1\}$ is a part for w_0, $|w_0| = 1$, and every point of T^2 is a trivial part ([11], p. 89).

These are special cases of a typical example, $P(Z)$. Let Z be a compact subset of $\boldsymbol{C}^2$, the product of two complex planes, and let $P(Z)$ be the algebra of all functions on Z which are uniform limits on Z of polynomials in z and w ([11]). If $Z = X \times Y$, X, Y subsets of $\boldsymbol{C}$, then $P(Z) = P(X) \hat{\otimes} P(Y)$. $P(X) = C(X)$ if and only if $X^i = \emptyset$ and X^c is connected ([11], Theorem 7.3). The analogous result holds componentwise for $P(X \times Y)$, that is, $P(X \times Y) = C(X \times Y)$ if and only if $X^i = Y^i = \emptyset$ and X^c, Y^c are connected. In the case of one complex variable, the theorem of Walsh states that if X is a compact subset of $\boldsymbol{C}$ with X^c connected and if X_0 is the boundary of X then every continuous real function on X_0 can be uniformly approximated by real parts of polynomials in z, so $P(X_0)$ becomes a dirichlet algebra. That this does not hold in the case of two complex variables can be seen as follows: Let $Z = D \times D$ and Z_0 be the boundary of Z. Since $P(T) = A_0$, we have $P(T^2) = \boldsymbol{A}_0$, hence $P(Z_0) = \boldsymbol{A} | Z_0$. But A_0 is not dirichlet, hence $\boldsymbol{A} | Z_0$ is also not dirichlet. Besides these, there are properties of A_0 which does not remain to hold in the case of $\boldsymbol{A}_0$. A_0 is maximal, but $\boldsymbol{A}_0$ is no longer. The Rudin-Carleson theorem holds for A_0, that is, if E is a closed subset of T of Lebesque measure zero then $A_0 | E = C(E)$. This fails to hold in $\boldsymbol{A}_0$. In fact, let x_0 be a fixed point of

T and $K=T$. Then $K\times\{x_0\}$ is a closed subset of T^2 and $(\mu\times\mu)(K\times\{x_0\})=0$, where $d\mu = d\theta/2\pi$. Let K_0 be a closed subset of K such that $K_0 \subsetneqq K, \mu(K_0) > 0$. Let f be a continuous function on $K\times\{x_0\}$ such that $f\neq 0, f|(K_0\times\{x_0\}) = 0$. For any $F\in \boldsymbol{A}_0$, F_{x_0} belongs to A_0. Hence, if $F_{x_0}=f$, then F_{x_0} must vanish identically on $K\times\{x_0\}$ by the theorem of F. and M. Riesz, which contradicts the choice of f. Thus, $\boldsymbol{A}_0|(K\times\{x_0\})\neq C(K\times\{x_0\})$.

On the other hand, the F. and M. Riesz theorem holds in the following sense ([12]), p. 321).

THEOREM 8. *If $F\in \boldsymbol{A}_0$ vanishes on a closed subset of the torus of positive measure, then $F=0$.*

PROOF. Let K be a closed subset of T^2, $(\mu\times\mu)(K)>0$, and let F vanish on K. We denote by K_x the x-section of K, i.e., $K_x = \{y|(x,y)\in K\}$. Let $E=\{x\in T|\mu(K_x)>0\}$, then $\mu(E)>0$. By regularity of μ, we may assume that E is closed. Every K_x is closed. For arbitrarily fixed $x\in E$, $F_x\in A_0$ and $F_x(y)=0$ for $y\in K_x$. By the F and M. Riesz theorem for $A_0, F_x=0$ on T. Thus, $F(x,y)=0$ for $(x,y)\in E\times T$. Hence, $F_y=0$ on E, for every $y\in T$, so $F_y=0$ on T. Thus, $F(x,y)=0$ on T^2.

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